Chevalley's trick

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1 Introduction

Let G be an algebraic group over a field k with closed subgroup H (not necessarily normal). We would like to come up with a space G/H, called a *homogeneous space*, that is algebraic and somehow corresponds to the idea of a coset space. It should be equipped with a transitive G-action whose points correspond to cosets of H in G. We should be more scheme-theoretically precise: when we are speaking of points, transitivity, and cosets, we mean on the set of k-valued points. In general, constructing such a space is subtle—it is not even a scheme in general, but rather an algebraic space.

Fortunately, there are many special cases where this construction is easier.

Example 1.1. Let $G = \operatorname{Spec} A$ be an affine commutative group variety with subgroup $H = \operatorname{Spec} B$. Then we may define G/H to $\operatorname{Spec} C$, where

$$C = \{a \in A : \tau_h^* a = a \ \ \forall h \in H(k)\}.$$

That is, we take C to be the subring of A that is fixed by translation by H. $\operatorname{Spec} C$ inherits the structure of a commutative group variety over k via restricting the comultiplication on A to C, and it satisfies the categorical properties of the cokernel of the map $H \to G$.

Example 1.2. If A is an abelian variety over k, then one may apply the theory of fpqc descent to meaningfully treat A as a quotient $A/A[n]$, or more generally build A/B for a finite subgroup B.

Example 1.3. If G is a linear algebraic group and H is a closed subgroup, then there is a trick to giving G/H the structure of a quasiprojective variety, called the *Chevalley trick*.

We will discuss this last example. The main idea is to construct a finite dimensional G-representation W such H is the stabilizer of some subspace of W. Thus H is the stabilizer of a point in a Grassmannian, and we define G/H to be the orbit of this point, which is locally closed. We will assume some background on the Grassmannian; I have also written an essay about this topic.

The space G/H extremely useful in geometric representation theory, especially in the case where H is a Borel or parabolic subgroup. Its existence is the starting point for flag varieties, the Borel-Weil-Bott theorem, and the theory of L-packets, to name a few.

Most of these basic results can be found in [Hum75].

2 Faithful representations of affine algebraic groups

Lemma 2.1. *Let G be an affine algebraic group over a field k, and let $f_1, \ldots, f_n$ be a finite subset of $\Gamma(G, \mathcal{O}(G))$. Then there exists a finite dimensional subrepresentation of $\Gamma(G, \mathcal{O}(G))$ (equipped with either the left or right regular action) containing the f_i.*

Proof. We adapt [Dem72]. It suffices to prove this for $n = 1$, say $f_1 = x$, and by symmetry we need only prove the case of the right regular representation. Say $G = \operatorname{Spec} A$. The group law on G corresponds to a comultiplication law $\Delta : A \to A \otimes_k A$. Choose a k-basis $\{a_i\}$ of A and write

$$\Delta(x) = \sum_i x_i \otimes a_i$$

Let F be the k-vector space spanned by x and the x_i. Then F is finite-dimensional, and we claim that F is invariant under right G-translation, hence satisfying our criteria.

Let $\epsilon : A \to k$ be the counit map induced by the unit map $e : \operatorname{Spec} k \to G$. Then

$$((\mathbf{id} \otimes \epsilon) \circ \Delta)(x) = \sum_i x_i \epsilon(a_i),$$

so this element lies in F. Comultiplication is coassociative, which means

$$\sum_i (\Delta x_i) \otimes a_i = ((\Delta \otimes \mathbf{id}) \circ \Delta)(x) = ((\mathbf{id} \otimes \Delta) \circ \Delta)(x) = \sum_i x_i \otimes \Delta(a_i).$$

Write $\Delta a_i = \sum_j b_{ij} \otimes a_j$ for finitely many nonzero b_{ij}. We have

$$\sum_i (\Delta x_i) \otimes a_i = \sum_{i,j} (x_i \otimes b_{ij}) \otimes a_j$$

so matching coordinates (using the fact that the a_i form a basis for A) shows that $\Delta x_i = \sum_{i,j} x_j \otimes b_{ji}$. Hence $\Delta x_i \in F \otimes A$, and by definition $\Delta x \in F \otimes A$.

The right regular representation is defined by $(R_g f)(h) = f(hg)$, which translates to $R_g f = ((\mathbf{id} \otimes \operatorname{ev}_g) \circ \Delta) f$, where ev_g is the functional $A \to k$ induced by the k-point $g \hookrightarrow G$. Hence, with x_i as above,

$$R_g x_i = ((\mathbf{id} \otimes \operatorname{ev}_g) \circ \Delta) x_i \in F \otimes \operatorname{ev}_g(A) \simeq F.$$

Likewise $R_g x \in F$. Thus F is stable under this action, as desired. ■

I think this lemma is a compelling example of the power of Hopf algebras. It also implies the very important:

Theorem 2.2. *Affine group varieties have a faithful finite-dimensional representations.*

Proof. If G is affine, let $f_1, \ldots, f_n$ generate its coordinate ring, and let V be a finite dimensional subrepresenation of the right regular representation containing the f_i by the previous lemma. Then V is faithful, since if R_g fixes each f_i, then it fixes the entire coordinate ring and induces the identity map on G, so $g = e$. ■

This also implies that affine algebraic groups are the same as linear alebgraic groups.

