Arborescences of Covering Graphs

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Introduction

An *arborescence* of a directed graph Γ rooted at a vertex v is a directed spanning tree with edges directed toward v. We denote the sum of the weights of all arborescences rooted at v by $A_v(\Gamma)$.

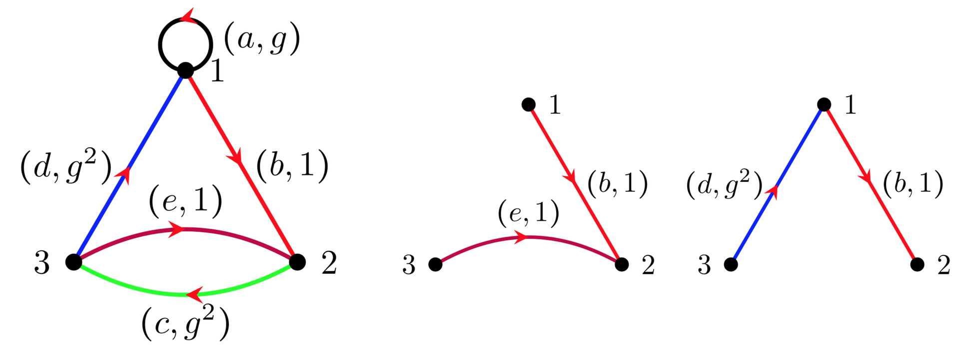

Fig. 1: A $\mathbb{Z}/3\mathbb{Z}$-voltage graph Γ (left) and its two arborescences rooted at vertex 2. $A_2(\Gamma) = bd + be$

A k-fold covering graph $\tilde{\Gamma}$ of Γ is a graph equipped with a k-to-1 quotient map onto Γ.

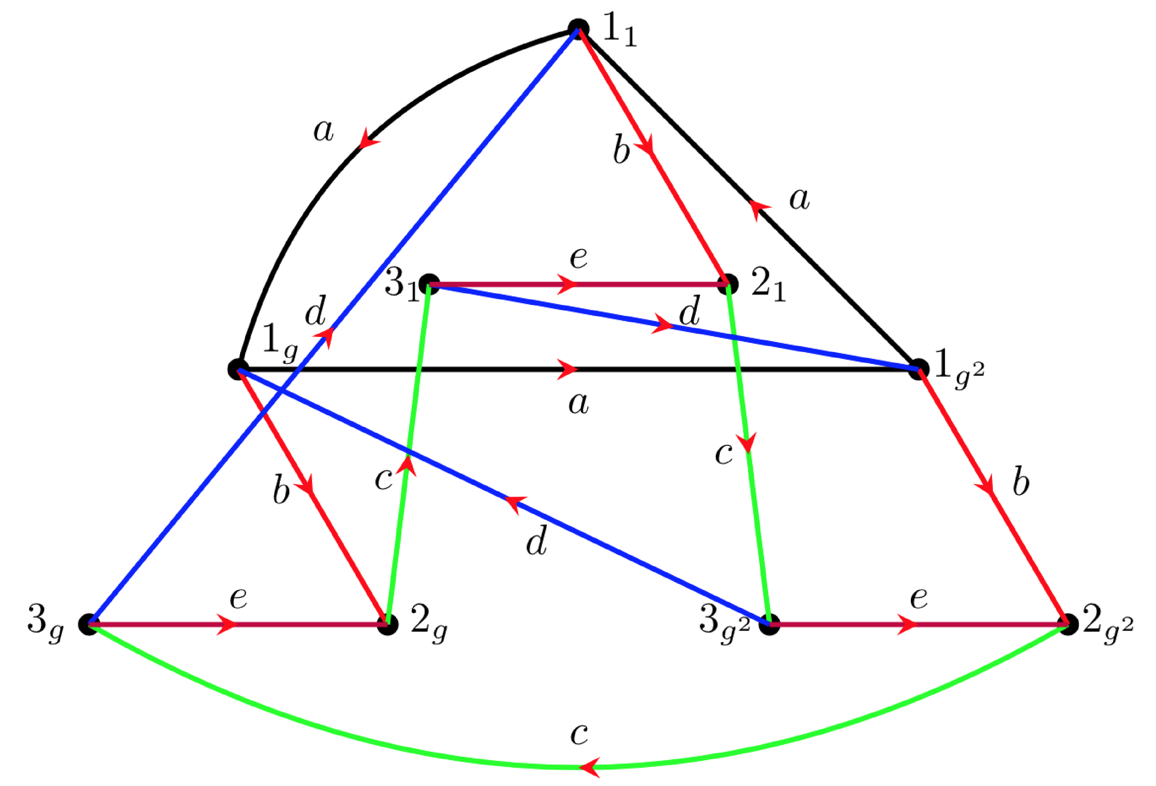

Fig. 2: The derived covering graph $\tilde{\Gamma}$ of Γ. How many arborescences can you find rooted at vertex 2_1?

If a vertex $\tilde{v} \in \tilde{\Gamma}$ is a lift of the vertex $v \in \Gamma$, then Galashin-Pylyavsky [2] showed that the ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ does *not* depend on the choice of v for strongly connected Γ, but did not compute this ratio.

Main Question

How are the arborescences of a covering graph related to the arborescences of the base graph? Can we find an explicit formula for the ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$?

