Notes on Haar measures on Lie groups

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April 6, 2023

1 Introduction

Haar measures can often seem mysterious. From what I've heard from my peers, a proper study of the Haar measure is notorious for being omitted from most courses. It is a useful construction with many relevant applications, but instead of developing any of the theory, the instructor cites whatever facts are relevant for the application at hand and then moves on. Omitting the existence and uniqueness proof is probably a correct decision, as it requires some genuinely involved measure theory that is difficult to condense. However, plenty of other very useful facts about the Haar measure are straightforward to prove.

My interest in this topic is in its role in the theory of automorphic forms and representaions. I am currently reading on Bump's *Automorphic Forms and Representations*, Chapters 2 and 3, which make use primarily of the Lie group case for $\mathrm{GL}_2(\mathbb{R})$ or one of its subgroups. My goal with this short set of notes is to flesh out some simple results and computations about the Haar measure, focusing about providing basic results about Haar measures on Lie groups, since the Haar measure is easier to think about when it is given by a volume form, and I don't do differential geometry very often. There will be a sequel to this when I reach Chapter 4, which focuses on the p-adic side of the GL_2 representation theory.

Like most other things I post, I'm writing this to solidify my own understanding, so don't expect an incredible exposition here. But if you read this and find it helpful please let me know, likewise if you find any errors or typos.

2 Facts about the Haar measure

2.1 Computing the Haar measure

Theorem 2.1.1: Existence and uniqness of Haar measure

Let G be a locally compact Hausdorff topological group. Then there exists a nontrivial regular locally finite left-invariant Borel measure on G. This measure, which we call the left *Haar measure*, is unique up to multiplication by a positive constant.
Analogously, there exists a unique right Haar measure up to a constant.

Upacking some of the terminology:

- A locally finite measure is one that assigns a finite measure to any compact set.
- Nontrivial simply means that not every Borel set has measure 0.
- Left-invariant means that for any $g \in G$ and Borel set $U \subseteq G$, the measure μ satisfies $\mu(U) = \mu(gU)$. Right-invariant means $\mu(U) = \mu(Ug)$.
- Regularity means that that for any Borel set S we have

$$\mu(S) = \inf\{\mu(U) : S \subseteq U \text{ open}\} = \sup\{\mu(K) : \text{ compact } K \subseteq U\}$$

We won't discuss this in general here, but we can identify the Haar measure explicitly for Lie groups. A Haar measure on a Lie group is given by some volume form, so we can work in coordinates to determine it.

Let G be a Lie group. If we choose local coordinates around some $g \in G$, the Haar measure is given by some volume form $d_L g$. Let $\lambda(h)$ denote left-translation by a fixed element $h \in G$. This is a smooth automorphism on G. Left-invariance of the Haar measure means that it is invariant under pullback by $\lambda(h)$, up to signs: $|\lambda(h)^* d_L g)_{h'}| = |(d_L g)_{h'}|$ for any $h' \in G$. (The absolute value bars are necessary since a measure is positive, whereas a volume form may be negative.) This is, in particular, true when h' is the identity, so this computes the Haar measure at h relative to the Haar measure at e.

To make this more explicit, let $\mathbf{x}$ be a system of local coordinates around e and $\mathbf{y}$ a system of local coordinates around h, with respective local volume forms $d\mathbf{x}$ and $d\mathbf{y}$. The Haar measure is locally of the form $E(\mathbf{x})d\mathbf{x} = d_L g$ near, and of the form $H(\mathbf{y})\, d\mathbf{y} = d_L g$ near h. We may normalize the Haar measure so that $E(e) = 1$. In these coordinates, map $\lambda(h)$ has differential that looks like

$$d\lambda(h) = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}$$

We identify $T_e^* G \cong \mathbb{R}^n$ and $T_h^* G \cong \mathbb{R}^n$ via their respective coordinate systems. Under these identifications, the pullback $\lambda(h)^* : \bigwedge^n T_h^* G \to \bigwedge^n T_e^* G$ on volume forms is computed in local coordinates via multiplication by the determinant $\det d\lambda(h)$ of this matrix. This determinant is some function depending on the element h. Therefore, left-invariance $|\lambda(h)^* d_L g| = |d_L g|$ means that we must have