3 Chevalley's trick and applications

Theorem 3.1. (Chevalley's Trick.) *Let G be a linear algebraic group with closed subgroup H. Then there exists a finite dimensional representation $\rho : G \to \mathrm{GL}(V)$ and $x \in \mathbb{P}(V)$ such that $H = \operatorname{Stab}_G(x)$ under the induced action on $\mathbb{P}(V)$.*

Proof. Consider the left regular representation $G \curvearrowright H^0(G, \mathcal{O}_G)$, defined on k-points by $(L_g f)(x) = f(g^{-1}x)$. Let I_H be the ideal of the subgroup H. Then H is exactly the stablizer of I_H: we have $H = \{g \in G : L_g(I_H) = I_H\}$. Elements of I_H are characterized by the fact that $f(h) = 0$ for all $h \in H(k)$ if and only if $f \in I_H$, and $H(k)$ is characterized by $h \in H(k)$ if and only if $f(h) = 0$ for all $f \in I_H$. It is clear that H is contained in this stabilizer. Conversely, if $g \notin H(k)$, then there exists $f \in I_H$ such that $f(g) = 1$. Then $(L_g^{-1} f)(e) = f(g) = 1$, which implies that g does not stabilize I_H—otherwise, $L_g^{-1} f \in I_H$, but any such element must kill e.

Write $I_H = (f_1, \ldots, f_n)$. By Lemma 2.1, we can always find a G-stable finite dimensional subspace $W \subseteq \Gamma(G, \mathcal{O}(G))$ with $f_1, \ldots, f_n \in W$. Let $L = W \cap I_H$. Then $f_1, \ldots, f_n \in L$, so if $g \in G(k)$ stabilizes L, it also stabilizes I_H, hence $g \in H(k)$. We conclude that $\operatorname{Stab}_G(L) = H$.

Let $n := \dim_k W$ and $d := \dim_k L$. Then the action $G \curvearrowright W$ yields an action on the Grassmannian $\operatorname{Gr}(d, n)$. The subspace L corresponds to a point x in this Grassmannian, so we have $\operatorname{Stab}_G(x) = H$. Let $V = \bigwedge^d W$. Then V inherits the structure of a G-representation, and the Plücker embedding $\operatorname{Gr}(d, n) \hookrightarrow \mathbb{P}(V)$ is compatible with the G-action. Hence $x \hookrightarrow \mathbb{P}(V)$ is stabilized by H, as desired. ∎

Corollary 3.2. *Let G be a linear algebraic group with closed subgroup H. Then there exists a smooth quasiprojective variety, denoted G/H, equipped with a transitive G-action with kernel H.*

Proof. Let $x \in \mathbb{P}(V)$ be as in Chevalley's trick. Then the orbit $Y := G \cdot x$ is constructible by Chevalley's theorem (see [Har77, Exercise II.3.19], since it is the image of a finite type morphism of noetherian schemes defined by $G \to \mathbb{P}(V) : g \mapsto g \cdot x$. Then $\overline{Y}$ is stable under the action of G by a density argument, and Y contains an open subset U of $\overline{Y}$, again by [Har77, Exercise II.3.19(b)]. The action of G is transitive on Y, and each translation is a homeomorphism of $\overline{Y}$. Therefore, let $y_0 \in U$ and let $y \in Y$ be some other point. Choosing $g \in G$ with $g \cdot y_0 = y$, we also have $y \in g \cdot U \subseteq Y$, so we conclude that y is an interior point of Y inside $\overline{Y}$. Hence Y is open in $\overline{Y}$, so Y is locally closed in $\mathbb{P}(V)$ and may therefore be endowed with the structure of a quasiprojective variety. Transitivity of the G-action implies that all stalks of Y are isomorphic, hence Y is smooth. Thus we may define $G/H := Y$. ∎

Corollary 3.3. *Let G be a linear algebraic group with* normal *closed subgroup H. Then there exists a representation $\rho : G \to \operatorname{GL}(W)$ such that $\ker \rho = H$. Hence we may define G/H to be the image of G in $\operatorname{GL}(W)$, which inherits the structure of a group variety.*

