The Grassmannian and the 27 lines

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Note to reader: *This essay was originally written as a term paper for a second course in scheme theory. I found these topics enlightening mainly for two reasons:*

- *Illustrating the utility of representable functors: in the 27 lines, it is essential that we may treat the Grassmannian both as a smooth projective variety and as a moduli space.*
- *A reason to care about fibers of morphisms and flatness: keeping track of the fibers in the incidence variety is the main idea of the proof in the 27 lines. We apply Miracle Flatness in a concrete scendario, and we learns about subtleties that must be addressed in these types of applications (e.g. nonreduced fibers).*

If these are topics that seem mysterious to you, then this essay might be worth a read. This paper mostly follows Vakil's outline of the proof, completing exercises that he sets out, so there is not much new content if you are familiar with that already.

1 Introduction

Grassmannians are ubiquitous because they are moduli spaces of linear subspaces, which are of obvious importance in many areas of mathematics. In geometric representation theory, they are a central object of study—as a homogeneous space for GL_n, the representation theory of algebraic groups may be studied through the Grassmannian [Bri04]. In algebraic combinatorics, the Schubert calculus arises from the study of the cohomology ring of Schubert varieties, which are subvarieties of the Grassmannian describing more specialized subspace data. More classically, the Grassmannian is indispensable in incidence algebraic geometry: the study of the containment and configuration of linear spaces inside an algebraic variety.

In Section 2, we define the Grassmannian and discuss it in the scheme-theoretic setting, with the main goal of proving its representability as a regular projective scheme. To accomplish this, we discuss the Zariski criterion for representability and the Plücker embedding. We conclude the section by giving an explicit description of $\mathrm{Gr}(2, 4)$, which is the first Grassmannian not isomorphic to projective space. In Section 3, we prove the celebrated theorem of the 27 lines on a smooth cubic surface. The Grassmannian $\mathrm{Gr}(2, 4)$ plays a central role in the proof, and indeed the fact that this moduli space may be represented as a smooth projective variety is critical to be able to apply the machinery of flat morphisms. This justifies the discussion in the previous section and illustrates the utility of representable functors generally.

2 Representability of the Grassmannian

2.1 The Zariski criterion for representability

Definition 2.1.1. A functor $F : \mathbf{Sch}^{\mathrm{opp}} \to \mathbf{Set}$ is said to be a *Zariski sheaf* if it defines a sheaf on the Zariski site of any scheme S. In more detail, this means that for any open cover $\{S_i\}$ of S in the Zariski topology, we have an equalizer exact sequence

$$\{*\} \longrightarrow F(S) \longrightarrow \prod F(S_i) \rightrightarrows \prod F(S_i \cap S_j).$$

If X is a scheme, then $\mathrm{Hom}(-, X)$ is a Zariski sheaf because *morphisms glue*. That is, if S has an open cover $\{S_i\}$ and there exists morphisms $f_i : S_i \to X$ such that f_i agrees with f_j on $S_i \cap S_j$, then there is a unique morphism $f : S \to X$ that restricts to the f_i. Hence, a necessary criterion for a functor to be representable is that it is a Zariski sheaf.

Definition 2.1.2. Let $F : \mathbf{Sch}^{\mathrm{opp}} \to \mathbf{Set}$ be a functor. An *open subfunctor* of F is a functor F' equipped with a natural transformation $F' \to F$ satisfying the following axiom: for any scheme S and any natural transformation $\mathrm{Hom}(-, S) \to F$, the fibered product $F' \times_F \mathrm{Hom}(-, X)$ is representable by some open subscheme $U \subseteq S$. That is, the following is a Cartesian diagram:

$$\begin{array}{ccc} \mathrm{Hom}(-, U) & \longrightarrow & \mathrm{Hom}(-, S) \\ \downarrow & & \downarrow \\ F' & \longrightarrow & F \end{array}$$

(The fibered product of functors $F' \times_F \mathrm{Hom}(-, S)$ can be defined at the level of fibered products of sets. That is, $(F' \times_F \mathrm{Hom}(-, S))(Y) = F'(Y) \times_{F(Y)} \mathrm{Hom}(Y, S)$.) A collection of open subfunctors F_i are said to *cover* F if for every natural transformation of the form $\mathrm{Hom}(-, S) \to F$, the corresponding U_i form a Zariski open cover of S.

Any representable functor $\mathrm{Hom}(-, X)$ is covered by itself. By the Yoneda lemma, any natural transformation $\mathrm{Hom}(-, S) \to \mathrm{Hom}(-, X)$ is induced by a morphism of schemes $X \to S$, so the following diagram is Cartesian:

$$\begin{array}{ccc} \mathrm{Hom}(-, S) & \longrightarrow & \mathrm{Hom}(-, S) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(-, X) & \longrightarrow & \mathrm{Hom}(-, X), \end{array}$$

as this is the Cartesian diagram obtained by applying Hom to the Cartesian diagram

$$\begin{array}{ccc} S & \longrightarrow & S \\ \downarrow & & \downarrow \\ X & \longrightarrow & X. \end{array}$$

Hence we have demonstrated that a representable functor is a Zariski sheaf that can be covered by representable open subfunctors. A useful criterion in proving representability is that the converse also holds:

Theorem 2.1.3. [Sta23, Tag 01JJ] *A functor $F : \mathbf{Sch}^{\mathrm{opp}} \to \mathbf{Set}$ is representable if and only if it is a Zariski sheaf that has an open cover by representable functors.*

2.2 The Grassmannian

Definition 2.2.1. [Sta23, Tag 089R] The Grassmannian functor $\mathrm{Gr}(d, n) : \mathbf{Sch}^{\mathrm{opp}} \to \mathbf{Set}$ is a contravariant functor given on objects by sending a scheme S to the set $\mathrm{Gr}(d, n)(S)$ of isomorphism classes of surjections

$$\mathscr{O}_S^{\oplus n} \to \mathscr{F}.$$

where $\mathscr{F}$ is locally free of rank $n - d$. Here, a morphism between two surjections $\alpha : \mathscr{O}_S^{\oplus n} \to \mathscr{F}$ and $\alpha' : \mathscr{O}_S^{\oplus n} \to \mathscr{F}'$ is a morphism $\beta : \mathscr{F} \to \mathscr{F}'$ making the following diagram commute:

$$\begin{array}{ccc} \mathscr{O}_S^{\oplus n} & \xrightarrow{\alpha} & \mathscr{F} \\ \downarrow{\scriptstyle \mathbf{id}} & & \downarrow{\scriptstyle \beta} \\ \mathscr{O}_S^{\oplus n} & \xrightarrow{\alpha'} & \mathscr{F}'. \end{array}$$

