Wrapped sheaves expository

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Abstract

This is the notes for the talk given in the topology seminar in University of Southern California in Fall 2022. The goal of this talk is to give a rough picture of [3] for general audience. Thus, we will focus mostly on examples and give only definitions and facts which are indispensable.

1 Introduction

The project for this talk is a continuation of a previous research by Ganatra, Pardon, and Shende in [1]. Let $M$ be a real analytic manifold and $\Lambda$ a closed subanalytic (singular) isotropic, i.e., a union of isotropic submanifolds in $S^*M$. The result we care from that paper is an equivalence of categories

$$W(T^*M, \Lambda) \cong Sh_{\Lambda}(M)^c.$$

We will give more information about these categories later. For the moment, we just mention that $W(T^*M, \Lambda)$ is a defined through Floer theory while $Sh_{\Lambda}(M)^c$ can be understood combinatorially.

One fact to know is that this equivalence is not proven by defining a functor and then proving that it is indeed an equivalence. Instead, the authors of [1] find a class of generators on both sides and show that these generators match functorially an hence prove the equivalence. Now, there is an equivalence established earlier in [5, 4] between the infinitesimal Fukaya category of $T^*M$ and the category of constructible sheaves on $M$. It is expected that all the categories mentioned above can be fit into the following diagram:

$$\begin{array}{c}
\text{Fuk}_\epsilon(T^*M)^{op} \supseteq \text{Fuk}_\epsilon(T^*M, \Lambda)^{op} & \overset{I_{\text{Fuk}}}{\longrightarrow} & W(T^*M, \Lambda)^{op} \\
\downarrow{[5, 4]} & & \downarrow{[5, 4]} \\
\text{Sh}_{\text{constr}}(M) \supseteq \text{Sh}_{\text{constr}, S^*M\setminus\Lambda}(M) & \overset{I_{\text{Sh}}}{\longrightarrow} & \text{wsh}_{\Lambda}(M) & \overset{\mathcal{W}_\Lambda^-}{\longrightarrow} & \text{Sh}_{\Lambda}(M)^c \\
\end{array}$$

That is, it is expected the categories considered by Ganatra, Pardon, and Shende can be obtained by localizations from those considered by Nadler and Zaslow. What is done in [3]
is to define the localized category \( \mathfrak{wsh}_\Lambda(M) \) and a comparison functor \( \mathfrak{W}^+\Lambda \), which exhibits that it’s the same as the more classical category \( \text{Sh}_\Lambda(M)^c \).

The plan for this talk is to provide descriptions of what those categories are, in the order of \( \mathcal{W}(T^*M, \Lambda) \), \( \text{Sh}_\Lambda(M)^c \), and finally \( \mathfrak{wsh}_\Lambda(M) \), as well as the functor \( \mathfrak{W}^+\Lambda \). The last two construction depends heavily from the sheaf quantization construction by Guillermou, Kashiwara, and Schapira in [2]. To make the talk simple, we use \( \mathbb{Z} \) as our coefficient.

## 2 Wrapped Fukaya categories

Without going into too much details, we mention that the wrapped Fukaya category \( \mathcal{W}(T^*M, \Lambda) \) is an \( A_\infty \)-category, which means in addition to composition there are higher morphisms encoding the associativity of composition, and all such operation comes from pseudo-holomorphic disk counting.

We will only look at its homotopy category, i.e., we look at its morphisms after passing to cohomology. Now the basic objects of \( \mathcal{W}(T^*M, \Lambda) \) are given Lagrangians with structures as usual. To have a well-functioning Floer theory, a Lagrangian \( L \) should be compact horizontally and conic, i.e., invariant with respect to the scaling of \( T^*M \), at \( \infty \). This way, we can talk about the corresponding Legendrian \( \partial_\infty L \) in \( S^*M \) and we require it to be away from \( \Lambda \).

