MATH 104 HOMEWORK 8
DUE FRIDAY OCT 19 AT 12PM

1. For each nonzero rational \( x \in \mathbb{Q} \), let \( x = \frac{p(x)}{q(x)} \) be its representation such that \( p(x) \in \mathbb{Z} \), \( q(x) \in \mathbb{N} \), and \( \gcd(p(x), q(x)) = 1 \). Note in particular that \( p(0) = 0, q(0) = 1 \). Define \( f : \mathbb{R} \rightarrow \mathbb{R} \) by

\[
f(x) = \begin{cases} \frac{1}{q(x)^{\alpha}}, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}
\]

Prove that \( f \) is continuous on \( \mathbb{R} \setminus \mathbb{Q} \) and not continuous at any \( x \in \mathbb{Q} \).

2. Rudin 4.4

3. Let \( X \) be a metric space, \( E \subset X \) be a dense subset.
   (a) \( f : E \rightarrow \mathbb{R} \) be uniformly continuous. Prove that \( f \) has a unique continuous extension to \( X \); that is, prove there exists a unique continuous function \( \tilde{f} : X \rightarrow \mathbb{R} \) which agrees with \( f \) on \( E \).
   (b) Show that if \( X \) is compact, the converse is true: \( f : E \rightarrow \mathbb{R} \) has a continuous extension to \( X \) if and only if \( f \) is uniformly continuous.
   Also show, by giving a counterexample, that the statement is false without the compactness assumption.

4. A fixed point of a mapping \( f : X \rightarrow X \) is an element \( x \in X \) such that \( f(x) = x \). Show that if \( f : [0, 1] \rightarrow [0, 1] \) is continuous, then \( f \) has a fixed point.

5. Fix \( 0 < \alpha \leq 1 \). A function \( f : [a, b] \rightarrow \mathbb{R} \) is Hölder continuous or \( \alpha \)-Hölder continuous if there exists \( M > 0 \) such that

\[
|f(x) - f(y)| \leq M |x - y|^\alpha \quad \text{for all } x \neq y \in [a, b].
\]

Show that the set of \( \alpha \)-Hölder continuous functions \([a, b] \rightarrow \mathbb{R} \) form a vector space, denoted \( C^{0, \alpha}([a, b]) \), under the usual addition and scalar multiplication of functions. Also show that

\[
\|f\|_{C^{0, \alpha}} := \sup_{x \in [a, b]} |f(x)| + \sup_{x \neq y \in [a, b]} \frac{|f(x) - f(y)|}{|x - y|^\alpha}
\]

is a norm on \( C^{0, \alpha}([a, b]) \).