Hence isomorphism classes of surjections are in bijection with the possible kernels of α. Given a morphism of schemes $f : T \to S$, we obtain a morphism $\mathrm{Gr}(d,n)(S) \to \mathrm{Gr}(d,n)(T)$ sending $\alpha : \mathscr{O}_S^{\oplus n} \to \mathscr{F}$ to the induced morphism $\mathscr{O}_T^{\oplus n} \to f^*\mathscr{F}$, making $\mathrm{Gr}(d,n)$ a contravariant functor. Note that f^* is a right-exact functor that preserves locally free modules and their rank, and that $f^*\mathscr{O}_S = \mathscr{O}_T$.

While we tend to think of the Grassmannian as parametrizing subspaces, this definition instead appears to parametrize quotients. But since the isomorphism class of a surjection $\mathscr{O}_S^{\oplus n} \to \mathscr{F}$ is uniquely determined by its kernel, we recover the original meaning. In particular, when $S = \operatorname{Spec} k$ for a field k, we may canonically identify $\mathrm{Gr}(d,n)(k)$ with the collection of d-dimensional subspaces of k^n, since a locally free sheaf on $\operatorname{Spec} k$ of rank $n-d$ is just k^{n-d} and a surjection $k^n \to k^{n-d}$ is specified up to isomorphism by a d-dimensional subspace of k^n.

Theorem 2.2.2. $\mathrm{Gr}(d,n)$ *is a representable by a scheme of finite type over* $\mathbb{Z}$ *of relative dimension* $d(n-d)$.

Proof. Suppose we are given an open cover $\{S_i\}$ of a scheme S and a collection of surjections $\alpha_i : \mathscr{O}_{S_i}^{\oplus n} \to \mathscr{F}_i$ with $\mathscr{F}_i$ locally free of rank $n-d$. If α_i and α_j define isomorphic surjections when restricted to $S_i \cap S_j$, then we may think of

$$\mathscr{F}_i|_{S_i\cap S_j} \cong \mathscr{F}_j|_{S_i\cap S_j} \cong \mathscr{O}_{S_i\cap S_j}^{\oplus n} / \ker \alpha_i|_{S_i\cap S_j} \cong \mathscr{O}_{S_i\cap S_j}^{\oplus n} / \ker \alpha_j|_{S_i\cap S_j}.$$

This means that we may glue the $\mathscr{F}_i$ to get a locally free sheaf $\mathscr{F}$ defined on all of S, and moreover we obtain a globally defined surjection $\alpha : \mathscr{O}_S^{\oplus n} \to \mathscr{F}$ by gluing all the α_i. Thus the Grassmannian is a Zariski sheaf.

We define open subfunctors $\mathrm{Gr}(d,n)_I$ of $\mathrm{Gr}(d,n)$ for each index set $I \subseteq \{1,\dots,n\}$ of size $n-d$ by setting

$$\mathrm{Gr}(d,n)_I(S) = \{\alpha \in \mathrm{Gr}(d,n) : \mathscr{O}_S^{\oplus n-d} \to_\iota \mathscr{O}_S^{\oplus n} \to_\alpha \mathscr{F} \text{surjective}\}$$

where $\iota_I : \mathscr{O}_S^{\oplus d} \to \mathscr{O}_S^{\oplus n}$ is the inclusion onto the direct summands indexed by I. The intuition from linear algebra is that $\mathrm{Gr}(d,n)_I$ parametrizes d-dimensional subspaces of n-dimensional space that are complementary to the $(n-d)$-dimensional space spanned by standard basis elements $\{e_i\}_{i\in I}$. It is reasonable to think of this condition as being open: perturbing a subspace slightly does not change the fact that it is complementary to another subspace.

To show that the $\mathrm{Gr}(d,n)_I$ are indeed open subfunctors, note that we have a natural transformation $\mathrm{Gr}(d,n)_I \to \mathrm{Gr}(d,n)$ given by inclusion of sets. If S is any scheme, a natural transformation $\mathrm{Hom}(-,S) \to \mathrm{Gr}(d,n)$ yields a distinguished element $\alpha \in \mathrm{Gr}(d,n)$ corresponding to the identity in $\mathrm{Hom}(S,S)$. Let $\alpha \circ \iota_I : \mathscr{O}_S^{d-k} \to \mathscr{F}$. The locus where this composition is surjective is open: given $x \in S$ such that $(\alpha \circ \iota_I)_x$ is a surjection, we may choose an open neighborhood $U \ni x$ on which $\mathscr{F}$ is free, say generated by $s_1, \dots, s_{n-d}$. By the definition of surjectivity at a stalk, after possibly shrinking U we conclude that there exist $t_I \in \mathscr{O}_S^{\oplus n-d}(U)$ such that $(\alpha \circ \iota_I|_U)(t_i) = s_i$, yielding surjectivity in a neighborhood of x.

Let S_I denote the open locus of surjectivity, and suppose $f : T \to S$ is a morphism of schemes. A point $x \in S$ lies in S_I if and only if $(\alpha \circ \iota_I)|_x : k(x)^{n-d} \to \mathscr{F}_x \otimes k(x)$ is a surjection on fibers, which follows by Nakayama's lemma. Given a point $y \in T$, the map $(f^*\alpha \circ f^*\iota_I)|_t$ is the base change of $(\alpha \circ \iota_I)|_x$ by the extension $k(x) \hookrightarrow k(t)$, so by flatness of field extensions we obtain surjectivity at t if and only if we have surjectivity at x. This means that $f^*\alpha \in \mathrm{Gr}(d,n)_I(T)$ if and only if f factors through the open subscheme S_I. Hence, for all morphisms $f : T \to S$, the following is a Cartesian diagram:

$$\begin{array}{ccc} \mathrm{Hom}(T,S_i) & \longrightarrow & \mathrm{Hom}(T,S) \\ \downarrow & & \downarrow \\ \mathrm{Gr}(d,n)_I(T) & \longrightarrow & \mathrm{Gr}(d,n)(T). \end{array}$$

Thus each $\mathrm{Gr}(d,n)_I$ is an open subfunctor. To show that these subfunctors cover $\mathrm{Gr}(d,n)$, we need to show that the S_I as above cover S. Again, by Nakayama, surjectivity of $\alpha \circ \iota_I$ at $x \in S$ can be checked on fibers, so this statement amounts to showing that if $k^n \to k^{n-d}$ is a surjection, then the corresponding $(n-d) \times n$ matrix has some subset of $n-d$ columns that are linearly independent, with the choice of subset corresponding to a choice of I. But this is clearly true from linear algebra, so we conclude that the $\mathrm{Gr}(d,n)_I$ form an open cover of $\mathrm{Gr}(d,n)$.