$$|H(\mathbf{y}) \cdot \det d\lambda(h)| = E(\mathbf{x}).$$

In particular, if we've normalized the Haar measure so that $E(e) = 1$, this means that we may take the Haar measure at h to be given in local coordinates by

$$|\det d\lambda(h))^{-1}| d\mathbf{y} = d_L g.$$

We may do the same thing for right-translation: letting ρ denote right-multiplication, we have

$$|\det d\rho(h)|^{-1} d\mathbf{y} = d_L g$$

where $d\rho(h)$ is some matrix that we can identify explicitly after taking local coordinates. Instead of using the identity as our base point, we could just as well use any other element of G.

Computing a Haar measure explicitly amounts to:

1. Finding a coordinate system for G and translating the group law into this coordinate system.
2. Doing some multivariable calculus. The left (or right) Haar measure at $h \in G$ is given in locally coordiantes by $|\det \lambda(g)|^{-1}$ (or $|\det \rho(g)|^{-1}$) times the chosen local volume form.

If you are someone who does more differential geometry than me, all of these things are probably obvious to you. But if you're like me and you spend your days mucking about with objects that are primarily algebraic or arithmetic, thinking things through and working them out explicitly in terms of local coordinates can be helpful.[1] We will do many examples in the next section.

2.2 Unimodular groups

Definition: Unimodularity

We say that a locally compact group G is *unimodular* if its left and right Haar measures agree.

[1] And considering I have a qualifying exam coming with Lie groups as a minor topic, I need all the practice I can get...

Unimodularity is nice to have since thinking about right vs. left Haar measures is a headache. Our previous discussion about local coordinates gives us the following nice criterion:

Proposition 2.2.1: Criterion for unimodularity

A Lie group G is unimodular if and only if

$$|\det \operatorname{Ad} g| = 1,$$

for all $g \in G$, where $\operatorname{Ad} : G \to \operatorname{End}(\mathfrak{g})$ is the adjoint representation of G. If G is connected, we can remove the absolute value bars, and this is also equivalent to $\operatorname{tr} \operatorname{ad} X = 0$ for all $X \in \mathfrak{g}$.

Proof. Recall that the adjoint $\operatorname{Ad} g$ is given by taking the differential of the map $h \mapsto ghg^{-1}$ at the identity, so $\det \operatorname{Ad} g = \pm 1$ is equivalent to saying $\det \operatorname{Ad} g = (\det \operatorname{Ad} g)^{-1} = \det \operatorname{Ad} g^{-1}$. Taking advantage of the fact that left and right multiplication commute with each other, we can compute the adjoint action in different ways—one by doing the left multiplication first, and the other by doing right multiplication first. This means we can equate

$$\det \operatorname{Ad} g = \det d\rho(g^{-1})_g \cdot \det d\lambda(g)_e = \det \lambda(g^{-1})_e \cdot \det d\rho(g)_e = \det \operatorname{Ad} g^{-1}.$$

Noting that we must have $\det d\lambda(g^{-1})_g = (\det d\lambda(g^{-1})_g)^{-1}$, this is saying that

$$(\det d\rho(g)_e)^2 = (\det d\lambda(g)_e)^2,$$

We conclude that the above is equivalent to $\det d\rho(g)_e = \pm \det d\lambda(g)_e$, and this is equivalent unimodularity base on our previous discussion.

