1. General Cauchy Theorems

The Cauchy theorems were previously proved for functions holomorphic on discs and other star-shaped domains. In this section we consider more general domains, such as annuli, by decomposing such domains into elementary pieces.

**Question:** Let $\Omega \subset \mathbb{C}$ be an open set. For which closed curves $\gamma$ in $\Omega$ do we have

$$\int_{\gamma} f \, dz = 0$$

for all holomorphic $f : \Omega \to \mathbb{C}$?

The idea is that $\gamma$ should not be allowed to “wind around” any points in the complement of $\mathbb{C}$. Otherwise, functions of the form $\frac{1}{z - z_0}$, with $z_0 \in \mathbb{C} \setminus \Omega$, would be holomorphic on $\Omega$ but would have nonzero integral.

**Definition 1.1** (Winding number of a curve). Let $\gamma$ be a closed curve. For each $z \in \mathbb{C} \setminus \gamma$, define the winding number of $\gamma$ around $z$ by

$$W_\gamma(z) := \frac{1}{2\pi i} \int_{\gamma} \frac{dw}{w - z}.$$ 

**Lemma 1.1.** The winding number is continuous in $z$ and takes values in the integers, hence is constant on each connected component of $\mathbb{C} \setminus \gamma$.

For the proof of this lemma, see Stein-Shakarchi Appendix B.

To formulate the general Cauchy integral theorem, we need to slightly generalize integration to integrating over a collection of curves.

**Definition 1.2.** Given a region $\Omega \subset \mathbb{C}$, a **chain** is a collection of curves $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$, where $\gamma_j \subset \Omega$. Repetitions are allowed (so for instance $\{\gamma\}$ and $\{\gamma, \gamma\}$ are distinct chains). A **cycle** is a chain where all the $\gamma_j$ are closed curves.

If $f : \Omega \to \mathbb{C}$ is continuous, define the integral of $f$ over a chain $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ as

$$\int_{\Gamma} f \, dz := \sum_{j} \int_{\gamma_j} f \, dz.$$ 

**Definition 1.3.** Two chains $\Gamma_1$ and $\Gamma_2$ are **equivalent** (denoted $\Gamma_1 \sim \Gamma_2$) if

$$\int_{\Gamma_1} f \, dz = \int_{\Gamma_2} f \, dz$$

for all continuous functions $f$.

In particular, $\Gamma_1 \sim \Gamma_2$ if they differ by a collection of curves $\gamma_1, \gamma_1, \ldots, \gamma_n, \gamma_n$.

In view of these definitions, the following notation is both extremely convenient and suggestive.

- If $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ is a chain, write $\Gamma = \gamma_1 + \cdots + \gamma_n$. The definition of integrating over a chain then reads

  $$\int_{\gamma_1 + \cdots + \gamma_n} f \, dz := \int_{\gamma_1} f \, dz + \cdots + \int_{\gamma_n} f \, dz.$$ 

- Let $\gamma$ be a curve. Write $-\gamma = \gamma^-$ for the curve with the opposite orientation.

  If $n$ is a positive integer, write $n \cdot \gamma$ for the chain $\gamma, \ldots, \gamma$ ($n$ times). If $n < 0$, is a negative integer, write $n \cdot \gamma := |n| \cdot (-\gamma)$. 

If $\Gamma$ and $\Lambda$ are two chains, write

\[ n\Gamma := \sum_j n\gamma_j \]

\[ -\Gamma := \sum_j (-\gamma_j) \]

\[ \Gamma + \Lambda := \sum_j \gamma_j + \sum_k \lambda_k \]

\[ \Gamma - \Lambda := \Gamma + (-\Lambda). \]

In this notation, we trivially have the following identities for any continuous $f : \Omega \to C$. Let $\Gamma$ and $\Lambda$ be chains.

\[ \int_{n\Gamma} f = n \int_{\gamma} f. \]

\[ \int_{\Gamma + \Lambda} f \, dz = \int_{\Gamma} f \, dz + \int_{\Lambda} f \, dz. \]

HW: simple examples of cycles and integration.

**Definition 1.4.** Let $\Gamma = \gamma_1 + \cdots + \gamma_n$ be a cycle (so that each $\gamma_j$ is a closed curve). For each $z \in C \setminus \Gamma$, define the winding number of $\Gamma$ around $z$ by

\[ W_{\Gamma}(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{dw}{w - z} = \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{\gamma_j} \frac{dw}{w - z} = \sum_j W_{\gamma_j}(z). \]

As before, we have

**Lemma 1.2.** The winding number is continuous in $z$ and takes values in the integers, hence is constant on each connected component of $C \setminus \Gamma$.