Finally, we show that the $\mathrm{Gr}(d,n)_I$ are representable. If $\alpha \in \mathrm{Gr}(d,n)_I(S)$, the composition $\alpha \circ \iota_I$ defines a surjective morphism between the two locally free modules $\mathscr{O}_S^{\oplus n-d} \to \mathscr{F}$ of the same rank. This means that, on sufficiently small affine opens, this morphism is induced by a surjection $A^{n-d} \to A^{n-d}$ for a ring A. But any such surjection is an isomorphism [Sta23, Tag 05G8]! We conclude that $\alpha \circ \iota_I$ is an isomorphism globally, so we may split α as

$$\mathscr{O}_S^{\oplus n-d} \oplus \mathscr{O}_S^{\oplus d} \to \mathscr{O}_S^{\oplus n-d}$$

where the first component maps identically onto the image. Such morphisms are represented by $\mathbb{A}_{\mathbb{Z}}^{d(n-d)}$, since a choice of α corresponds to an arbitrary choice of map $\mathscr{O}_S^{\oplus d} \to \mathscr{O}_S^{\oplus n-d}$, which is given by d choices of $(n-d)$-tuples of sections. Equivalently, we are choosing the entries of the right portion of the following $(n-d) \times n$ matrix:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & a_{11} & a_{12} & \dots & a_{1d} \\ 0 & 1 & \dots & 0 & a_{21} & a_{22} & \dots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & a_{n-d,1} & a_{n-d,2} & \dots & a_{n-d,d} \end{pmatrix}$$

Hence, by Theorem 2.1.3, $\mathrm{Gr}(d,n)$ is representable. Since it has a finite open subcover by schemes of finite type over $\mathbb{Z}$, we conclude that it, too, is of finite type over $\mathbb{Z}$, and has relative dimension $d(n-d)$. ■

In fact, $\mathrm{Gr}(d,n)$ is representable by a projective scheme over $\mathbb{Z}$. We will show this in a moment by describing the Plücker embedding, but we first illustrate with the following special case.

Lemma 2.2.3. *For $n \geq 1$, $\mathrm{Gr}(n,n+1)$ is represented by $\mathbb{P}_{\mathbb{Z}}^n$.*

Proof. If S is a scheme, then by definition $\mathbb{P}_{\mathbb{Z}}^n(S)$ consists of morphisms $S \to \mathbb{P}_{\mathbb{Z}}^n$, and all such morphisms are given by the data of an invertible sheaf $\mathscr{L}$ on S along with a set of $n+1$ global sections $s_0, \dots, s_n$ that globally generate $\mathscr{L}$. To these data we associate the surjection

$$\mathscr{O}_S^{\oplus n+1} \to \mathscr{L} : (a_0, \dots, a_n) \mapsto \sum a_i s_i.$$

Conversely, an element of $\mathrm{Gr}(n,n+1)(S)$ is a surjection $\alpha : \mathscr{O}_S^{n+1} \to \mathscr{L}$ for some locally free sheaf $\mathscr{L}$ of rank 1, i.e. an invertible sheaf. The image of the $n+1$ generators $(0,\dots,0,1,0,\dots,0) \in \Gamma(S, \mathscr{O}_S^{n+1})$ map to global sections of $\mathscr{L}$, and these global sections define a morphism $S \to \mathbb{P}_{\mathbb{Z}}^n$. These two constructions define mutually inverse natural transformations of functors. ■

This corresponds to the intuition of n-dimensional subspaces of an $(n+1)$-dimensional space as being parametrized by lines, i.e. by taking the annihilator of such a subspace in the dual space.

2.3 The Plücker embedding

Given a surjection $\mathscr{O}_S^{\oplus n} \to \mathscr{F}$ for $\mathscr{F}$ locally free of rank $n-d$ on S, taking exterior powers gives a surjection

$$\alpha : \mathscr{O}_S^{\oplus \binom{n}{d}} \to \det \mathscr{F}$$

onto an invertible sheaf on S. The data of such a surjection is equivalent to a collection of $\binom{n}{d}$ globally generating sections of $\det \mathscr{F}$, so we associate to $\alpha \in \mathrm{Gr}(d,n)(S)$ a morphism $S \to \mathbb{P}_{\mathbb{Z}}^{\binom{n}{d}-1}$, i.e. an element of $\mathbb{P}_{\mathbb{Z}}^{\binom{n}{d}-1}(S)$. That is, we have a map $\mathrm{Gr}(d,n)(S) \to \mathbb{P}_{\mathbb{Z}}^{\binom{n}{d}-1}(S)$ for every scheme S. It can be checked that this is natural in S, so by the Yoneda lemma we conclude that we have a morphism of schemes $\mathrm{Gr}(d,n) \to \mathbb{P}_{\mathbb{Z}}^{\binom{n}{d}-1}$. In fact:

Proposition 2.3.1. *The morphism* $\mathrm{Gr}(d,n) \to \mathbb{P}_{\mathbb{Z}}^{\binom{n}{k}-1}$ *is a closed embedding, known as the* Plücker embedding.