If G is connected, then since $\det \operatorname{Ad} g$ is a continuous function of g, $|\det \operatorname{Ad} g| = 1$ is equivalent to $\det \operatorname{Ad} g = 1$ since $\det \operatorname{Ad} e = 1$, so we may remove the absolute value bars in this case. If $\det \operatorname{Ad} g = 1$ for g in some open neighborhood of the identity, then since any open neighborhood of the identity generates G, we conclude the same for all $g \in G$. Hence the criterion need only be checked on a neighborhood of e. Since $\det \operatorname{Ad} e = 1$, this is equivalent to $\det \operatorname{Ad}$ being locally constant near e, or equivalently that its derivative is 0. The formula for the derivative of $\det \operatorname{Ad}$ is the $\operatorname{tr}\ \operatorname{ad}$ (trace of the adjoint) on the Lie algebra. ■

Note that it is possible for $\det \operatorname{Ad} g$ to be negative, which is equivalent to having right and left multiplication by g having different orierntations. Since this is a continuous homomorphism $G \to \mathbb{R}^\times$, this can only happen if G is disconnected. An example is the group of 1-dimensional linear affine transformations $\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}$, which seems to be a very typical example of a non-unimodular group. Off the top of my head I can't think of a unimodular group where $\det \operatorname{Ad} g = -1$ for some g.

As an immediate application we deduce a number of classes of unimodular Lie groups.

Corollary 2.2.1: Types of unimodular Lie groups

A Lie group is unimodular if it is any one of the following:

- Compact
- Discrete
- Abelian
- Connected and semisimple
- Connected and reductive
- Connected and nilpotent

In fact any compact, discrete, or abelian locally compact group is unimodular.

Proof. For abelian groups, this is obvious since the left- and right-translation actions are the same. For discrete groups the left- and right-invariant Haar measures must both be the counting measure.

If G is a compact Lie group, then we may treat $|\det \mathrm{Ad}\, g|$ as a continuous function of $g \in G$—in fact, it is a group homomorphism $G \to \mathbb{R}_+^\times$. Using the criterion for unimodularity, suppose that $|\det \mathrm{Ad}\, g| \neq 1$ for some $g \in G$. Then WLOG $|\det \mathrm{Ad}\, g| > 1$, so $|\det \mathrm{Ad}(g^n)| = |\det \mathrm{Ad}\, g|^n \to \infty$ as $n \to \infty$. This contradicts the fact that $|\det \mathrm{Ad}\, g|$ attains a maximum on G.

For the remaining statements, connectedness means that we can use the criterion for unimodularity on the Lie algebra. If $\mathfrak{g}$ is nilpotent, then all linear transformations $\mathrm{ad}\, X$, $X \in \mathfrak{g}$ are nilpotent elements of $\mathrm{End}(\mathfrak{g})$. A nilpotent linear transformation always has trace 0, so we conclude that we have unimodularity.

If $\mathfrak{g}$ is reductive, then its Lie algebra is the direct sum of a semisimple ideal and an abelian one, and $\mathrm{tr}\,\mathrm{ad}\, X$ is the sum of its restriction onto these two factors. $\mathrm{ad}\, X = 0$ in any abelian Lie algebra, so it suffices to prove the claim for a semisimple Lie algebra. This follows from the fact that in a semisimple Lie algebra, every element can be expressed as a bracket of two other elements $X = [Y, Z]$. (One equivalent definition of being semisimple is that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.) The adjoint map is a Lie algebra homomorphism $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, so we have $\mathrm{ad}\, X = \mathrm{ad}\, Y \circ \mathrm{ad}\, Z - \mathrm{ad}\, Z \circ \mathrm{ad}\, Y$. But the commutator of any two linear transformations has trace 0, so $\mathrm{tr}\,\mathrm{ad}\, X = 0$. ■

Example 1: Examples of unimodular groups

The following Lie groups are unimodular:

