(1) (a) Show that
\[ ab \leq \frac{a^2 + b^2}{2} \] for all \( a, b \in \mathbb{R} \).

(b) Let \( x_1 := 1 \), and for \( n \geq 1 \) define
\[ x_{n+1} := \frac{x_n}{2} + \frac{2}{x_n}. \]
Prove that \( \{x_n\}_{n=1}^{\infty} \) converges. [Caution: taking the limit of both sides as \( n \to \infty \) is only justified if you already know that the limit exists.]

(2) Let \( \{x_n\}_{n=1}^{\infty} \) be a sequence in \( \mathbb{R} \). Suppose there exists \( z \in \mathbb{R} \) such that every subsequence \( \{x_{n_k}\}_k \) has a further subsequence \( \{x_{n_{k\ell}}\}_\ell \) which converges to \( z \). Prove that \( \{x_n\} \) converges.

*Suggestion.* Suppose for contradiction that \( \{x_n\} \) does not converge. Let \( x, y \) be distinct subsequential limits....

(3) Show that if \( s_n \leq t_n \), then \( \lim \sup s_n \leq \lim \sup t_n \) and \( \lim \inf s_n \leq \lim \inf t_n \).
Deduce from this the so-called *squeeze lemma* that you likely encountered in single-variable calculus: Let \( \{a_n\}, \{b_n\}, \{c_n\} \) be sequences in \( \mathbb{R} \) such that \( a_n \leq b_n \leq c_n \) for all \( n \). If \( \lim a_n \) and \( \lim c_n \) both exist and are equal, then \( \{b_n\} \) converges.

(4) Rudin 3.2
(5) Rudin 3.4
(6) Rudin 3.6abc
(7) Rudin 3.7