Proof. We first show that the open subscheme $\mathrm{Gr}(d,n)_I \simeq \mathbb{A}_{\mathbb{Z}}^{d(n-d)}$ identified in the proof of representability is the preimage of $D_+(x_I)$ under this map, where we treat the coordinates of $\mathbb{P}_{\mathbb{Z}}^{\binom{n}{d}-1}$ as indexed by $(n-d)$-element subsets $I \subset \{1,\dots,n\}$. This can be checked via the functor of points: if $\alpha \in \mathrm{Gr}(d,n)_I(S)$, then the composition

$$\mathscr{O}_S^{\oplus(n-d)} \xrightarrow{\iota_I} \mathscr{O}_S^{\oplus n} \xrightarrow{\alpha} \mathscr{F}$$

is surjective, so it remains surjective after taking $(n-d)$-th wedge powers, yielding a surjective composition

$$\mathscr{O}_S \to \mathscr{O}_S^{\oplus\binom{n}{d}} \to \det \mathscr{F}.$$

Let s_i be the images of the coordinate section $e_i \in \Gamma(S, \mathscr{O}_S^{\oplus n})$ in $\mathscr{F}$, for each $i \in I$. Surjectivity of the above composition means means that the wedge product $s_1 \wedge \cdots \wedge s_{n-d}$ does not vanish anywhere. By the construction of the map to projective space, this means that the morphism $S \to \mathbb{P}_{\mathbb{Z}}^{\binom{n}{d}}$ lies in the chart $D_+(x_I)$, as desired. Conversely, nonsurjectivity of $\alpha \circ \iota_I$ implies that the wedge product $s_1 \wedge \cdots \wedge s_{n-d}$ vanishes somewhere on S, so that point of S must be mapped to $V(x_I)$. Hence we conclude that the preimage of $D_+(x_I)$ is exactly $\mathrm{Gr}(d,n)_I$.

Hence, the morphism $\mathrm{Gr}(d,n) \to \mathbb{P}_{\mathbb{Z}}^{\binom{n}{d}-1}$ is obtained by gluing together the morphisms $\mathbb{A}_{\mathbb{Z}}^{d(n-d)} \simeq \mathrm{Gr}(d,n)_I \to D_+(x_I) \subset \mathbb{P}_{\mathbb{Z}}^{\binom{n}{d}-1}$ for various I. Since the distinguished opens $D_+(x_I)$ cover projective space, it suffices to prove that each of the morphisms $\mathrm{Gr}(d,n)_I \to D_+(x_I)$ are closed embeddings. Without loss of generality, take $I = \{1,\dots,d\}$. The morphism $\mathrm{Gr}(d,n)_I \to D_+(x_I)$ is given by a ring homomorphism

$$\mathbb{Z}\left[\frac{x'_I}{x_I}\right]_{I'} \to \mathbb{Z}[\{a_{ij}\}]_{1\le i\le n-d<j\le n}$$

where the subsets I' range over all d-element subsets of $\{1,\dots,n\}$. We must show that this ring homomorphism is surjective. As we did when proving the $\mathrm{Gr}(d,n)_I$ is representable, we organize the variables a_{ij} into a matrix, though we change the indexing slightly:

$$\begin{pmatrix} 1 & 0 & \dots & 0 & a_{1,n-d+1} & a_{1,n-d+1} & \cdots & a_{1n} \\ 0 & 1 & \dots & 0 & a_{2,n-d+1} & a_{2,n-d+2} & \cdots & a_{2,n-d+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & a_{n-d,n-d+1} & a_{n-d,n-d+2} & \cdots & a_{n-d,n} \end{pmatrix}$$

The identification $\mathrm{Gr}(n,d)_I \simeq \mathbb{A}_{\mathbb{Z}}^{d(n-d)}$ was made by identifying a surjection of the form $\mathscr{O}_S^{\oplus n} \to \mathscr{O}_S^{\oplus d}$ given in the matrix form above with the coordinates a_{ij}. Taking $(n-d)$-th exterior powers and letting $x_{I'}$ be the wedge of the coordinates in I', we find the new morphism $\mathscr{O}_S^{\oplus\binom{n}{d}} \to \det \mathscr{F}$ is given by the row vector $(M_{I'})_{I'}$, where $M_{I'}$ is the minor of the matrix above given by the columns indexed by I'. Hence, we identify the ring homomorphism as the one given by sending $\frac{x'_I}{x_I}$ to the minor $M_{I'}$. It follows that the ring homomorphism is surjective, since $M_{I'} = a_{ij}$ for $I' = \{1,\dots,\hat{i},\dots,d,j\}$. ∎

Corollary 2.3.2. *For any field k, $\mathrm{Gr}(d,n)_k$ is a smooth projective variety of dimension $d(n-d)$.*

Proof. The Plücker embedding shows that the Grassmannian is projective. The relative dimension, irreducibility, and smoothness follow from the fact that $\mathrm{Gr}(d,n)_k$ can be covered by copies of finitely many copies of affine space $\mathbb{A}_k^{d(n-d)}$. ∎

The *Plücker equations* are the polynomials that cut out the Grassmannian as a projective variety. The $(n-d)\times(n-d)$ minors of any $(n-d)\times n$ matrix satisfy a quadratic relationship, so by our construction of the Plücker embedding the Grassmannian will lie in the vanishing locus of the corresponding quadratic relations among the x_I. It turns out that these relations are sufficient; we omit the proof, instead turning to the following example.

2.4 Gr(2, 4)

We have already seen that $\mathrm{Gr}(n, n+1) \simeq \mathbb{P}^n$ for any $n \geq 1$; by reasons of dimension, the Plücker embedding also gives an isomorphism $\mathrm{Gr}(1, n+1) \simeq \mathbb{P}^n$. (In general, $\mathrm{Gr}(d, n) \simeq \mathrm{Gr}(n-d, n)$, but we will not show this here.) Therefore, the first geometrically interesting Grassmannian is $\mathrm{Gr}(2, 4)$.

