**Worksheet #5: The Heroic Age of Polar Exploration**
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**Math 53: Fall 2022**
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**Problem 1.** By now you've seen plane curves described in a number of different ways (as functions $y = f(x)$, as solutions to more general Cartesian equations, as parametric curves, as polar curves), and often you are asked to graph these curves. One of the most common questions I get asked is how one should go about graphing a curve by hand. Sometimes it's hard to see the forest through the trees.

(Discussion.) Why do we care about graphing curves? If you're given equations (rectangular, parametric, polar, etc) for a curve that you've never seen before, how should you go about graphing it by hand? What should your graph use as guidelines, and what should it emphasize?

For graphing, here are some points that people thought were useful/important:

1. Making a table of values and plotting a bunch of points usually gives you a rough idea of the shape of the graph. This always works when one variable is a function of another variable, but it tends to take a long time and can miss out on some important features.

2. Any symmetries of the defining function should be reflected somehow in the graph. By "symmetry," I mean any type of transformation under which the function/graph remains unchanged, or changed in a simple way. For example, if $f(x) = y$ for an even function $f$, then the graph should be symmetric with respect to reflection across the $y$-axis, so you only need to figure out how it looks for $x \geq 0$. Another example of a symmetry: if $r = f(\theta)$ and $f$ has period $\theta_0$, then the (Cartesian) graph should look the same when you rotate it by angle $\theta_0$. $f$ could also be *anti-periodic* with anti-period $\theta_0$, which means that $f(\theta + \theta_0) = -f(\theta)$; in this case, rotating the graph by angle $\theta_0$ gives you the same thing as rotating by 180 degrees. For example, $\cos(4\theta)$ has anti-period $\theta_0 = \pi/4$, so you should be able to rotate it by $\pi + \pi/4$ and get the same thing. This means that it suffices to just draw a single "leaf" of the graph, which can be rotated around by multiples of $5\pi/4$ to get the entire group. Understanding how to translate the symmetries of a function into the symmetries of the graph (rectangular or polar) will go a long way towards drawing good graphs and understanding these functions.

3. Give emphasis to points are somehow distinguished or special. These might include: points where the graph intersects itself, the axes, or another graph; points where the tangent is horizontal or vertical; inflection points; points of non-differentiability. It's good to have first and second derivatives handy in order to be able to find these points.

4. Asymptotes or other types of limits at infinity or points where the function is not defined.

5. Using inflection points, you can determine where the graph should be concave up or concave down.

There are more things to add to this list. Ultimately, we draw graphs of functions and curves because it can give us a lot of important information about this function/curve at a glance. Important information might include any of the points above, which is why they are important to emphasize while drawing. It also helps to have a visual aid to determine where pieces of a graph are relative to each other, e.g. when trying to find the area bounded by given curves. Having a sort of sanity check is also important: graphing might allow you to see that the result of a computation is obviously incorrect, so you can check your work.

As with other things in math, graphing requires practice, especially under the pressure of an exam. You won't always have the time or energy to check for all of the items mentioned in the list, but instead certain things (e.g. a symmetry, an asymptote) should jump out at you once you have enough practice, and you can focus on those.

**Problem 2.** Give an informal explanation why the length form is $ds = \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta$ in polar coordinates. In other words, why should we expect this to be the right formula for the infinitesimal arc length assocated to an infinitesimal change in $\theta$?

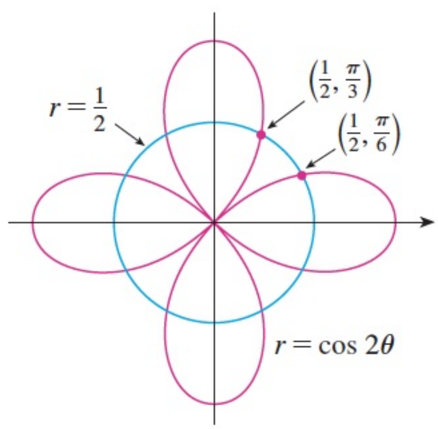
At a given point $(r, \theta)$ on the curve, changing $\theta$ by a small amount $d\theta$ moves the curve approximately $rd\theta$ in the direction of rotation and $\frac{dr}{d\theta}d\theta$ in the radial direction. These two directions are perpendicular, so drawing a right triangle the length of the tiny piece of the curve coming from this change $d\theta$ is $\sqrt{r^2 + \frac{dr}{d\theta}} d\theta$.

**Problem 3.** (Based on Stewart, Example 3 in Chapter 10.4)

(a) Sketch graphs of the two curves $r = \cos 2\theta$ and $r = \frac{1}{2}$ on top of one another. (Hint: the curve $r = \cos 2\theta$ is the "four-leaf rose," which you saw in lecture yesterday.)

(b) Find all points of intersection between these two curves. (Why do you get more than you might expect from just solving algebraically?)

(c) Find the area of the intersection of the regions enclosed by these two curves. (Hint: symmetry. You'll still need to compute the area of two separate pieces and add them up. Justify your bounds of integration.)




(a)

(b) $\frac{1}{2} = \cos 2\theta$ has solutions $\theta = \pi/6, 5\pi/6, 7\pi/6$ and $11\pi/6$, which in polar coordinates gives points of intersection $(1/2, \pi/6)$, $(1/2, 5\pi/6)$, $(1/2, 7\pi/6)$ and $(1/2, 11\pi/6)$. We also get solutions for $\theta$ differing from the given ones by a multiple of $2\pi$, but these represent the same point in polar coordinates. However, there is another way two polar coordinates can represent the same point: recall that $(-r, \theta)$ represents the same point as $(r, \theta + \pi)$. So we need to also solve $-1/2 = \cos 2\theta$, which has solutions $\theta = \pi/3, 2\pi/3, 4\pi/3, 5\pi/3$ corresponding to the points $(-1/2, \pi/3), (-1/2, 2\pi/3), (-1/2, 4\pi/3, (-1/2, 5\pi/3)$ in polar coordinates, which is the same as

$(1/2, 4\pi/3), (1/2, 5\pi/3), (1/2, \pi/3), (1/2, 2\pi/3)$. We don't get any new points by considering multiples of $2\pi$, so there are 8 points of intersection in total.

(c) By symmetry, the total area is 8 times the area between $0 \le \theta \le \pi/4$. On $0 \le \theta \le \pi/6$, the circle is shorter than the rose, and on $\pi/6 \le \theta \le \pi/4$, the rose is shorter than the circle. This means we should use the circle for the radius on $0 \le \theta \le \pi/6$ and the rose for the radius on $\pi/6 \le \theta \le \pi/4$ (where there is only a tiny sliver of area!). Therefore, using the formula for area in polar coordinates, the area is given by

$$\begin{aligned}
\frac{A}{8} &= \int_0^{\pi/6} \frac{1}{2}(1/2)^2 d\theta + \int_{\pi/6}^{\pi/4} \frac{1}{2}(\cos^2(2\theta))d\theta \\
&= \frac{\pi}{48} + \frac{1}{4}\int_{\pi/6}^{\pi/4} (1+\cos(4\theta))d\theta \\
&= \frac{\pi}{48} + \frac{1}{4}[\theta + \frac{1}{4}\sin(4\theta)]\Big|_{\pi/6}^{\pi/4} \\
&= \frac{\pi}{48} + \frac{1}{4}\left[\frac{\pi}{12} - \frac{1}{4}\frac{\sqrt{3}}{2}\right] \\
&= \frac{\pi}{24} - \frac{\sqrt{3}}{32}.
\end{aligned}$$

Therefore $A = \pi/3 - \sqrt{3}/4$.