Voltage graphs

One convenient way to construct a covering graph is to fix a *voltage group* G and to label each edge e of the base graph with an element $\nu(e)$ of G. Then the derived covering graph $\tilde{\Gamma}$ is defined to be a $|G|$-fold cover with edges defined by the group law. The *voltage Laplacian matrix* $\mathscr{L}(\Gamma)$, due to Chaiken [1], is given by

$$\ell_{ij} = \delta_{ij} \sum_{e=(v_i,w)} \mathrm{wt}(e) - \sum_{e=(v_i,v_j)} \nu(e)\mathrm{wt}(e)$$

e.g.

$$\mathscr{L}(\Gamma) = \begin{bmatrix} (1-\zeta_3)a+b & -b & 0 \\ 0 & c & -\zeta_3^2 c \\ -\zeta_3^2 d & -e & d+e \end{bmatrix}$$

Formula for ratio of arborescences

Theorem (Dowd-Zhang-Zhang): The ratio $\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)}$ may be expressed in terms of the determinant of a matrix:

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} = \frac{1}{k}\det[\mathscr{L}(\Gamma)]_{\mathbb{Z}}$$

When $\tilde{\Gamma}$ is a regular cover, the matrix $[\mathscr{L}(\Gamma)]_{\mathbb{Z}}$ may be realized as the $\mathbb{Z}$-linearization of the voltage Laplacian matrix $\mathscr{L}(\Gamma)$ via restriction of scalars. As a corollary, if $\tilde{\Gamma}$ is a regular cover of prime degree p, then this determinant may be expressed as a field norm of the determinant of the voltage Laplacian:

$$\frac{A_{\tilde{v}}(\tilde{\Gamma})}{A_v(\Gamma)} = \frac{1}{p}\mathrm{N}_{\mathbb{Q}(\zeta_p):\mathbb{Q}}(\det\mathscr{L}(\Gamma)) = \frac{1}{p}\prod_{\sigma\in\mathrm{Gal}(\mathbb{Q}(\zeta_p):\mathbb{Q})}\sigma(\det\mathscr{L}(\Gamma))$$

Proof Sketch

Let $L(\Gamma)$ be the Laplacian matrix of Γ. The Matrix Tree Theorem says that $A_{v_i}(\Gamma) = L_i^i(\Gamma)$, the minor of $L(\Gamma)$ obtained by removing the row and column corresponding to v_i. We found that $L(\tilde{\Gamma})$ may be triangularized nicely under a particular change of basis S:

$$S^{-1}L(\tilde{\Gamma})S = \begin{bmatrix} L(\Gamma) & * \\ 0 & [\mathscr{L}(\Gamma)]_{\mathbb{Z}} \end{bmatrix}$$

Thus, we need to compare the minor of $L(\tilde{\Gamma})$ before and after the change of basis. It turns out that the minor at any lift $\tilde{v}$ of $v \in \Gamma$ after the change of basis is $\sum_{i=1}^k A_{\tilde{v}_i}(\Gamma)$, the sum of all arborescences rooted at *any* lift of v. By symmetry, this is $kA_{\tilde{v}}(\Gamma)$; the main result follows.

Vector fields

Arborescences are closely related to *vector fields*, which are subgraphs such that every vertex has outdegree 1.

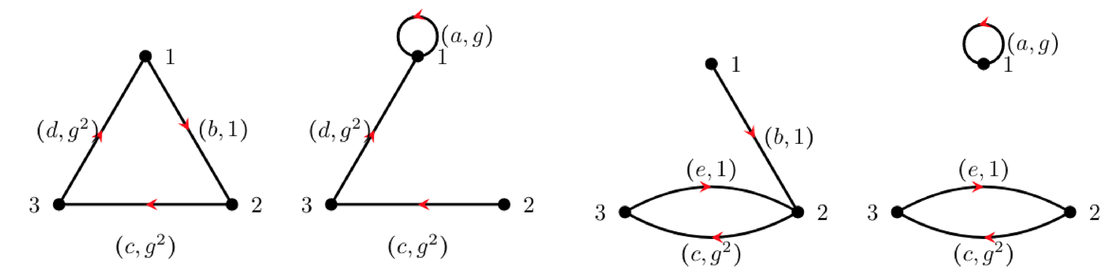

Fig. 3: The four vector fields of Γ from Figure 1.

Via deletion-contraction, we derived a novel proof of the following formula, originally due to Chaiken [1]:

Let G be an abelian group, and let Γ be an edge-weighted G-voltage graph. Then

$$\sum_{\gamma\subseteq\Gamma}\left[\mathrm{wt}(\gamma)\prod_{c\in C(\gamma)}(1-\nu(c))\right] = \det\mathscr{L}(\Gamma)$$

where the sum ranges over all vector fields γ of Γ, $C(\gamma)$ is the set of cycles in γ, and $\nu(c)$ is the product of the voltages of the edges of the cycle c. The Matrix Tree Theorem is an easy corollary of this result.

In the case $k = 2$, we have $[\mathscr{L}(\Gamma)]_{\mathbb{Z}} = \mathscr{L}(\Gamma)$, which leads to a combinatorial interpretation of the right-hand side of the main theorem in terms of vector fields. Does a similar combinatorial interpretation exist for covers of all degrees? If so, can we come up with an explicit combinatorial bijection yielding the main theorem?

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References

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