- All finite groups (discrete)
- All finitely generated groups with their discrete topology
- $\mathbb{R}, \mathbb{R}^\times, \mathbb{C}, \mathbb{C}^\times, \mathbb{R}/\mathbb{Z}$ (abelian)
- The general linear group with elements of positive determinant $\mathrm{GL}_n(\mathbb{R})^+$ (connected reductive)
- The general linear group $\mathrm{GL}_n(\mathbb{R})$. This follows from the $\mathrm{GL}_n(\mathbb{R})^+$ case after checking that left and right translation by some element in $\mathrm{GL}_n(\mathbb{R})^-$ map have the same determinant, e.g. the identity matrix with the first entry swapped to -1. We'll also compute this explicitly in coordinates soon.
- The classical semisimple Lie groups $\mathrm{SL}_n(\mathbb{R}), \mathrm{SO}_n(\mathbb{R}), \mathrm{Sp}_n(\mathbb{R})$ (connected semisimple, or in the case of $\mathrm{SO}_2(\mathbb{R})$, abelian)
- The orthogonal group (compact)
- The sporadic semisimple Lie groups
- The unitary groups U_n (compact)
- Covering spaces of the above, e.g. the universal cover of $\mathrm{SL}_n(\mathbb{R})$
- Complexified versions of the above
- All Fuchsian groups (discrete)
- The space of upper triangular matrices with 1's on the diagonal (nilpotent)

2.3 Subgroups and quotients

We ask whether we can "assemble" the Haar measure of a group from a set of spanning subgroups. Part (ii) of the following gives conditions for doing so.

Proposition 2.3.1

(i) Let H be a locally compact Hausdorff group and M a compact subgroup. Then there exists a positive regular Borel measure on the quotient space H/M that is invariant under the action of H by left translation, unique up to constant multiple.

(ii) If G is a unimodular group containing subgroups P, K with $P \cap K$ compact and $G = PK$, then

$$dg = d_L p \, d_R k.$$

This version of the proposition and its proof are from Bump's *Automorphic Forms and Representations*, Proposition 2.1.5, though I've modified part (ii) since his presentation is a bit handwavy. In part (i), note that we do not require M to be a normal subgroup; the quotient space H/M always makes sense as a topological space. However, since this isn't a group, we can't call the measure a "Haar measure," so instead we ask for left invariance under the natural action of H. A key point of the proof it to make use of the Riesz representation theorem to freely translate between measures and linear functionals.

Proof. (i) Let $X = H/M$. Since M is compact, we may normalize the Haar measure on M so that M has volume 1, i.e. $\int_M dm = 1$. (Compactness means M is unimodular, so we need not specify whether the Haar measure is right or left invariant.)

We define a map $\sigma : C_c(H) \to C_c(X)$ by letting

$$[\sigma(\phi)](hM) = \int_M \phi(hm) \, dm.$$

This is independent of the choice of coset representative h for hM since the Haar measure dm is left-invariant. The map σ has a right inverse $\tau : C_c(X) \to C_c(H)$ given by descending to the quotient, i.e. $(\tau(\psi))(h) = \psi(hM)$, so σ is surjective. Note that applying τ to a compactly supported function on X does give a compactly supported function on H because compactness of M implies that the quotient map $H \to X$ is proper.

To construct the measure on X, we define the linear functional on $C_c(X)$:

$$f \mapsto \int_H [\tau f](h) \, dh$$

where dh is the left Haar measure on h. This is a positive linear functional on $C_c(X)$, so by the Riesz rpresentation theorem it corresponds to a positive Borel measure on X. The linear functional above is invariant with respect to the left action of H, since for any $g \in H$ we have

$$(\lambda(g) \cdot f) \mapsto \int_H f(ghM) dh = \int_H f(hM) \, d(g^{-1}h) = \int_H f(hM) \, dh$$

since the Haar measure on H is left-invariant.[2] (Here $\lambda(g) \cdot f$ the function $x \mapsto f(gx)$, given by the action of g in the left regular representation.)

Suppose that another left-invariant positive Borel measure exsts on X with corresponding positive invariant linear functional $\Lambda : C_c(X) \to \mathbb{C}$. We obtain another linear functional $\lambda = \Lambda \circ \sigma : C_c(H) \to \mathbb{C}$.

[2]Note here that it is important that X is a left coset space, since a right coset space would not be compatible with the left action of H here.

