Worksheet #4: Polarized
Date: 09/02/2022
Math 53: Fall 2022
Instructor: Norman Sheu
Section Leader: CJ Dowd

Problem 1. Determine polar equations for the following curves:

(a) The circle of radius a centered at the origin.

(b) The vertical line $x = 1$.

(c) The line $x = y$.

(d) The parabola $x^2 = y$.

(a) $r = a$ or $r = -a$

(b) $r = \sec\theta$

(c) $\theta = \pi/4$, or $\theta = 5\pi/4$.

(d) $r = \tan\theta\sec\theta$.

Problem 2. Recall our favorite parametric curve, the astroid $x = \cos^3 t, y = \sin^3 t$. Since $r = \sqrt{x^2 + y^2}$, you might think that polar coordinates for the astroid are given by

$$r = \sqrt{\cos^6\theta + \sin^6\theta}$$

but this is in fact *false*. Why is this incorrect?

Despite being an input to a trig function, t isn't actually the same as the angle θ. The actual relationship between t and θ is

$$\theta = \tan^{-1}(\tan^3 t).$$

Problem 3. When we have rectangular coordinates for a curve, it's relatively easy to shift the curve vertically or horizontally: to shift the curve h units to the right and k units upward, just substitute $x - h$ for x and $y - k$ for y. In parametric coordinates $(x, y) = (f(t), g(t))$, this shift is given by $(x, y) = (f(t) + h, g(t) + k)$. Think about why it is, in general, difficult to describe a translation in polar coordinates. If you want something more concrete to chew on, try coming up with polar coordinates for the parabola $y = x^2 + 1$ and see how messy it gets compared to your answer to 1(d).

The main issue is that constant changes in x and y lead to variable changes in r and θ that are tricky to describe for the whole plane simulteneously. You're trying to solve for a new angle φ and a new radius ρ in terms of θ, r such that

$$\rho\cos\varphi = h + r\cos\theta$$
$$\rho\sin\varphi = k + r\cos\theta,$$

which looks very unpleasant for arbitrary values of h and k.

Problem 4. Derive the formula for the slope of a line tangent to a polar curve $r = f(\theta)$:

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

(Hint: start by writing $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ using the chain rule, and then use the formulas for x and y in terms of r and θ.) This is an example of a formula I'd recommend knowing how to re-derive if necessary rather than memorizing it.

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} \\ &= \frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} \\ &= \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta},\end{aligned}$$

where in the last step we use the product rule and remember that we are treating r as a function of θ.