Worksheet #1: Review of Calculus
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Problem 1. What is the definition of a derivative? If $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable functions, why is it true that

$$(f+g)'(x) = f'(x) + g'(x)?$$

The formal defintion of the derivative is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

A function f is said to be differentiable at x when this limit exists. For the second part, write

$$\begin{aligned}(f+g)'(x) &= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x),\end{aligned}$$

where the step between the first and second lines is justified by the fact that the sum of limits is the limit of the sum, assuming all limits exist.

Problem 2. Compute the derivatives with respect to x of the following functions:

(a) $\sin(x)\cos(x)$ Use the product rule to obtain $\cos(x)\cos(x) + \sin(-\sin(x)) = \cos^2(x) - \sin^2(x)$. You can also use trig identities: $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$, which has derivative $\cos(2x)$, which is equal to the previous answer by the double angle formula.

(b) 2^{x^2} Use the chain rule and the fact $\frac{d}{dx}2^x = \ln(2)2^x$ to obtain the derivative $\ln(2)2^{x^2} \cdot 2x$.

(c) $\frac{x}{\sqrt{x^2+1}}$ Use the quotient rule to get

$$\frac{d}{dx}\frac{x}{\sqrt{x^2+1}} = \frac{\sqrt{x^2+1} - x \cdot \frac{x}{\sqrt{x^2+1}}}{x^2+1}.$$

Multiply numerator and denominator by $\sqrt{x^2+1}$ to simplify the answer to $(x^2+1)^{-3/2}$.

Problem 3. What is the definition of the definite (Riemann) integral $\int_a^b f(x)dx$? What is the definition of the indefinite integral $\int f(x)dx$?

The indefinite integral $\int f(x)dx$ is a class of functions $F(x)$ such that $F'(x) = f(x)$; there is more than one such function since changing F by a constant does not change its derivative, so we usually add a $+C$ to account for all possible antiderivatives.

The definite integral $\int_a^b f(x)dx$ is the signed area bounded by the x-axis and the vertical lines $x = a, x = b$ (here "signed area" means that area is considered negative if it is below the x-axis). Formally, this integral is the limit of a Riemann sum

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f\left(a + (b-a)\frac{i}{n}\right).$$

Technically, the above is the left Riemann sum; there are many ways to partition the interval into infinitesimal pieces and evaluate the area of each piece. However, if f is integrable, all methods will lead to the same result.

Problem 4. State both parts of the Fundamental Theorem of Calculus and conceptually/pictorially explain why they are true.

The first part of the FToC states that

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

To make intuitive sense of this, imagine the function $F(x) = \int_a^x f(t)dt$ scanning out the area under the curve as x increases at a constant rate. The rate at which new area is added to the curve at any given moment is given by the height of f, i.e. $f(x)$, so this i

The second part of the FToC states that if $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

When you take the definite integral of f, you are adding up the infinitesimal change of F as you go travel through the interval $[a, b]$. The net change should be the sum of all the little changes, so it makes sense that the integral turns out to be the net change of F from point a to point b, i.e. $F(a) - F(b)$.

Problem 5. Compute the following indefinite integrals:

(a) $\int \frac{dx}{x\ln(x)}$ Perform a u-substitution with $u = \ln(x), du = dx/x$:

$$\int \frac{dx}{x\ln(x)} = \int \frac{du}{u} = \ln(u) + C = \ln(\ln(x)) + C.$$

(b) $\int ye^y dy$ Integrate by parts with $u = y, dv = e^y dy$:

$$\int ye^y dy = ye^y - \int e^y dy = ye^y - e^y + C.$$

(c) $\int \frac{1}{1+u^2} du$ The integrand is the derivative of $\arctan(u)$.

(d) $\int \frac{\sqrt{z^2-1}}{z} dz$ (Hint: Trig substitution)

Substitute $\sec\theta = z, dz = \tan\theta \sec\theta d\theta$. The integral becomes

$$\begin{aligned}
\int \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta} \tan\theta \sec\theta d\theta &= \int \sqrt{\tan^2\theta} \tan\theta d\theta \\
&= \int \tan^2\theta d\theta \\
&= \int (\sec^2\theta - 1) d\theta \\
&= \tan(\theta) - \theta + C \\
&= \tan(\operatorname{arcsec}(x)) - \operatorname{arcsec}(x) + C.
\end{aligned}$$

You can simplify $\tan(\operatorname{arcsec}(x))$ by considering the right triangle with hypotenuse x and side adjacent to the angle θ of length 1, so that $\sec(\theta) = x$. Tangent is opposite over adjacent, which by the Pythagorean theorem is $\sqrt{x^2-1}/1$. So the simplified answer is $\sqrt{x^2-1} - \operatorname{arcsec}(x) + C$.