Endomorphism algebras of the simple F-isocrystals

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This exercise is Problem 9 from Problem Set 5 of Lassina Dembélé's course in the Preliminary Arizona Winter School 2023. I decided to do a writeup of this problem (among the many great exercises from this course) because I found the result interesting, and also because the outline given in the problem set was easier to follow than another source I had been reading ([Dem72, §IV.3]).

Let k be an algebraically closed field of characteristic p and let W(k) be the Witt ring of k. The associated Dieudonné ring is the noncommutative ring $D_k = W(k)[T, V] / \sim$, with relations TV = VT = p, $F\lambda = \lambda^{\sigma} F$, and $V\lambda^{\sigma} = \lambda V$ for $\lambda \in W(k)$, where σ is the natural Frobenius automorphism on W(k). The rational Dieudonné ring is the division is $D_k[1/p]$. Modules over this ring are known as *F*-isocrystals.

Let m, n be a pair of nonnegative coprime integers. Then we define the (left) $D_k[1/p]$ -module

$$N_{m,n} := D_k[1/p]/D_k[1/p](F^m - V^n),$$

treating $(F^m - V^n)$ as a left ideal.

The Dieudonné-Manin theorem states that the category of F-isocrystals over k is semisimple, and moreover that the simple objects in this category are the $N_{m,n}$, which are pairwise non-isomorphic. That is, every F-isocrystal can be expressed as a direct sum of the $N_{m,n}$, uniquely up to reordering. Algebraic closure of k is necessary here. There are more isocrystals than just the $N_{m,n}$ if we allow arbitrary perfect k, and moreoever the category of F-isocrystals may no longer be semisimple. However, even in this case, we still get *isoclinic* decompositions of F-isocrystals. If M is an F-isocrystal over k, and $M_{\overline{k}} = \bigoplus_{m,n} N_{m,n}^{\oplus i_{m,n}}$ is the base change to an F-isocrystal over \overline{k} (via $W(k) \hookrightarrow W(\overline{k})$), then M decomposes as $\bigoplus_{m,n} \mathcal{N}_{m,n}$ as an F-isocrystal over k, where $\mathcal{N}_{m,n}$ is some F-isocrystal over k that base changes to $N_{m,n}^{\oplus i_{m,n}}$ over \overline{k} . That is, we can still uniquely decompose M into components determined by slope, although these individual components may not be semisimple. The decomposition $M = \bigoplus_{m,n} \mathcal{N}_{m,n}$ is known as the *isoclinic* decomposition.

This exercise proves:

Theorem 1. Let k be algebraically closed. Then $\operatorname{End}_{D_k[1/p]}(N_{m,n})$ is a finite-dimensional central division algebra over \mathbb{Q}_p with Hasse invariant n/(m+n).

Left multiplication by F^n is an automorphism of $D_k[1/p]$, so we may alternatively identify

$$N_{m,n} \simeq D_k[1/p]/D_k[1/p](F^{m+n} - p^n).$$

Thus the powers $1, F, F^2, \ldots, F^{m+n-1}$ form a W(k)[1/p]-basis for $N_{m,n}$. Let φ be a $D_k[1/p]$ -endomorphism of $N_{m,n}$, and write

$$\varphi(1) = \sum_{i=0}^{m+n-1} a_i F^i$$

with $a_i \in W(k)[1/p]$. Since φ is a $D_k[1/p]$ -endomorphism, we must have

$$(F^{m+n} - p^n)\varphi(1) = \varphi(F^{m+n} - p^n) = 0.$$

We expand

$$\begin{split} (F^{m+n} - p^n)\varphi(1) &= \sum_{i=0}^{m+n-1} F^{m+n} a_i F^i - \sum_{i=0}^{m+n-1} p^n a_i F^i \\ &= \sum_{i=0}^{m+n-1} a_i^{\sigma^{m+n}} F^{m+n+i} - \sum_{i=0}^{m+n-1} p^n a_i F^i \\ &= \sum_{i=0}^{m+n-1} p^m a_i^{\sigma^{m+n}} F^i - \sum_{i=0}^{m+n-1} p^n a_i F^i. \end{split}$$

Since the F^i are a W(k)[1/p]-basis, we conclude that all of the a_i are fixed by the p^{m+n} -th power Frobenius, i.e. they lie in $\mathbb{Q}_{p^{m+n}} \subset W(k)[1/p]$ (the maximal unramified degree m + n extension of \mathbb{Q}_p). Conversely, if $\varphi(1)$ is of this form, then left multiplication by $\varphi(1)$ is a well-defined endomorphism of $N_{m,n}$, so we conclude $\dim_{\mathbb{Q}_p} \operatorname{End}_{D_k}[1/p](N_{m,n}) = (m+n)^2$.

If φ is central in $\operatorname{End}_{D_k[1/p]}(N_{m,n})$, then φ must, in particular, commute with left multiplication by F, so in this case all of the a_i lie in \mathbb{Q}_p . But φ must also commute with left multiplication by any element of W(k), which shows that $a_i = 0$ for all i > 0. Hence the center of $\operatorname{End}_{D_k[1/p]}(N_{m,n})$ is \mathbb{Q}_p (via left multiplication). Since $N_{m,n}$ is simple, $\operatorname{End}_{D_k}[1/p](N_{m,n})$ is a division algebra, so it is a (finite-dimensional) central division algebra over \mathbb{Q}_p .

For any central division algebra D over \mathbb{Q}_p with maximal subfield L and $[D:\mathbb{Q}_p]^{1/2} = n$, there exists an element α such that the Frobenius endomorphism of L is given by $\beta \mapsto \alpha \beta \alpha^{-1}$, and α is uniquely determined up to multiplication by L^{\times} . We may always write $\alpha^n = up^r$ for some $u \in \mathcal{O}_L^{\times}$, and the Hasse invariant of D is defined to be $r/n \in \mathbb{Q}/\mathbb{Z}$, which is well-defined and independent of the choice of α .

In our case with $D = \operatorname{End}_{D_k[1/p]}(N_{m,n})$, we may take $L = \mathbb{Q}_{p^{m+n}}$, since any maximal subfield always has degree $[\operatorname{End}_{D_k[1/p]}(N_{m,n}) : \mathbb{Q}_p]^{1/2} = m + n$, and $\alpha = F$, since $F\beta F^{-1} = \beta^{\sigma} FF^{-1} = \beta^{\sigma}$. Since $F^{m+n} = p^n$, we conclude that the Hasse invariant is n/(m+n).

References

[Dem72] Michel Demazure. Lectures on p-divisible groups, volume 302 of Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1972.