MATH 104 HOMEWORK 1
DUE FRIDAY AUGUST 31 AT 12PM

(1) This problem has two parts.
   (a) Prove that \( \sqrt{5} \) is irrational.
   (b) Suppose you attempt to “prove” that \( \sqrt{4} \) is irrational by mimicking
       your argument for \( \sqrt{5} \). Retrace your steps in the previous part and
       identify where the arguments break down. Moral: if you find that you
       have proved a false statement, you have made a logical error some-

(2) Write down the negation of each of the following statements. Let \( x_0, x_1, x_2, \ldots \)
    be a sequence of real numbers.
   (a) “There exists \( A > 0 \) such that \( |x_n| \leq A \) for all \( n \).”
   (b) “For any \( \varepsilon > 0 \) there exists \( N > 0 \) such that \( |x_m - x_n| < \varepsilon \) for all
       \( m, n > N \).”

(3) Write down the converse and contrapositive of each of the following con-
    ditional statements. Here \( f : \mathbb{R} \to \mathbb{R} \) denotes a function and \( \{a_n\}_{n=1}^\infty \)
    a sequence of real numbers. The properties “Cauchy”, “converges”, and
    “continuous” will be defined precisely later in this course.
   (a) “If \( f \) is differentiable then \( f \) is continuous.”
   (b) “If \( \{a_n\}_{n=1}^\infty \) is Cauchy then \( \{a_n\}_{n=1}^\infty \) converges.”

(4) Prove that \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \) for any positive integer \( n \).
    Solution. Base case: When \( n = 1 \), \( 1 = \frac{1 \cdot 2 \cdot 3}{6} \).
    Inductive step: Assume that \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \). Then
    \[
    \sum_{i=1}^{n+1} i^2 = \left( \sum_{i=1}^{n} i^2 \right) + (n+1)
    = \frac{n(n+1)(2n+1)}{6} + (n+1)^2 = \frac{(n+1)[n(2n+1) + 6(n+1)]}{6}
    = \frac{(n+1)[2n^2 + 7n + 6]}{6} = \frac{(n+1)(n+2)(2n+3)}{6}
    = \frac{(n+1)(n+2)(2(n+1)+1)}{6}.
    
    Hence by induction the identity holds for all \( n = 1, 2, \ldots \).

(5) Show that for each integer \( n \geq 0 \) and real number \( x > -1 \), the inequality
    \( (1 + x)^n \geq 1 + nx \) holds.

(6) Prove that if \( 0 \neq a \) is rational and \( b \) is irrational, then both \( a + b \) and \( ab \)
    are irrational.
    Solution. If \( a + b \) is rational, then \( b = a + b - a \) is rational. Similarly, if
    \( ab \) is rational then \( b = a^{-1}ab \) is rational.

(7) Prove the reverse triangle inequality: \( ||a| - |b|| \leq |a - b| \) for all \( a, b \in \mathbb{R} \).
Solution. For any \( x, y \in \mathbb{R} \), \(|x| = |x - y + y| \leq |x - y| + |y|\), we have \(|x| - |y| \leq |x - y|\). Interchanging \( x \) and \( y \) yields \(|y| - |x| \leq |x - y|\). Thus \(||x| - |y|| \leq |x - y|\).

(8) For each of the following subsets of \( \mathbb{R} \), determine its sup, inf, min, max, or indicate that the quantity does not exist.

(a) \((-5, 6] = \{x \in \mathbb{R} : -5 < x \leq 6\}\)
(b) \(\{0, 1, 2, \ldots\}\)
(c) \(\{\frac{n-1}{n} : n = 1, 2, \ldots\}\)

Solution. (a) inf = -5, sup = 6 = max, and min does not exist.
(b) min = inf = 0, neither sup nor max exist.
(c) inf = min = 0, sup = 1, max does not exist.

(9) (Rudin 1.4) Let \( E \) be a nonempty subset of an ordered set. Suppose \( \alpha \) is a lower bound of \( E \) and \( \beta \) is an upper bound of \( E \). Prove that \( \alpha \leq \beta \).

Solution. Fix \( x \in E \), which exists since \( E \) is nonempty. By hypothesis \( \alpha \leq x \) and \( x \leq \beta \), hence \( \alpha \leq \beta \).

(10) (Rudin 1.5) Suppose \( A \) is a nonempty subset of \( \mathbb{R} \) which is bounded below. Let \(-A := \{-x : x \in A\}\). Prove that
\[
\inf(A) = -\sup(-A).
\]

Solution. Let \( \alpha = \inf(A) \). By definition \( \alpha \) is a lower bound of \( A \), so \( \alpha \leq x \) for all \( x \in A \). Thus \(-x \leq -\alpha \) for all \( x \in A \), so \(-\alpha \) is an upper bound of \(-A\). Now suppose \( \gamma \) is another upper bound of \(-A\). Then \(-\gamma \) is a lower bound of \( A \), so \(-\gamma \leq \alpha \), thus \(-\alpha \leq \gamma \). Therefore \(-\alpha \) is the least upper bound of \(-A\).