**Example 1.** Let $\gamma_1$ and $\gamma_2$ be the counterclockwise circles of radius 1 and 2, respectively, centered at the origin. Then $3\gamma_1 - 2\gamma_2$ is the cycle which traverses the circle $\gamma_1$ counterclockwise thrice and the circle $\gamma_2$ clockwise twice. Suppose $f$ is holomorphic on $C$. Then

\[ W_{3\gamma_1 - 2\gamma_2}(0) = W_{3\gamma_1}(0) + W_{-2\gamma_2}(0) = 3W_{\gamma_1}(0) - 2W_{\gamma_2}(0) = 1. \]

**Definition 1.5.** Let $\Omega \subset C$ be open. A cycle $\Gamma \subset \Omega$ is nullhomologous in $\Omega$ or homologous to 0 in $\Omega$ if $W_{\Gamma}(z) = 0$ for all $z \in C \setminus \Omega$.

This makes precise the notion that a system of oriented closed curves do not wind around any “holes” in the region.

**Example 2.** Let $\Omega = C \setminus \{0\}$ be the punctured plane. Let $\gamma_1$ and $\gamma_2$ denote the counterclockwise circles of radius 1 and 2, respectively, centered at 0. Then $\gamma_2 - \gamma_1 \sim_\Omega 0$. Indeed, $C \setminus \Omega = \{0\}$, and

\[ W_{\gamma_2 - \gamma_1}(0) = W_{\gamma_2}(0) - W_{\gamma_1}(0) = 1 - 1 = 0. \]

**Theorem 1.3** (General CIT). Let $f$ be holomorphic on some open set $\Omega \subset C$. Then

\[ \int_{\Gamma} f = 0 \]

for all cycles $\Gamma \subset \Omega$ with $\Gamma \sim_\Omega 0$. 

Proof. The following proof is taken from *Complex Analysis* by Ahlfors. We begin with a construction which is of independent interest; see also Stein-Shakarchi Chapter 2 Section 5.5 for essentially the same presentation.

**Lemma 1.4.** Let \( \Omega \subset \mathbb{C} \) be a bounded domain. For every \( \delta > 0 \), there exists a cycle \( \Gamma_\delta \subset \Omega \) such that:

- For any \( f \) holomorphic on \( \Omega \), one has
  \[
  f(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(w)}{w-z} \, dw \quad \text{for all } z \in \Omega_\delta.
  \]

- For any cycle \( \Gamma \subset \Omega_\delta \), \( W_\Gamma(w) = 0 \) for any \( w \in \Gamma_\delta \).

**Proof.** Tile the complex plane with a grid of rectangles \( \mathcal{C} = \bigcup_{Q \in \mathcal{Q}} Q \) with diameter \( \delta/2 \). Let \( Q' \subset Q \) be the rectangles contained in \( \Omega \), and observe that \( \Omega_\delta \subset \bigcup_{Q \in \mathcal{Q}'} Q \).

Define the cycle \( \Gamma'_\delta = \sum_{Q \in \mathcal{Q}'} \partial Q \).

Then \( \Gamma'_\delta \) is a sum of oriented edges of the rectangles in \( \mathcal{Q}' \).

Note that if a line segment belongs to two adjacent rectangles in \( \mathcal{Q}' \), it occurs twice in the cycle with opposite orientations. Thus, if we define \( \Gamma_\delta \) by deleting from \( \Gamma'_\delta \) all such pairs of line segments, then \( \Gamma_\delta \) is equivalent to \( \Gamma'_\delta \).

For \( z \in \Omega_\delta \), suppose that \( z \) belongs to the interior of some \( Q_z \in \mathcal{Q}' \). For any other \( Q' \in \mathcal{Q} \), Cauchy's integral theorem implies that

\[
\int_{\partial Q'} \frac{f(w)}{w-z} \, dw = 0.
\]

Therefore, by CIF for rectangles,

\[
\int_{\Gamma_\delta} \frac{f(w)}{w-z} \, dw = \int_{\partial Q_z} \frac{f(w)}{w-z} \, dw + \sum_{Q \neq Q_z} \int_{\partial Q} \frac{f(w)}{w-z} \, dw = 2\pi i f(z),
\]

so

\[
f(z) = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(w)}{w-z} \, dw
\]

for all \( z \in \Omega_\delta \) belonging to the interior of a rectangle. However, both sides of the above equation are well-defined and continuous for all \( z \in \Omega_\delta \). Consequently the equation holds for all \( z \in \Omega_\delta \), which verifies the first property claimed in the Lemma.

Now fix a cycle \( \Gamma \subset \Omega_\delta \) with \( W_\Gamma(z) = 0 \) for all \( z \in \mathbb{C} \setminus \Omega \); we check that \( W_\Gamma(w) = 0 \) for all \( w \in \Gamma_\delta \).