The Plücker embedding realizes $\mathrm{Gr}(2, 4)$ as a hypersurface in $\mathbb{P}^5$, so it is cut out by one equation. Let $x_{12}, x_{13}, \ldots, x_{34}$ be the six coordinates on $\mathbb{P}^5$ corresponding to the six choices of 2-element subsets of $\{1, 2, 3, 4\}$. We claim that $\mathrm{Gr}(2, 4)$ is the vanishing locus

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0$$

This polynomial is irreducible, so it suffices to show that it vanishes on $\mathrm{Gr}(2, 4)$—if $\mathrm{Gr}(2, 4)$ were smaller than the vanishing locus, then it would have codimension greater than 1 in $\mathbb{P}^5$. We claim that the corresponding relation among 2×2 minors of a 2×4 matrix is the relevant Plücker equation. That is, we claim that

$$M_{12}M_{34} - M_{13}M_{24} + M_{14}M_{23} = 0$$

where M_{ij} is the minor obtained from retaining columns i and j of the 2×4 matrix. If such a matrix is given by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix},$$

then the purported identity is

$$\begin{aligned} 0 = (a_{11}a_{22} - a_{12}a_{21})(a_{13}a_{24} - a_{14}a_{23}) - (a_{13}a_{23} - a_{13}a_{21})(a_{12}a_{24} - a_{14}a_{22}) \\ + (a_{11}a_{24} - a_{14}a_{21})(a_{12}a_{23} - a_{13}a_{22}) \end{aligned}$$

which is true. By the construction of the Plücker embedding, the image of $\mathrm{Gr}(2, 4)$ satisfies any relations that the minors satisfy in the variables x_{ij}, so we conclude that $\mathrm{Gr}(2, 4)$ is indeed cut out by the given polynomial.

3 The 27 lines on a cubic

One classical application of the Grassmannian is its use in *incidence geometry*. To determine the configuration of lines, hyperplanes, or some other type of linear subspace on an algebraic variety, working over the Grassmannian is very useful. The Grassmannian does much more than merely providing a rigorous way of keeping track of the data of linear subspaces: the fact that $\mathrm{Gr}(d, n)$ *is itself a projective variety* grants us the full power of projective geometry. This is one illustration of why representability of the Grassmannian, and indeed representability in general, is an important notion.

Let k be an algebraically closed field of characteristic not 3.[1] We will prove the following celebrated theorem, known informally as "the 27 lines on a cubic surface":

There are 27 distinct lines on any smooth cubic surface in $\mathbb{P}^3_k$.

[1] The theorem remains true even in characteristic 3, but in the proof the Fermat cubic surface must be replaced by some other less pleasant smooth cubic surface. We omit this case for the sake of cohesion.

The wording is literal: we obtain 27 genuine distinct lines on the smooth cubic surface, with no caveats such as having to "count lines with correct multiplicity." We complete Vakil's outline of the proof ([Vak23], Chapter 27). There is more to say beyond the mere existence of the 27 lines—the configuration of the 27 lines is related to the Lie algebra E_6, and every smooth cubic surface is a plane blown up at 6 points—but we limit ourselves to discussing just the existence statement above.

3.1 Most surfaces of degree more than 3 have no lines

We illustrate the general setup by first showing what happens in degree > 3. This introduces the Grassmannian's typical role in an incidence correspondence.

Theorem 3.1.1. *A generic surface of degree $d > 3$ in $\mathbb{P}^3_k$ contains no lines.*

A degree d surface in 3-space is specified by the $\binom{d+3}{3}$ coefficients of a degree d homogeneous polynomial in 4 variables, up to rescaling. Hence, $\mathbb{P}_k^{\binom{d+3}{3}-1}$ parametrizes degree d surfaces. The precise claim of the theorem is that a Zariski dense open subset of $\mathbb{P}_k^{\binom{d+3}{3}-1}$ has closed points corresponding to surfaces with no lines. Note that we may interpret $\mathrm{Gr}(2,4)$ as parametrizing the set of lines in $\mathbb{P}^3_k$, since this is equivalent to parametrizing 2-dimensional subspaces of 4-dimensional space.

Proof. Let $N = \binom{d+3}{3} - 1$. We construct a closed subvariety X of $\mathbb{P}^N_k \times \mathrm{Gr}(2,4)$ whose closed points measure the incidence

$$X = \{(S, \ell) : \ell \text{ line in } \mathbb{P}^3_k, S \text{ surface of degree } d, \ell \subset S\}.$$

To make this explicit, we consider the open coordinate patches $\mathrm{Gr}(2,4)_I \simeq \mathbb{A}^4_k$ that cover the Grassmannian. Take without loss of generality $I = \{1, 2\}$. Recall that, in these coordinates, a point of $\mathrm{Gr}(2,4)$ is given by a matrix of the form

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

and the corresponding 2-dimensional subspace of k^4 is the kernel of this matrix. This kernel is spanned by

$$\begin{pmatrix} a \\ c \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} b \\ d \\ 0 \\ -1 \end{pmatrix}.$$

Hence, the set of degree d surfaces containing the line corresponding to this space in $\mathbb{P}^3_k$ are those given by degree d polynomial $f(x_0, x_1, x_2, x_3)$ satisfying

$$f(\lambda a + \mu b, \lambda c + \mu d, -\lambda, -\mu) \equiv 0 \tag{1}$$

identically as a cubic polynomial in λ and μ. Thus, we obtain $d+1$ relations involving the $N+1$ variables of $\mathbb{P}^N_k$ and a, b, c, d—one relation for each of the four possible monomials $\lambda^i \mu^j$ with $i + j = d$. These $d+1$ relations cut out X. Hence we realize an open subset of X as a closed subvariety of $\mathbb{P}^N_k \times \mathrm{Gr}(2,4)_{12}$. Patching these open subsets together over the six open subschemes $\mathrm{Gr}(2,4)_I$ that cover $\mathrm{Gr}(2,4)$ realizes X as a projective variety.

In fact, the projection $X \to \mathrm{Gr}(2,4)$ realizes X as a $\mathbb{P}^{N-d-1}$-bundle over $\mathrm{Gr}(2,4)$. Fix a specific line ℓ, say identified with (a, b, c, d) with coordinates in $\mathrm{Gr}(2,4)_I \simeq \mathbb{A}^4_k$ as above. Then the four relations that descend from equation (1) are all linear equations in the $N+1$ variables corresponding to the coefficients of a degree d in 4 variables.

