Worksheet #2: Call the parametrics
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Instructor: Norman Sheu
Section Leader: CJ Dowd

Problem 1. For many parametric curves $(x, y) = (f(t), g(t))$, we cannot express y as a function of x or vice versa; give an example. Despite this, the derivative $\frac{dy}{dx}$ at a given point $(x, y) = (a, b)$ on the curve usually makes sense if f, g are differentiable unless $\frac{dx}{dt} = f'(t) = 0$ at $x = a$. Discuss how this is possible.

Taking $(x, y) = (\cos(t), \sin(t))$ gives a circle, which fails both the vertical and horizontal line test. However, we can always "zoom in" on the parametric curve to look at a small piece of it to try to treat y locally as a function of x, ignoring the parts of the graph that cause it to fail the vertical line test. This is what we mean when we say dy/dx in this context, and the resulting tangent line coincides with the one we get from considering the tangent vector $(f'(t), g'(t))$. However, if $f'(t) = 0$, then the tangent line might be vertical, and we still might not be able to express y as a function of x no matter how far we zoom in. If you want to see this for yourself, think about the points on the circle where the tangent is vertical.

Problem 2. An ant is crawling on a table. Its position at time t is given by $(x, y) = (f(t), g(t))$. What is the ant's speed at any given time?

The velocity vector of the ant is $(f'(t), g'(t))$, so to get speed we take the magnitude of this, which is $\sqrt{f'(t)^2 + g'(t)^2}$.

Problem 3. For a parametric curve $(x, y) = (f(t), g(t))$, $t \in \mathbb{R}$, describe another parameterization that gives the same curve. And another. And another...

Replacing t by any function of t that takes on all real values gives the same curve. Some examples:

- $(f(at), g(at))$ for any nonzero real number a. This gives the same parametric curve, but it is traced out $|a|$ times as fast, and backwards if a is negative.
- $(f(t+a), g(t+a))$ for any real number a. This is the same as starting to draw earlier or later.
- $(f(t^3), g(t^3))$ is a more complicated example. We are drawing the same curve, but the rate varies as t does: it will be a lot faster when $|t|$ is large, and when t gets close to zero it slows down. At $t = 0$, we "instantenously pause," which you can verify by computing the velocity vector using the chain rule.

Some non-examples:

- $(f(t) + 1, g(t) + 1)$ shifts the entire curve up and to the right by 1 unit each, which usually gives a different curve.
- If we parameterize with $(f(t^2), g(t^2))$, then the inputs to f and g no longer range over all real numbers, but only positive real numbers. This means that we might not get the entire curve. For example, $f(t) = g(t) = t$ yields the line $x = y$, but $f(t) = g(t) = t^2$ only gives the portion of this line contained in the first quadrant (and the origin).

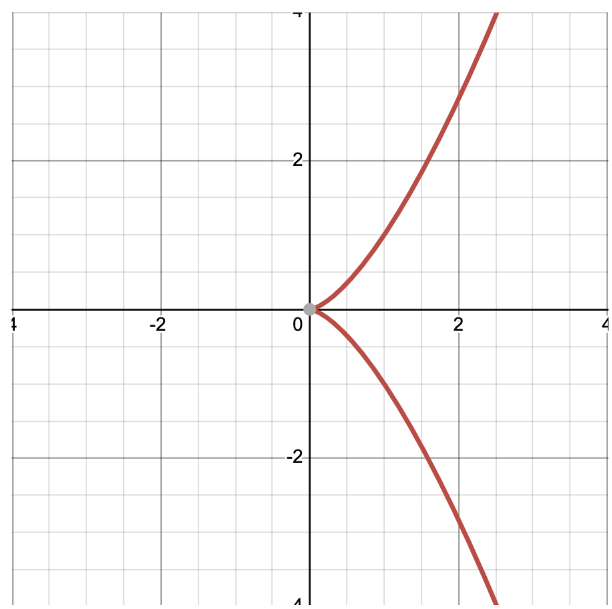
In the situation where t ranges over some interval $[a, b]$, a reparameterization needs to stay within the interval and take on every value in the interval. For example, if t rangest over $[0, 1]$, then t^2 also ranges over $0, 1$, so in this case $(f(t), g(t))$ *does* give the same curve as $(f(t^2), g(t^2))$. However, replacing t with $t+1$ gives something different.

Problem 4. The *cuspidal cubic* is the curve $\{x^3 = y^2 : x, y \in \mathbb{R}\}$, i.e. the set of points (x, y) for which $x^3 = y^2$ and x and y are real numbers.

(a) Draw the cuspidal cubic. (Hint: take a square root.) Is y a function of x?

(b) Come up with functions $x = f(t), y = g(t)$ to parameterize the cuspidal cubic. (You might want to try part (b) first if you are struggling with part (a).)

(c) Does $\frac{dy}{dx}$ exist at $(x, y) = (0, 0)$? What about $\frac{dx}{dy}$? How is this reflected visually in the graph?

(a) 

To figure out how to draw this without using a graphing calculator or plotting many points, you can take square roots to write this curve as $\pm x^{3/2} = y$. The graph of $x^{3/2} = y$ looks similar to the parabola $x^2 = y$, but it's a little shallower and we can't have negative values of x. The graph of $-x^{3/2} = y$ is the same thing mirrored over the y-axis. Combining the two pieces gives the curve.

(b) Just by guessing, you might be able to come up with $x = t^2, y = t^3$. Alternatively, you can note that x is a function of y, so take cube roots to get $x = y^{2/3}$, so setting $y = t, x = t^{2/3}$ gives our curve. As we noted in Problem 3, replacing t by t^3 gives the same curve, so $x = t^{2/3}, y = t$ is the same curve as $x = t^2, y = t^3$.

(c) Neither derivative exists. You should expect this visually by noting the distinctive cusp in the graph. If you try to do this algebraically using a parameterization, you will either get $\frac{dy/dt}{dx/dt} = 0/0$ or one of the functions won't be differentiable.

Problem 5. Let $(x, y) = (f(t), g(t))$ be a parametric curve. Is it possible for $\frac{dy}{dx}$ to exist even at a point where $\frac{dx}{dt} = 0$?

Yes. Consider $x = t^3, y = t^3$. The corresponding curve is the line $y = x$, and clearly $\frac{dy}{dx}\Big|_{(x,y)=(0,0)} = 1$. What is happening is that the parameterization slows down and "stops instantaneously" when it reaches the origin, so it's not meaningful to extract information about the tangent line by considering the velocity vector.