Worksheet #3: Hearts and Diamonds
Date: 08/26/2022
Math 53: Fall 2022
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Problem 1. True or false: if $x(t)$ and $y(t)$ are twice-differentiable and $x''(t) \neq 0$, then

$$\frac{d^2y}{dx^2} = \frac{d^2y/dt^2}{d^2x/dt^2}.$$

This is false. In general you cannot simply "cancel" derivatives like fractions; this happens to work out in the case of applying the chain rule for a first derivative, but not in general. The correct expression is

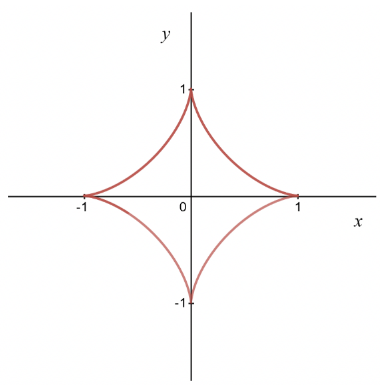
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

which you can verify by noting that $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$ (by definition) and then applying the chain rule. You can also expand the expression above using the formula for $\frac{dy}{dx}$ to get an expression entirely in terms of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

Problem 2.

(a) Express the length of the curve $(x(t), y(t)), t \in [a, b]$ as an integral, i.e. give a general formula for computing arc length of a parametric curve.

(b) (Stewart 10.2.54.) Compute the arc length of the curve $(x, y) = (\cos^3(t), \sin^3(t)), t \in [0, 2\pi]$. This shape is known as an *astroid*, and it looks like this:



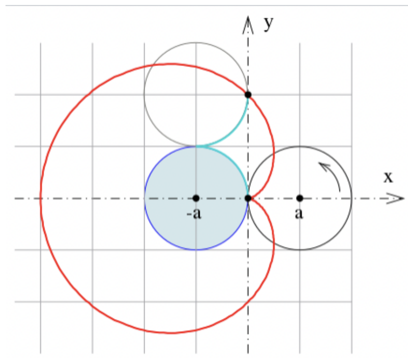
(a) The length formula is $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$. You should think of $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$ as the "infinitesimal length;" make sure this interpretation makes sense to you.

(b) The length is 6. This computation was done as part of the lecture on 08/30/2022, so I won't repeat it here; see the lecture notes.

Problem 3. Recall from lecture that a *cycloid* is curve that can be described parametrically as the motion of a point on the circumference of a wheel rolling along the ground. A *cardioid* is what you get if the wheel rolls along another circle instead of the ground.

(a) Suppose that both circles have radius a, and their centers start at $(-a, 0)$ and $(a, 0)$ as shown in the diagram (so the parametric curve starts at the origin). Derive parametric equations for the cardioid. (Hint: think about the outer circle's rotation in a vacuum. It rotates at a constant rate; how does this rate compare to its rate of revolution about the inner circle?)

(b) (Messy, optional.) Prove that the arc length of the cardioid is $16a$.



(a) To break down this problem into manageable pieces, it's best to think the revolution of the outer circle about the inner circle and the rotation of the outer circle around itself separately. We'll find parametric equations for the center of the outer circle, and then we'll find parametric equations for the displacement of the curve relative to this center, and so adding these two together should give parametric equations for the net motion.

The center of the outer circle moves in a circle of radius $2a$ about the point $(-a, 0)$, so it has parametric equations given by

$$x_{\text{center}} = -a + 2a\cos t$$
$$y_{\text{center}} = 2a\sin t,$$

with the parameter t corresponding to the angle the center has orbited around the inner circle. Observe that when the center has revolved angle t, the outer circle has rotated by angle $2t$: looking at the picture, when the outer circle has made a quarter revolution about the inner circle, it has already rotated by 180 degrees. Therefore, the displacement of the curve relative to the center $(x_{\text{center}}, y_{\text{center}})$ is given by a circular motion of radius a and twice the rotational speed as before. Noting that the starting point starts at angle π in the outer circle, this gives parametric equations

$$x_{\text{rel}} = a\cos(2t + \pi) = -a\cos(2t)$$
$$y_{\text{rel}} = a\sin(2t + \pi) = -a\sin(2t).$$

Therefore, the parametric equations for the cardioid are given by

$$x = x_{\text{center}} + x_{\text{rel}} = -a + 2a\cos t - a\cos(2t)$$
$$y = y_{\text{center}} + y_{\text{rel}} = 2a\sin t - a\sin(2t).$$

Using the double angle formulas, this can be also be written as

$$x = 2a(1 - \cos t)\cos(t)$$
$$y = 2a(1 - \cos t)\sin(t).$$

(b) We need to integrate over one revolution, which corresponds to $0 \leq t \leq 2\pi$. We have

$$\frac{dx}{dt} = -2a\sin t + 2a\sin(2t)$$
$$\frac{dy}{dt} = 2a\cos t - 2a\cos(2t).$$

The integral is essentially one big exercise in applying the double angle formulas:

$$\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
&= \int_0^{2\pi} \sqrt{(-2a\sin t + 2a\sin(2t))^2 + (2a\cos t - 2a\cos(2t))^2} dt \\
&= 2a\int_0^{2\pi} \sqrt{\sin^2 t - 2\sin(t)\sin(2t) + \sin^2(2t) + \cos^2 t - 2\cos(t)\cos(2t) + \cos^2(2t)} dt \\
&= 2a\int_0^{2\pi} \sqrt{2 - 2\sin(t)\sin(2t) - 2\cos(t)\cos(2t)} dt \\
&= 2a\int_0^{2\pi} \sqrt{2 - 4\sin^2(t)\cos(t) - 2\cos^3(t) + 2\cos(t)\sin^2(t)} dt \\
&= 2a\int_0^{2\pi} \sqrt{2 - 2(\sin^2(t) + \cos^2(t))\cos(t)} dt \\
&= 2a\int_0^{2\pi} \sqrt{2 - 2\cos(t)} dt.
\end{aligned}$$

To evaluate this last integral, we again apply the double angle formula to write $1 - \cos(t) = 2\sin^2(t/2)$:

$$\begin{aligned}
&= 2a\int_0^{2\pi} \sqrt{4\sin^2(t/2)} dt \\
&= 4a\int_0^{2\pi} |\sin(t/2)| dt \\
&= 4a\int_0^{2\pi} \sin(t/2) dt
\end{aligned}$$

noting that $\sin(t/2)$ is nonnegative for $0 \leq t \leq 2\pi$,

$$\begin{aligned}
&= -8a\cos(t/2)\Big|_0^{2\pi} \\
&= 16a.
\end{aligned}$$