Lemma 3.1.2. *The $d+1$ relations described by (1) are always linearly independent with respect to the $N+1$ variables corresponding to the $N+1$ coefficients of a degree d surface.*

Proof. For each of the $d+1$ relations, there is a variable that appears only in that relation and in no others. These are the variables corresponding to the coefficients of $x_2^i x_3^j$, $i+j=d$; each of them appears in one equation with coefficient $(-1)^d$ and coefficient 0 in all others. The independence of these variables implies the independence of all the relations. ■

Hence X is a $\mathbb{P}_k^{N-d-1}$ bundle over $\mathrm{Gr}(2,4)$, since the lemma shows that this is true fiberwise. In particular, X is a smooth projective variety. Since $\dim \mathrm{Gr}(2,4) = 4$, this implies that X has dimension $N-d+3$. Hence the image of the projection $X \to \mathbb{P}_k^N$ has positive codimension as long as $d > 3$, so the complement of this image contains a Zariski dense open set. But the image of this projection parametrizes degree d surfaces containing at least one line, yielding the theorem. ■

We summarize the important facts about X we found during the course of this proof, since they will be relevant later for the 27 lines:

Proposition 3.1.3. *For $d \geq 1$, the incidence variety $X \subset \mathbb{P}_k^{\binom{d+3}{3}-1} \times \mathrm{Gr}(2,4)$ is a $\mathbb{P}_k^{\binom{d+3}{3}-d-2}$-bundle over $\mathrm{Gr}(2,4)$. Hence X is a smooth projective variety of dimension $\binom{d+3}{3} - d + 2$.*

3.2 The Fermat cubic surface

In the proof of the 27 lines, we will bootstrap up from the assertion that *at least one* smooth cubic surface has 27 lines. Therefore, we first construct one such surface explicitly.

Definition 3.2.1. The *Fermat cubic surface* is the surface cut out by the polynomial

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$$

in $\mathbb{P}_k^3$. In characteristic not 3, this is easily checked to be a smooth surface over k.

Proposition 3.2.2. *The Fermat cubic surface has exactly 27 distinct lines.*

Proof. Let S denote the Fermat cubic surface. We claim that the lines contained in S are

$$x_0 + \omega x_i = x_j + \omega' x_k = 0$$

where $\{i,j,k\}$ is some permutation of $\{1,2,3\}$ and ω, ω' are two cube roots of unity (possibly equal to 1, distinct, or the same). In all cases, we can rewrite the polynomial defining S as

$$(x_0 + x_i)(x_0 + \omega x_i)(x_0 + \omega^2 x_i) + (x_j + x_k)(x_j + \omega' x_k)(x_j + {\omega'}^2 x_k)$$

This polynomial lies in the ideal $(x_0 + \omega x_i, x_j + \omega' x_k)$, so we conclude that S contains the corresponding line.

Before proving that these are the only lines lying on S, we count how many distinct lines we have described. Recall that we are assuming k algebraically closed of characteristic not 3, so there are three distinct cube roots of unity. There are 9 choices for the plane $x_0 + \omega x_i$ corresponding to the three choices for ω and x_i, respectively. For each choice, there are exactly three distinct planes of the form $x_j + \omega' x_k = 0$: there are two ways to order x_j and x_k and three choices for ω', but $x_j + \omega' x_k = 0$ defines the same plane as $x_k + {\omega'}^2 x_j = 0$. Hence there are 27 distinct pairs of planes of the given form. The line given by the intersection of such a pair of planes uniquely determines this pair: on this line, there is a relation $x_0 = -\omega x_i$ but no relation between x_0 and x_j or x_k, so ω and x_i are determined, and likewise x_j, x_k, ω' are determined up to the symmetry we have already observed. Hence there are exactly 27 distinct lines of the given form.

To show that no other lines are contained in S, let ℓ be some line in $\mathbb{P}^3$. By eliminating parameters and possibly permuting coordinates (which leaves S unaffected), we may assume without loss of generality that ℓ is given by $x_0 = ax_2 + bx_3, x_1 = cx_2 + dx_3$ for some $a,b,c,d \in k$, so that the line ℓ is parametrized by the free variables x_2, x_3. Therefore, in order to have $\ell \subset S$, we must have

$$(ax_2 + bx_3)^3 + (cx_2 + dx_3)^3 + x_2^3 + x_3^3 = 0.$$

for all values of $x_2, x_3 \in k$, and over an algebraically closed field this means that this polynomial must be the zero polynomial. Taking coefficients of each monomial term, we obtain the relations

$$\begin{aligned} a^3 + c^3 + 1 &= 0 \\ 3a^2b + 3c^2d &= 0 \\ 3ab^2 + 3cd^2 &= 0 \\ b^3 + d^3 + 1 &= 0. \end{aligned}$$

We claim that at least one of the variables a, b, c, d must be zero in order for this to be solvable. Else, we deduce $\left(\frac{a}{c}\right)^2 = -\frac{d}{b}$ and $\frac{a}{c} = -\left(\frac{d}{b}\right)^2$ simultaneously, hence both $\frac{a}{c}$ and $\frac{d}{b}$ are cube roots of -1. But we need

$$a^3 + c^3 + 1 = 0 \iff 2 + \left(\frac{a}{c}\right)^3 = 0,$$

contradiction. Hence, without loss of generality $d = 0$. We deduce from the fourth equation that b is a cube root of -1. Then the second equation gives $a = 0$, and first shows that c is also a cube root of -1. Thus the line is of the form $x_0 = -\omega x_3, x_1 = -\omega' x_2$, which up to permutation of coordinates is one of the lines already described. ∎

3.3 Proof of the 27 lines

We repeat the setup in the proof of Theorem 3.1.1. Let $X \subset \mathbb{P}^{19}_k \times \mathrm{Gr}(2,4)$ be the incidence variety for cubic surfaces containing lines in $\mathbb{P}^3_k$. We have already shown that X is an irreducible smooth variety of dimension 19, being a $\mathbb{P}^{15}$-bundle over $\mathrm{Gr}(2,4)$.

Lemma 3.3.1. *The projection $\pi : X \to \mathbb{P}^{19}_k$ is surjective:* every cubic surface contains at least one line*. Moreover, there exists an open set in $\mathbb{P}^{19}_k$ on which the fibers are 0-dimensional:* most cubic surfaces contain finitely many lines*.*

Proof. X and $\mathbb{P}^{19}_k$ are projective varieties over k, so by the cancellation property for projective morphisms we conclude that $\pi : X \to \mathbb{P}^{19}_k$ is a projective morphism. A projective morphism is closed, so the image $\pi(X)$ is a closed subset of $\mathbb{P}^{19}_k$. This image is irreducible since X is irreducible.

