Endomorphism algebras of the simple rational Dieudonné modules

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This exercise is Problem 9 from Problem Set 5 of Lassina Dembélé's course in the Preliminary Arizona Winter School 2023. I decided to do a writeup of this problem (among the many great exercises from this course) because I found the result interesting, and also because the outline given in the problem set was easier to follow than another source I had been reading ([Dem72, §IV.3]).

Let k be an algebraically closed field of characteristic p and let $W(k)$ be the Witt ring of k. The assoicated Dieudonné ring is the noncommutative ring $D_k = W(k)[T,V]/\sim$, with relations $TV = VT = p$, $F\lambda = \lambda^\sigma F$, and $V\lambda^\sigma = \lambda V$ for $\lambda \in W(k)$, where σ is the natural Frobenius automorphism on $W(k)$. The rational Dieudonné ring is the division is $D_k[1/p]$.

Let m, n be a pair of nonnegative coprime integers. Then we define the (left) $D_k[1/p]$-module

$$N_{m,n} := D_k[1/p]/D_k[1/p](F^m - V^n),$$

treating $(F^m - V^n)$ as a left ideal.

Fact: this module is a simple a simple object in the category of $D_k[1/p]$-modules, or equivalently in the isogeny category of Dieudonné modules over k. This exercise proves:

Theorem 1. $\mathrm{End}_{D_k[1/p]}(N_{m,n})$ *is a finite-dimensional central division algebra over* $\mathbb{Q}_p$ *with Hasse invariant* $n/(m+n)$.

Left multiplication by F^n is an automorphism of $D_k[1/p]$, so we may alternatively identify

$$N_{m,n} \simeq D_k[1/p]/D_k[1/p](F^{m+n} - p^n).$$

Thus the powers $1, F, F^2, \ldots, F^{m+n-1}$ form a $W(k)[1/p]$-basis for $N_{m,n}$. Let φ be a $D_k[1/p]$-endomorphism of $N_{m,n}$, and write

$$\varphi(1) = \sum_{i=0}^{m+n-1} a_i F^i$$

with $a_i \in W(k)[1/p]$. Since φ is a $D_k[1/p]$-endomorphism, we must have

$$(F^{m+n} - p^n)\varphi(1) = \varphi(F^{m+n} - p^n) = 0.$$

We expand

$$\begin{aligned}
(F^{m+n} - p^n)\varphi(1) &= \sum_{i=0}^{m+n-1} F^{m+n} a_i F^i - \sum_{i=0}^{m+n-1} p^n a_i F^i \\
&= \sum_{i=0}^{m+n-1} a_i^{\sigma^{m+n}} F^{m+n+i} - \sum_{i=0}^{m+n-1} p^n a_i F^i \\
&= \sum_{i=0}^{m+n-1} p^m a_i^{\sigma^{m+n}} F^i - \sum_{i=0}^{m+n-1} p^n a_i F^i.
\end{aligned}$$

Since the F^i are a $W(k)[1/p]$-basis, we conclude that all of the a_i are fixed by the p^{m+n}-th power Frobenius, i.e. they lie in $\mathbb{Q}_{p^{m+n}} \subset W(k)[1/p]$ (the maximal unramified degree $m+n$ extension of $\mathbb{Q}_p$). Conversely, if $\varphi(1)$ is of this form, then left multiplication by $\varphi(1)$ is a well-defined endomorphism of $N_{m,n}$, so we conclude $\dim_{\mathbb{Q}_p} \mathrm{End}_{D_k}[1/p](N_{m,n}) = (m+n)^2$.

If φ is central in $\mathrm{End}_{D_k[1/p]}(N_{m,n})$, then φ must, in particular, commute with left multiplication by F, so in this case all of the a_i lie in $\mathbb{Q}_p$. But φ must also commute with left multiplication by any element of $W(k)$, which shows that $a_i = 0$ for all $i > 0$. Hence the center of $\mathrm{End}_{D_k[1/p]}(N_{m,n})$ is $\mathbb{Q}_p$ (via left multiplication). Since $N_{m,n}$ is simple, $\mathrm{End}_{D_k}[1/p](N_{m,n})$ is a division algebra, so it is a (finite-dimensional) central division algebra over $\mathbb{Q}_p$.

For any central division algebra D over $\mathbb{Q}_p$ with maximal subfield L and $[D : \mathbb{Q}_p]^{1/2} = n$, there exists an element α such that the Frobenius endomorphism of L is given by $\beta \mapsto \alpha\beta\alpha^{-1}$, and α is uniquely determined up to multiplication by $L^\times$. We may always write $\alpha^n = up^r$ for some $u \in \mathcal{O}_L^\times$, and the Hasse invariant of D is defined to be $r/n \in \mathbb{Q}/\mathbb{Z}$, which is well-defined and independent of the choice of α.

In our case with $D = \mathrm{End}_{D_k[1/p]}(N_{m,n})$, we may take $L = \mathbb{Q}_{p^{m+n}}$, since any maximal subfield always has degree $[\mathrm{End}_{D_k[1/p]}(N_{m,n}) : \mathbb{Q}_p]^{1/2} = m+n$, and $\alpha = F$, since $F\beta F^{-1} = \beta^\sigma F F^{-1} = \beta^\sigma$. Since $F^{m+n} = p^n$, we conclude that the Hasse invariant is $n/(m+n)$.

References

[Dem72] Michel Demazure. *Lectures on p-divisible groups*, volume 302 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1972.