Qualifying Exam Syllabus

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Committee: Sug Woo Shin (Advisor), Mark Haiman (Exam Chair), Martin Olsson, David Nadler

1 Major topic: Algebraic Geometry (Algebra)

Reference: Hartshorne II.1-8, III.1-10, IV.1-5

- Schemes and morphisms: Affine, projective, reduced, irreducible, regular, and Noetherian schemes. Fiber products, varieties, and blowups. Open and closed embeddings, affine, finite, finite-type, separated, proper, projective, rational, dominant, flat, and smooth morphisms. Valuative criteria.
- Sheaves: Presheaves and sheaves. Quasicoherent, coherent, locally free, invertible, ample, very ample, flasque, twisting sheaves. Relationship between Weil divisors, Cartier divisors, invertible sheaves, and maps to $\mathbb{P}^n$. Sheaves of differentials: relative cotangent, conormal, and Euler sequences. Functors $f_*, f^*, f^!, f_!, \mathscr{H}om, \otimes$.
- Cohomology: Derived functor cohomology, Čech cohomology, $H^i(\mathbb{P}^r, \mathcal{O}(d))$, Grothendieck's vanishing theorem, Serre's criterion for affineness, Serre duality (statement).
- Curves: Riemann-Roch, Hurwitz, embeddings of curves, elliptic curves, Clifford's theorem.

2 Major topic: Algebraic Number Theory (Algebra)

References: Neukirch I, II; Milne CFT I, V.

- Number fields: Integrality, norm, trace, Dedekind domains, factorization of ideals, splitting behavior of primes in extensions. Ideal class groups and Dirichlet's unit theorem. Discriminant and different. Minkowski bound and class number formula. Orders of algebraic number fields.
- Local theory: p-adic numbers, completions, valuations and absolute values, unique extension of valuation, Hensel's lemma, Krasner's lemma, Ostrowski's theorem, local fields. Ramification, tame and wild ramification, higher ramification groups, decomposition and inertia groups, Frobenius elements. Relationship between local and global fields.
- Class field theory: adèles and idèles, product formula. Statements of local and global CFT, statement of Artin reciprocity, statement of existence theorem, statement of Chebotarev density.
- Examples: Cyclotomic fields, quadratic number fields, Hilbert class fields.

3 Minor topic: Lie Theory (Geometry/Algebra)

Reference: Varadarajan *Lie Groups, Lie Algebras, and Their Representations*

- Lie groups: Basic differential geometry, smooth manifolds, tangent and cotangent bundles, vector fields. Classical Lie groups, relationship between Lie group and Lie algebra, exponential map, Baker-Campbell-Hausdorff formula.
- Lie algebras: Solvable, nilpotent, semisimple Lie algebras, Lie and Engel's theorems, Killing form, Cartan's criteria, universal enveloping algebra, PBW theorem, Casimir element. Ext, Whitehead's theorem.
- Root systems: Cartan subalgebras, roots and root spaces. Root systems, Weyl groups, Cartan matrices, Dynkin diagrams. Statement of classification of semisimple Lie algebras over $\mathbb{C}$.
- Representations of semisimple Lie algebras: Weyl complete reducibility. Highest weight and Verma modules. Representation theory of $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{sl}_3(\mathbb{C})$. Weyl character formula.