Let $P \in \mathbb{P}^{19}_k$ be the point corresponding to the Fermat cubic surface. We have seen that the fiber $\pi^{-1}(P)$ is 0-dimensional. We cite:

Theorem 3.3.2 ([Vak23], Theorem 12.4.3)**.** *For any closed morphism of varieties, the dimension of the fibers is upper-semicontinuous on both the source and the target.*

This tells us that the set $V = \{Q \in \mathbb{P}^{19}_k : \dim \pi^{-1}(Q) \geq 1\} \subset \pi(X)$ is closed. This set is a proper subset of the irreducible set $\pi(X)$ since it does not contain P, so we conclude that it has codimension at least 1 inside $\pi(X)$.

Suppose π is not surjective, so that $\pi(X)$ is a proper subvariety in $\mathbb{P}^{19}_k$. Then the map $X \to \pi(X)$ is a morphism from a 19-dimensional variety to a variety of dimension at most 18, but a dense open subset of $\pi(X)$ is supposed to have nonempty finite fibers. That this is a contradiction is made precise via:

Theorem 3.3.3 ([Vak23], Theorem 12.4.1)**.** *Suppose $\pi : X \to Y$ is a finitely presented dominant morphism of irreducible schemes such that $K(X)/K(Y)$ has transcendence degree r. Then there exists a nonempty open subset $U \subseteq Y$ such that for almost all $q \in U$, the fiber over q has pure dimension r, or is empty.*

None of the fibers over $\pi(X)$ are empty, so if $\dim \pi(X) \leq 18$, hence $K(X)/K(\pi(X))$ of transcendence degree at least 1, we must have a dense open subset of $\pi(X)$ with fibers of dimension at least 1, contradicting the fact that a dense open subset of $\pi(X)$ has fiber dimension 0. We conclude that π is surjective and that a dense open subset of $\mathbb{P}^{19}_k$ has 0-dimensional fibers. ∎

Thus far we have not used anything critical about *smooth* cubic surfaces. Let $\Delta \subset \mathbb{P}^{19}_k$ denote the (closed) locus of singular cubic surfaces. We wish to show that π is a finite flat morphism over $\mathbb{P}^{19}_k \setminus \Delta$. While this lets us deduce that all the fibers have height 27 (the height must be constant, so it must match the Fermat cubic surface), this alone is not yet enough to conclude the 27 lines. *A priori*, some of these fibers might be nonreduced, in which case the cubic surface would have fewer than 27 *distinct* lines (but would, in some sense, have a "multiple line" through it). The intuition is that lines might "come together" or "split apart" as we vary the cubic. Proving that this does not happen for smooth cubics is technical, so we delay its proof for the moment and assume the necessary result:

Lemma 3.3.4. *If ℓ is a line on a smooth cubic surface, then ℓ is a reduced isolated point in the fiber of π over $\mathbb{P}^{19}_k \setminus \Delta$.*

We may use this lemma to quickly conclude the proof of the 27 lines. Let $U = \mathbb{P}^{19}_k \setminus \Delta$ be the locus of smooth cubic surfaces, which is a dense open subset, hence of dimension 19. By surjectivity of π, the restriction $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is a morphism of regular varieties of the same dimension. Lemma 3.3.4 states that the fibers over closed points consist of isolated points, hence these fibers are 0-dimensional. By Miracle Flatness ([Har77], Exercise III.10.9), this is a flat morphism.[2] Therefore the height of the fibers of π over the closed points of U, i.e. the number

$$\dim_k \pi_*(\mathscr{O}_X) \otimes_{\mathscr{O}_{U,Q}} \kappa(Q)$$

for closed $Q \in U$, is constant ([Mum99], §III.10, p. 218). Since the Fermat cubic surface lies in U, this constant height must be 27. Since all fibers are reduced and consist of isolated points, we conclude that these fibers consist of 27 distinct closed points, corresponding to 27 distinct lines lying on a smooth cubic surface.

3.4 The points corresponding to line in the fibers are isolated and reduced

We conclude by proving Lemma 3.3.4. Let ℓ be a line in $\mathbb{P}^3_k$ and let S be a cubic surface containing it, say cut out by the cubic polynomial f in four variables. By change of coordinates, we may assume that ℓ corresponds to the point $(0,0,0,0)$ in the coordinate chart $\mathrm{Gr}(2,4)_{12} \simeq \mathbb{A}^4_k$ as described previously. Unraveling the construction of the Grassmannian, this means we assume $\ell \subset \mathbb{P}^3_k$ is cut out by $\{x_0 = x_1 = 0\}$. Let $P \in \mathbb{P}^{19}_k$ be the point corresponding to the cubic f. Based on our description of the incidence variety X, the portion of scheme-theoretic fiber $\pi^{-1}(P)$ lying over $\mathrm{Gr}(2,4)_{12}$ is cut out in X by the four equations that describe the relation

$$f(\lambda a + \mu b, \lambda c + \mu d, -\lambda, -\mu) \equiv 0,$$

vanishing identically as a polynomial in λ, μ. More precisely, we recall that the portion of X lying over $\mathrm{Gr}(2,4)_{12}$ is cut out as a subset of $\mathbb{P}^{19}_k \times \mathbb{A}^4_k$ by four equations involving the 20 coefficients of a cubic polynomial as well as the 4 variables corresponding to the coordinates of $\mathrm{Gr}(2,4)_{12}$. In the fiber above, we set each of the 20 coefficients to the value they take for f, so we may also think of this fiber as lying over $\mathrm{Gr}(2,4)_{12} \simeq \mathbb{A}^4_k$.

We need to show that ℓ corresponds to a reduced isolated point of this fiber. Letting g_1, g_2, g_3, g_4 be the four equations cutting out the fiber, this means that we need to show that the origin is isolated and reduced in

$$\operatorname{Spec} k[a,b,c,d]/(g_1,g_2,g_3,g_4).$$

That the origin is isolated is clear, since (a,b,c,d) is a maximal ideal, so it remains to show reducedness. If there are any nilpotents in the stalk at the origin, they will remain after quotienting by the ideal $(a,b,c,d)^2$, so we may instead prove reducedness of the origin in

$$\operatorname{Spec} k[a,b,c,d]/((a,b,c,d)^2 + (g_1,g_2,g_3,g_4)).$$

[2] Note that Miracle Flatness can be proven using only the condition on the fiber dimension at closed points, since then flatness at the nonclosed points follows from the fact that flatness is stable under localization.