Proof. Take a vector space V as in the Chevalley trick, so that there exists $v \in V$ with $\operatorname{Stab}_G(\langle v \rangle) = H$. Hence H acts on $\langle v \rangle$ by characters $H \to \mathbb{G}_m$. For a given character $\chi : H \to \mathbb{G}_m$, set $V_\chi = \{v \in : hu = \chi(h) u \ \forall h \in H(k)\}$, the χ-isotypic subspace of V. Write $V' = \bigoplus_\chi V_\chi$, ranging over all characters of χ; note that the space V_χ are linearly independent inside V. Since H is assumed to be normal, we have $g(V_\chi) = V_{\chi^g}$, where $\chi^g : H \to \mathbb{G}_m$ is the character defined on points by $\chi^g(h) = \chi(g^{-1} h g)$. This shows that V' is stable under the action of G, so without loss of generality we may shrink V to assume $V = V'$.

Define $W \subseteq V \otimes V^*$ by

$$W := \bigoplus_\chi V_\chi \otimes V_\chi^*.$$

Then H acts trivially on W, since it acts by $\chi \cdot \chi^{-1} = 1$ on each component. Conversely, a given $g \in G(k)$ acts trivially on W if and only if g acts by a scalar multiple on each V_θ. But since our original choice of v lies in one of the V_χ, this implies that $g \in H(k)$. ∎

Definition 3.4. A *Borel subgroup* of an algebraic group is a maximal solvable connected subgroup.

Theorem 3.5. *Let G be a linear algebraic group.*

(a) If B is a Borel subgroup, then G/B is a projective variety.

(b) All Borel subgroups are conjugate to each other.

Proof. Let B_0 be a Borel subgroup of maximal dimension. By Chevalley's trick, we may choose a representation $\rho : G \to \operatorname{GL}(V)$ such that $B_0 = \operatorname{Stab}_G V_1$ for a 1-dimensional subspace $V_1 \subseteq V$. By Lie's theorem, we may inductively extend to a full flag F given by $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = V$, where $\dim V = n$, that is stabilized by B_0. Letting $\mathcal{F}$ be full flag variety of V, the orbit $G \cdot F$ is isomorphic to G/B_0; note that $B_0 = \operatorname{Stab}_G(F)$ since $G - B_0$ does not preserve the V_1 part of the flag F.

We claim that $G \cdot F$ is an orbit of minimal dimension in $\mathcal{F}$. Let F' be some other flag. Then $B := \mathrm{Stab}_G(F')$ is also solvable: choosing a basis of V based on the flag F', the image $\rho(B)$ in $\mathrm{GL}(V)$ lies inside the upper triangular matrices. The kernel of ρ is contained in B_0, so $\ker \rho$ is also solvable; hence B fits into an exact sequence $0 \to \ker \rho \cap B \to B \to \rho(B) \to 0$ where the outer two groups are solvable, so B is solvable. Since B_0 is chosen to be a connected solvable subgroup of maximal dimension, we must have $\dim B_0 \geq \dim B^0 = \dim B$, where B^0 denotes the neutral component of B. Hence $\dim G \cdot F \leq \dim G \cdot F'$, so $G \cdot F = G/B_0$ is an orbit of minimal dimension.

$\overline{G/B_0} \setminus G/B_0$ is also a union of orbits of G and has dimension strictly less than $G \cdot F$, so by minimality we conclude that this complement is empty. Hence, G/B_0 is closed in the projective variety $\mathcal{F}$, hence projective itself, proving (a) assuming (b).

For a general Borel B, the fixed point theorem (see below) asserts that B always has a fixed point x on G/B_0. Since G/B_0 is the orbit of a point with stabilizer B_0, we conclude

$$B \subseteq \mathrm{Stab}_G(x) = gB_0g^{-1}.$$

But gB_0g^{-1} is a solvable connected subgroup, and B is supposed to be a maximal solvable connected subgroup. Hence $B = gB_0g^{-1}$, yielding (b). ■

Theorem 3.6. (Fixed point theorem.) *Let X be a nonempty projective variety, and let G be a connected solvable affine algebraic group that acts on X. Then G has a fixed point on X.*

Proof. Suppose first that G is commutative. Then take a closed orbit $Y \subseteq X$ (e.g. a minimal orbit), and let $H = \mathrm{Stab}_G\, y$ for some $y \in Y$. Then H is automatically normal, and $Y = G/H$ is affine but also projective, hence just a point.

In general, we induct on the dimension of G. Let $H = [G, G]$, so that $\dim H < \dim G$ and G/H is commutative. Then H has a fixed point on X by inductive hypothesis. Let Y be the nonempty set of all fixed points of H. Since H is normal in G, Y is G-stable, so G/H acts on Y. Hence, by the commutative case, $Y \subseteq X$ has a point fixed by G. ■

References

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