SOME NOTES ON THE COMPLEX LOGARITHM

1. THE COMPLEX LOG, BRANCH CUTS

Let $\Omega \subset \mathbb{C}$. A complex logarithm on $\Omega$ is a continuous function $\log : \Omega \to \mathbb{C}$ such that

$$\log(w) \in \log(w) := \exp^{-1}(w) = \{z : \exp(z) \in w\}$$

for all $z \in \Omega$. On any given $\Omega$ there may be more than one function that satisfies this property; each is called a branch of $\log$.

Recall that $\exp(x + iy) = w$ means that $|w| = e^x$ and $e^{iy} = \frac{w}{|w|}$, that is,

$$x = \log |w|, \quad y \in \text{arg}(w),$$

where $\log$ above denotes the usual real logarithm. Thus, the elements of $\log(z)$ all have the same real part $\log |w|$, but their imaginary parts differ by integer multiples of $2\pi$.

For any $z \in \mathbb{C}\{0\}$ we can choose an element $\log(z)$ from the set $\log(z)$; the question is whether we can make this choice in a continuous fashion as $z$ varies over the domain $\Omega$. The answer depends on the domain. For instance, we shall see in Example 1 that this is not possible for $\mathbb{C}\{0\}$.

More generally, if $f : \Omega \to \mathbb{C}\{0\}$ is a function, a complex logarithm of $f$ on $\Omega$ is a continuous function $\log(f) : \Omega \to \mathbb{C}$ such that

$$\log(f)(z) \in \log(f(z))$$

for all $z \in \Omega$, i.e.

$$\Re(\log(f)(z)) = \log |f(z)| \quad \text{and} \quad \Im(\log(f)(z)) \in \text{arg}(f(z)).$$

Each such function is called a branch of $\log(f)$.

The “complex logarithm” discussed previously implicitly took $f(z) = z$.

To define $\log(f)$ it may be necessary to make a branch cut i.e. to remove a subset $K \subset \mathbb{C}$ such that it is possible to choose $\log(f)(z) \in \log(f(z))$ in a continuous fashion for $z \in \Omega := \mathbb{C}\setminus K$.

**Example 1.** The function $f(z) = z$ maps $\mathbb{C}\{0\} \to \mathbb{C}\{0\}$. It is not possible to define $\log(z)$ continuously for all $z \in \mathbb{C}\{0\}$. To see this, suppose on the contrary that $g(z)$ is a continuous branch of $\log(z)$. Let $z(\theta) = e^{i\theta}$, $0 \leq \theta \leq 2\pi$, parametrize the unit circle centered at $-1$. From HW3 Problem 4, there exists an integer $k \in \mathbb{Z}$ such that

$$\log(z(\theta)) = \log(e^{i\theta}) = i\theta + 2\pi ik.$$  

In particular, $\log(1) = 2\pi k$. However, as $\theta \to 2\pi^-$, $z(\theta) \to 1$, therefore $\log(z(\theta)) \to 2\pi i(k + 1)i$. The continuity hypothesis then implies that $\log(1) = 2\pi i(k + 1)$, which is a contradiction.

This example shows that to define a branch of $\log(z)$, we need to ensure that no curve in the domain can wind fully around the point $z = 0$. One possible cut is $K = i(-\infty, 0]$, the negative imaginary axis. See Figure 1.

The preimage of $\Omega := \mathbb{C}\setminus K$ under the exponential map $z \mapsto \exp(z)$ is a disjoint union of horizontal strips

$$\Lambda_k := \{x + iy : -\infty < x < \infty, \quad -\frac{\pi}{2} + 2\pi k < y < \frac{3\pi}{2} + 2\pi k, \quad k \in \mathbb{Z}\},$$

each of which is mapped in a 1-1 fashion onto $\Omega$. Therefore, the restriction $\exp|_{\Lambda_k} : \Lambda_k \to \Omega$ is invertible, and $(\exp|_{\Lambda_k})^{-1} : \Omega \to \Lambda_k$ is a branch of $\log$. 
As shown in Example 1, we encounter a problem if we try to define \( \log(z) \) continuously on \( \mathbb{C} \setminus \{0\} \). As \( z \) moves along a closed curve enclosing the origin, \( \log(z) \) would seem to increase by \( 2\pi i \), and in particular does not return to its starting value. By removing the negative imaginary axis \( K = (-i\infty, 0] \), we prevent curves in \( \Omega = \mathbb{C} \setminus K \) from winding completely around 0. Note that this is far from the only choice for \( K \); other curves that connect 0 and \( \infty \) (for example, the positive real axis) would also admit a branch of \( \log(z) \) on their complement.

For instance, if we let \( \text{arg}_{(-\pi/2, 3\pi/2]}(z) \) be the unique representative of \( \text{arg}(z) \) belonging to the interval \( (-\pi/2, 3\pi/2] \), then

\[
g(z) = \log |z| + i \text{arg}_{(-\pi/2, 3\pi/2]}(z) = (\exp|\Lambda_0|)^{-1}(z)
\]

is a branch of \( \log(z) \) on \( \Omega \). See Figure 2.

2. Derivative of log

**Theorem 2.1** ("Inverse function theorem"). Let \( \Omega_0, \Omega_1 \subset \mathbb{C} \) be open and \( f : \Omega_0 \to \Omega_1 \) be holomorphic such that \( f' \) is never zero on \( \Omega_0 \). Suppose \( g : \Omega_1 \to \Omega_0 \) is a continuous inverse function.
Then $g$ is holomorphic on $\Omega_1$, and
\[ g'(w) = \frac{1}{f'(g(w))} \text{ for all } w \in \Omega_1. \]

Proof. Fix $w \in \Omega_1$. We need to evaluate
\[ \lim_{k \to 0} \frac{g(z + k) - g(w)}{k}. \]
By hypothesis, $w = f(z)$ for some $z \in \Omega_0$, and for all $k$ small enough so that $w + k \in \Omega_1$, we can write $w + k = f(z + h(k))$ for a unique $h(k)$. Since $g$ is continuous, as $k \to 0$ we have $z + h(k) = g(w + k) \to g(w) = z$, thus $h(k) \to 0$.

Now
\[ \frac{g(w + k) - g(w)}{k} = \frac{z + h(k) - z}{k} = \frac{h(k)}{k}. \]
Since $f$ is holomorphic it has a first order Taylor expansion
\[ f(z + h(k)) = f(z) + f'(z)h(k) + r(h(k)) = f(z) + h(k)\left[f'(z) + \frac{r(h(k))}{h(k)}\right] \]
for some remainder $r(h(k))$ such that $\frac{r(h(k))}{h(k)} \to 0$ as $h(k) \to 0$ (that is, $r(h(k)) = o(h(k))$). Taking $k \to 0$ we have
\[ \frac{h(k)}{k} = \frac{1}{f'(z) + \frac{r(h(k))}{h(k)}} \overset{k \to 0}{\to} \frac{1}{f'(z)} = \frac{1}{f'(g(w))}. \]

Corollary 2.2. If $g : \Omega \to \mathbb{C}$ is a branch of $\log(z)$, then $g$ is holomorphic, and
\[ g'(z) = \frac{1}{z}. \]

Proof. Since $\exp(g(z)) = z$, the inverse function theorem implies that $g$ is holomorphic and that
\[ g'(z) = \frac{1}{\exp'(g(z))} = \frac{1}{\exp(g(z))} = \frac{1}{z}. \]