Therefore we need only consider terms in the g_i of degree 0 or 1 in the variables a, b, c, d. The linear terms are those coming from the coefficients of the following monomials of a general cubic surface:

$$x_0x_2^2, x_0x_2x_3, x_0x_3^2, x_1x_2^2, x_1x_2x_3, x_1x_3^2.$$

Let the coefficients of these terms in f be α_{ijk}, e.g. α_{022} is the coefficient of $x_0x_2^2$. The constant terms are those coming from the monomials:

$$x_2^3, x_2^2x_3, x_2x_3^2, x_3^3.$$

However, the fact that ℓ lies on the surface S implies that $f(0, 0, -\lambda, -\mu) \equiv 0$. Therefore, the coefficients of $x_2^3, x_2^2x_3, x_2x_3^2$ and x_3^3 in f must all be 0—that is, f lies in the ideal (x_0, x_1) inside $k[x_0, x_1, x_2, x_3]$. We deduce that, modulo $(a, b, c, d)^2$, the four relations $g_i = 0$ corresponding to the vanishing of the $\lambda^3, \lambda^2\mu, \lambda\mu^2$ and μ^3 terms in order are

$$\begin{aligned} \alpha_{022}a + \alpha_{122}c &= 0 \qquad\qquad (*) \\ \alpha_{023}a + \alpha_{022}b + \alpha_{123}c + \alpha_{122}d &= 0 \\ \alpha_{033}a + \alpha_{023}b + \alpha_{133}c + \alpha_{123}d &= 0 \\ \alpha_{033}b + \alpha_{133}d &= 0 \end{aligned}$$

If we can show that the linear system $(*)$ in the variables a, b, c, d has full rank, the proof is complete, since this implies

$$k[a, b, c, d]/((a, b, c, d)^2 + (g_1, g_2, g_3, g_4)) \simeq k,$$

hence the origin is a reduced point of the fiber.

We finally invoke smoothness of the surface S. In particular, S is smooth at all points on $\ell = \{[0 : 0 : s : t]\}$. The derivatives $\frac{\partial f}{\partial x_2}$ and $\frac{\partial f}{\partial x_3}$ always vanish along this line, recalling that f lies in the ideal (x_0, x_1). Hence, the fact that S is smooth at the point $[0 : 0 : s : t] \in \ell$ implies that

$$\frac{\partial f}{\partial x_0} = \alpha_{022}s^2 + \alpha_{023}st + \alpha_{033}t^2$$

and

$$\frac{\partial f}{\partial x_1} = \alpha_{122}s^2 + \alpha_{123}st + \alpha_{133}t^2$$

do not simultaneously vanish for any $[s : t] \in \mathbb{P}^1_k$. That is, the two sets of roots of these two polynomials are disjoint. The linear system $(*)$ may be rewritten suggestively as

$$(as + bt)\frac{\partial f}{\partial x_0} + (cs + dt)\frac{\partial f}{\partial x_1} \equiv 0,$$

vanishing identically as a polynomial in $k[s, t]$.

First suppose that neither $\frac{\partial f}{\partial x_0}$ nor $\frac{\partial f}{\partial x_1}$ are perfect squares, so that each has two distinct roots. Let $[s' : t']$ be a root of $\frac{\partial f}{\partial x_0}$. Then we have

$$(cs' + dt')\frac{\partial f}{\partial x_1}(s', t') = 0,$$

but $[s' : t']$ is not a root of $\frac{\partial f}{\partial x_1}$, so we conclude that $cs' + dt' = 0$. If c and d are not both 0, this is equivalent to saying $[c : d] = [-t' : s']$. However, we may repeat this for the other root $[s'' : t'']$ of $\frac{\partial f}{\partial x_0}$ to conclude $[c : d] = [-t'' : s'']$ unless $c = d = 0$. Since we assume $[s' : t'] \neq [s'' : t'']$, we conclude that $c = d = 0$. The same argument applied to the roots of $\frac{\partial f}{\partial x_1}$ shows $a = b = 0$.

Now suppose that one of these derivatives, say $\frac{\partial f}{\partial x_0}$, has a double root, but $\frac{\partial f}{\partial x_1}$ does not. We may perform a change of coordinates in $\mathbb{P}^3_k$ to assume that the double root occurs at the point $[0 : 1]$, so that $\frac{\partial f}{\partial x_0} = \alpha_{022}s^2$, i.e. $\alpha_{023} = \alpha_{033} = 0$. (Note that a change of coordinates preserving the line ℓ does not affect any previous parts of the proof.) Since the two derivatives do not share roots, we conclude that $\alpha_{133} \neq 0$. The same argument as the previous case goes through to show that $a = b = 0$. From the linear system $(*)$, we deduce that $d = 0$, and since at least one of α_{123} or α_{133} is not zero, we conclude $c = 0$.

Finally, suppose that both derivatives have double roots. We may again make a change of variables in $\mathbb{P}^3_k$, scaling and shifting the line ℓ so that the double roots occur at $[0:1]$ and $[1:0]$, respectively. Hence all of the α_{ijk} vanish except for α_{022} and α_{133}, which do not vanish. From linear system $(*)$, we immediately conclude that $a = b = c = d = 0$.

In summary, our work shows that $a = b = c = d = 0$ always in $k[a,b,c,d]/((a,b,c,d)^2 + (g_1, g_2, g_3, g_4))$. Hence the origin is a reduced point, concluding the proof of Lemma 3.3.4.

References

Brion, Michel. *Lectures on the geometry of flag varieties.* 2004. arXiv: math/0410240 [math.AG].

Hartshorne, Robin. *Algebraic geometry.* Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496. ISBN: 0-387-90244-9.

Mumford, David. *The red book of varieties and schemes.* expanded. Vol. 1358. Lecture Notes in Mathematics. Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello. Springer-Verlag, Berlin, 1999, pp. x+306. ISBN: 3-540-63293-X. DOI: 10.1007/b62130. URL: https://doi-org.libproxy.berkeley.edu/10.1007/b62130.

Stacks project authors. *The Stacks project.* https://stacks.math.columbia.edu. 2023.

Vakil, Ravi. *The Rising Sea: Foundations of Algebraic Geometry.* 2023. URL: https://math.stanford.edu/~vakil/216blog/.