Fix \( w \in \Gamma_\delta \). By the definition of \( \Gamma_\delta \), there exists a rectangle \( Q \in \mathcal{Q} \) such that \( w \in \partial Q \) and \( Q \) also contains a point \( z_0 \in \mathbb{C} \setminus \Omega \). For every \( z \in Q \) one has \( |z - z_0| \leq \text{diam}(Q) = \frac{\delta}{2} \); it follows that \( Q \subset \mathbb{C} \setminus \Omega_\delta \). In particular, the line segment \( [w, z_0] \) is contained in \( \mathbb{C} \setminus \Omega_\delta \) and therefore belongs to a connected component of \( \mathbb{C} \setminus \Gamma \). As the winding number \( W_\Gamma(\cdot) \) is constant on connected components of \( \mathbb{C} \setminus \Gamma \), one has \( W_\Gamma(w) = W_\Gamma(z_0) = 0 \). \( \square \)
Given the lemma, we integrate over \( \Gamma \) and interchange the order of integration to obtain
\[
\int_{\Gamma} f(z) \, dz = \frac{1}{2\pi i} \int_{\Gamma} \int_{\Gamma} \frac{f(w)}{w - z} \, dw \, dz = -\int_{\Gamma} f(w)W_{\Gamma}(w) \, dw = 0.
\]

\[\square\]

2. Some applications

As an application, we prove a version of Cauchy’s integral formula on an annulus.

**Theorem 2.1 (CIF for an annulus).** Let \( f : \Omega \to \mathbb{C} \) be holomorphic and suppose a closed annulus \( A_{r,R}(z_0) := \{ z \in \mathbb{C} : r \leq |z - z_0| \leq R \} \subset \Omega \) is contained in \( \Omega \). Then
\[
f(z) = \frac{1}{2\pi i} \int_{\partial D_R(z_0)} f(\zeta) \frac{1}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_{\partial D_r(z_0)} f(\zeta) \frac{1}{\zeta - z} \, d\zeta
\]
for all \( z \in \text{int}(A_{r,R}(z_0)) \) (that is, satisfying \( r < |z - z_0| < R \)).

**Proof.** For fixed \( z \) with \( r < |z - z_0| < R \), the function \( F(\zeta) = \frac{f(\zeta)}{\zeta - z} \) is holomorphic on the domain \( \Omega_z := \Omega \setminus \{z\} \). Let \( D_\varepsilon(z) \) be a small disc centered at \( z \) contained in the annulus.

We claim the cycle \( \partial D_R(z_0) - \partial D_r(z_0) - \partial D_\varepsilon(z) \) is nullhomologous in \( \Omega_z \). Indeed, one has \( \Omega \setminus \Omega_z = (\mathbb{C} \setminus \Omega) \cup \{z\} \). By hypothesis each \( \zeta \in \mathbb{C} \setminus \Omega \) satisfies either \( |\zeta - z_0| < r \) or \( |\zeta - z_0| > R \). In the first case \( W_{\partial D_R}(\zeta) = W_{\partial D_r}(\zeta) = 1 \) and \( W_{\partial D_\varepsilon}(\zeta) = 0 \), while in the second case \( W_{\partial D_R}(\zeta) = W_{\partial D_r}(\zeta) = W_{\partial D_\varepsilon}(\zeta) = 0 \). On the other hand, \( W_{\partial D_R}(z) = W_{\partial D_r}(z) = 1 \) while \( W_{\partial D_\varepsilon}(z) = 0 \). In all cases we have \( W_{\partial D_R - \partial D_r - \partial D_\varepsilon}(\zeta) = W_{\partial D_r}(\zeta) - W_{\partial D_\varepsilon}(\zeta) = 0 \) for \( \zeta \in \mathbb{C} \setminus \Omega \) or \( \zeta = z \).

Applying the general CIT to the function \( F \) on the domain \( \Omega_z \), we obtain
\[
0 = \int_{\partial D_R - \partial D_r - \partial D_\varepsilon} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \int_{\partial D_R} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \int_{\partial D_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \int_{\partial D_\varepsilon} \frac{f(\zeta)}{\zeta - z} \, d\zeta,
\]
and using the Cauchy integral formula for a disc for the integral over \( \partial D_\varepsilon(z) \), we conclude that
\[
f(z) = \frac{1}{2\pi i} \int_{\partial D_\varepsilon} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \frac{1}{2\pi i} \int_{\partial D_R} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_{\partial D_r} \frac{f(\zeta)}{\zeta - z} \, d\zeta.
\]
\[\square\]