For $h \in H$ and $f \in C_c(H)$, let f' denote the function $x \mapsto f(hx)$. Then

$$(\Lambda \circ \sigma)(f') = \Lambda(xM \mapsto \int_M f(hxm)\, dm$$

Λ is assumed to be a left-invariant functional on $C_c(X)$; this means

$$\begin{aligned}\Lambda(xM \mapsto \int_M f(hxm)\, dm &= \Lambda(xM \mapsto \int_M f(xm)\, dm) \\ &= (\Lambda \circ \sigma)(f).\end{aligned}$$

Hence $\Lambda \circ \sigma$ is a left-invariant functional on $C_c(H)$, so by uniqness of the Haar measure and the Riesz representation theorem it must differ by a constant multiple from

$$f \mapsto \int_H f(h)\, dh,$$

which is the functional giving the left Haar measure on H. The fact that σ is surjective means that Λ is determined uniquely by $\Lambda \circ \sigma$—this is true at a set-theoretic level. Hence, up to constant multiple it must be given by the functional we defined before.

(ii) This can be deduced from (i) by setting $H = P \times K^{\mathrm{opp}}$ and letting M be the subgroup consisting of elements of the form (g, g^{-1}) with $g \in P \cap K$. Here K^{opp} denotes the opposite group of K; denote the opposite group law in G by $*$ and the standard group law by juxtaposition. Then M is a compact subgroup of H isomorphic to $P \cap K$ since

$$(g, g^{-1})(h, h^{-1}) = (gh, g^{-1} * h^{-1}) = (gh, (gh)^{-1}).$$

The map $H/M \to G$ given by $(p, k)M \mapsto pk$ is a well-defined homeomorphism that satisfies the compatibility condition

$$(p', k')(p, k)M \mapsto (p'p)(kk').$$

Since this map is a homeomorphism, it induces a linear isomorphism $C_c(G) \cong C_c(H/M)$. Hence the linear functional giving the Haar measure on G corresponds to some linear functional Λ on H/M. Since G is unimodular, its Haar measure is invariant under both left translation by P and right translation by K. Via the compatibility condition noted above, this means that Λ is invariant under the left translation action by H.

However, the linear functional $C_c(H/M) \to \mathbb{R}$

$$f \mapsto \int_P \int_K f((p, k)M)\, dp_L\, d_R k$$

also has this property (here $d_R k$ denotes the right Haar measure on K, which is the same as the left Haar measure on K^{opp}). By part (i), we conclude that Λ equals this functional up to scalars. Pushing forward, we conclude that $dp_L\, d_R k$ gives the Haar measure on G.

■

3 Examples

3.1 $\mathrm{GL}_n(\mathbb{R})$

The general linear group $\mathrm{GL}_n(\mathbb{R})$ has a global chart given by an open subset of $\mathbb{R}^{n^2}$ via its matrix entries, so let's compute the Haar measure with respect to this chart. For $h \in G$, we claim that $d\lambda(h)_e = \lambda(h)$, treated as linear transformation $\mathfrak{gl}_n(\mathbb{R}) \to \mathfrak{gl}_n(\mathbb{R})$ via left matrix multiplication. This is easy to check from the definitions: given $X \in \mathfrak{gl}_n(\mathbb{R})$ we have

$$\frac{d}{dt}(h \cdot (e + tX)) = h \cdot X,$$

so this differential must be given by left multiplication by h.

We claim that the determinant of $\lambda(h) : \mathfrak{gl}_n(\mathbb{R}) \to \mathfrak{gl}_n(\mathbb{R})$ is $(\det h)^n$. In fact, $\lambda(h)$ has block-matrix form

$$\begin{pmatrix} h & & & \\ & h & & \\ & & \ddots & \\ & & & h \end{pmatrix}$$

as an $n^2 \times n^2$ matrix with respect to the basis given by the elementary matrices e_{ij}, ordered lexicographically (i.e. $e_{11}, e_{12}, \ldots, e_{n,n-1}, e_{nn}$). Indeed,

$$\lambda(h)e_{ij} = \sum_{k=1}^{n} h_{jk} e_{ik}$$

so for any fixed i_0 we conclude $\langle e_{i_0 j} \rangle$ is an invariant subspace of $\lambda(h)$, and the matrix of the transformation restricted to this subspace is the same as the matrix of h. Hence, we conclude that the left Haar measure is given by $|\det h|^{-n}$.

The right Haar measure is a similar computation: the differential of $\rho(h)$ is given by right-multiplication by h in $\mathfrak{gl}_n(\mathbb{R})$. The transpose of this operation is left-multiplication by h^t. A matrix and its transpose has the same determinant, so we conclude that the right Haar measure is also given by $|\det h|^{-n}$.

3.2 Affine linear transformations

A prototypical example of a non-unimodular group is the group of 1-dimensional linear affine transformations

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R}^\times, b \in \mathbb{R} \right\}.$$

In the obvious coordinates, the group law is given by $(a, b) \cdot (c, d) = (ac, ad + b)$. Hence, left multiplication by (a, b) is the map given in coordinates by

$$(c, d) \mapsto (ac, ad + b)$$

and right multiplication is the map

$$(c, d) \mapsto (ac, bc + d).$$

The Jacobian matrix of the former is $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, with determinant a^2, and the Jacobian matrix of the latter is $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$, with determinant a. Therefore this group is not unimodular. Other things to note: $\det \operatorname{Ad} g = a$, so this can take on any real value, leading to a perhaps strange situation where left translation by an element is orientation preserving, but right translation is orientation reverse, e.g. by the element $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.[3]

[3]Thanks to Victor Ginsburg for point this example out to me.

3.3 $\mathrm{GL}_2(\mathbb{R})$ via the Iwasawa decomposition

$\mathrm{GL}_2(\mathbb{R})$ is an important special case for me; indeed it was my study of automorphic forms that motivated me to write this exposition. The Haar measure is better approach from a different perspective in this case.

Proposition 3.3.1: Iwasawa decomposition for $\mathrm{SL}_n(\mathbb{R})$

Let $K = \mathrm{SO}_n(\mathbb{R})$, let $A \subset \mathrm{SL}_n(\mathbb{R})$ be the subgroup of diagonal matrices of determinant 1, and let $N \subset \mathrm{SL}_n(\mathbb{R})$ denote the group of upper triangular matrices with 1's on the diagaonal. Then the map

$$\begin{aligned} A \times N \times K &\to \mathrm{SL}_n(\mathbb{R}) \\ (a, n, k) &\mapsto kan \end{aligned}$$

is a diffeomorphism (and definitely not a group homomorphism).

Proof. In the specific case of $\mathrm{SL}_n(\mathbb{R})$, this is essentially stating the existence and uniqueness of the QR decomposition from linear algebra. ■

The order ANK above is not standard—usually it's KAN—but it is the one we'll be using below. More general versions of this exist for arbitrary semisimple Lie groups. We can modify the decomposition to work for $\mathrm{GL}_n(\mathbb{R})$ by including the set of scalar matrices Z as a fourth component of the Iwasawa decomposition.

Setting $n = 2$, the Iwasawa decomposition means that we may write any element of $\mathrm{GL}_2(\mathbb{R})$ in the form

$$\begin{pmatrix} u & \\ & u \end{pmatrix} \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

for a unique choice of $u \in \mathbb{R}^\times, y > 0, x \in \mathbb{R}$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Thus u, x, y, θ give a global coordinate system for $\mathrm{GL}_2(\mathbb{R})$, and this shows that $\mathrm{GL}_2(\mathbb{R})$ is diffeomorphic to two disjoint copies of $\mathbb{R}^2 \times S^1$.) The perhaps strange choice of expression for the center matrix is will be justified shortly: the group law is quite pleasant in these coordinate.

Let B denote the group of 2×2 upper triangular invertible real matrices with positive determinant. Via the Iwasawa decomposition we may write any element in B uniquely in the form

$$\begin{pmatrix} u & \\ & u \end{pmatrix} \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix},$$

with $u \neq 0, y > 0$, and $x \in \mathbb{R}$, giving a global coordinate chart of B. Denote this element by the vector (u, x, y). We have

$$\begin{pmatrix} u & \\ & u \end{pmatrix} \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ & y^{-1/2} \end{pmatrix} \begin{pmatrix} v & \\ & v \end{pmatrix} \begin{pmatrix} w^{1/2} & zw^{-1/2} \\ & w^{-1/2} \end{pmatrix} = \begin{pmatrix} uv & \\ & uv \end{pmatrix} \begin{pmatrix} (yw)^{1/2} & (x+yz)(yw)^{-1/2} \\ & (yw)^{-1/2}, \end{pmatrix}$$

which translates to nice group law in these coordinates:

$$(u, x, y) \cdot (v, z, w) = (uv, x + yz, yw).$$

Let $\lambda(x, y, z)$ denote left-multiplication by the element given by coordinates (v, z, w), and let $\rho(v, z, w)$ similarly denote right-multiplication. We consider the two translation maps associated to a fixed group element (v, z, w): they are

$$\begin{aligned} \lambda(v, z, w) &: (u, x, y) \mapsto (vu, wx + z, wy) \\ \rho(v, z, w) &: (u, x, y) \mapsto (vu, x + yz, wy) \end{aligned}$$

Hence the differentials are given in coordinates by

$$d\lambda(x,y,z) = \begin{pmatrix} v & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w \end{pmatrix}$$
$$d\rho(x,y,z) = \begin{pmatrix} v & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & w \end{pmatrix}$$

so the respective Jacobian determinants are vw^2 (left) and vw (right). Hence left-invariant Haar measure is

$$d_L g = \frac{du}{|u|}\frac{dx\,dy}{y^2}$$

and the right-invariant Haar measure is

$$d_R g = \frac{du}{|u|}\frac{dx\,dy}{|y|}.$$

This proves that B is not unimodular.

As a Lie group, $K = \mathrm{SO}_2(\mathbb{R})$ is isomorphic to the circle group S^1 via $\theta \mapsto \kappa_\theta := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, and the Haar measure of S^1 is $d\theta$. Using Proposition 2.3.1 and the fact that $\mathrm{GL}_2(\mathbb{R}) = BK$, we conclude that the Haar measure on $\mathrm{GL}_2(\mathbb{R})$ with respect to its Iwasawa coordinates is

$$dg = \frac{du}{|u|}\frac{dx\,dy}{y^2}d\theta = d_L b\; d_R k.$$

Note that our coordinates for $\mathrm{GL}_2(\mathbb{R})$ depend on the ordering of the matrices in the Iwasawa decomposition. Indeed, we could reverse the order to express $\mathrm{GL}_2(\mathbb{R}) = KB$ and conclude that $dg = d_L k\; d_R b$. Despite the non-unimodularity of B, this would not contradict our previous computation of dg since the new coordinate system (u,x,y,θ) on $\mathrm{GL}_2(\mathbb{R})$ would be different from the old one, so these measures are not directly comparable.

Omitting the scalar factor u, we also obtain a Haar measure for $\mathrm{SL}_2(R)$ in Iwasawa coordinates, which is $\frac{dx\,dy}{y^2}d\theta$.

Aside: Those interested in hyperbolic geometry or number theory might notice that this y^{-2} term makes the Haar measure strikingly similar to the hyperbolic metric on the upper half plane. Indeed this is the starting point for connecting two perspectives on the spectral theory of $\mathrm{GL}_2(\mathbb{R})$: determining the spectrum of the non-Euclidean Laplacian on $\Gamma\backslash\mathcal{H}$ is very closely related to determining the irreducible subrepresentations of the right regular representation on $\Gamma\backslash\mathrm{SL}_2(\mathbb{R})$.