For such Lagrangians \( L, K \), we denote the morphisms between them by \( \mathcal{W}(L, K) \). Then \( H^*\mathcal{W}(L, K) \cong HW^*(L, K) := \colim_{K \to K^w} HF^*(L, K^w) \). Here, \( HF^* \) is the ordinary Floer cohomology define as chain complexes generated by Lagrangian intersections with differential given by disk counting. The focus today is the (positive) wrapping \( K \to K^w \), which means a isotopy \( K_t \) between \( K \) and \( K^w \) such that \( K_t \) are all conic near infinity, \( \partial_\infty K_t \) is an isotopy of Legendrians, and \( \alpha(\partial_t \partial_\infty K_t) \geq 0 \).

We illustrate it with an example: Consider \( M = S^1 \), \( \Lambda = S^*_0 S^1 \), the negative part of the fiber at the origin, and we would like to understand the Lagrangian \( L = T^*_\epsilon S^1 \) for some small \( \epsilon \). We draw in the following picture how a wrapping in this situation look like:

![Diagram](image)

One can see that for each wrapping, the generator are given by the number of times where \( L \) goes around the circle once and hit itself. It is an exercise that no disk with correct index goes between them so \( HW^*(L, L) = \colim_{n \to \infty} \mathbb{Z}^{\geq n} = \mathbb{Z}^{\geq 0} \). We mention that a better way
to express it is $HW^*(L,L) = \mathbb{Z}[t]$, the polynomial rings, and if we change $\Lambda$ to the whole fiber $S_0S^1$ or no fiber, then the same Lagrangians gives $\mathbb{Z}$ or $\mathbb{Z}[t, t^{-1}]$.

### 3 Sheaves with a fixed microsupport condition

We turn our attention to $\text{Sh}_\Lambda(M)$. We first recall that a sheaf $F \in \text{Sh}(M)$ consists of assignments $U \mapsto F(U)$, $(V \subseteq U) \mapsto F(U) \to F(V)$, and more, plus a gluing condition. For each sheaf $F$, there exists a conic, closed, coisotropic set $SS(F)$ in $T^*M$, called the microsupport. On the zero section it recovers the support $\text{supp}(F) = \{x | F_x \neq 0\}$ and away from the zero section this set roughly encodes the codirections where the restriction is not an isomorphism. In general, it is hard to compute $SS(F)$ but when $F$ is constructible, meaning that there is a stratification $S$ such that $F|_{X_\alpha} \in \text{Loc}(X_\alpha)$ for each strata $X_\alpha$ in $S$, $SS(F)$ is easier to compute and is always a singular Lagrangian if we know that $SS(F)$ is subanalytic.

Consider the one dimensional case below and call it $F$:

\[
\begin{array}{ccc}
B_- & \leftarrow & A & \rightarrow & B_+ \\
& 0 & & & \\
\end{array}
\]

Away from 0, the sheaf is just constant so there is no microsupport. To answer whether $(0, -1)$ is in the $SS(F)$, we consider the restriction $F((-1, 1)) \rightarrow F((0, 1))$, or whether the map $A \rightarrow B_+$ is an isomorphism, and similar for $(0, 1)$. In general, for a given covector, we roughly consider the following picture:

The symbol $\text{Sh}_\Lambda(M)$ then stands for the subcategory of $\text{Sh}(M)$ whose objects $F$ satisfies $SS^\infty(F) \subseteq \Lambda$. One property of subanalytic Legendrians $\Lambda$ in $S^*M$ is that there is always a (Whitney) triangulation $S$ so that $\Lambda \subseteq S^*S := \cup_{\alpha \in S} N^*_\infty X_\alpha$ so $\text{Sh}_\Lambda(M)$ can be seen as a subcategory of $S$-Mod where some arrows are required to be isomorphisms.

Consider again the example $M = S^1$, $\Lambda = S^{*}_{0, \leq}S^1$. We provide directly the answer that the sheaf $F$ corresponds to $L = T^*_0S^1$ is given by the constructible sheaves $F$ whose stalks are given by $\mathbb{Z}^{Z \geq 0}$, and near 0 where the picture has the from $B_-A \rightarrow B_+$, is given by

\[
\mathbb{Z}^{Z \geq 0} \xrightarrow{m} \mathbb{Z}^{Z \geq 0}
\]
where $m$ is the shifting $m(a_0, a_1, a_2, \cdots) = (0, a_0, a_1, \cdots)$. This sheaf has an alternative expression by $F = \pi_! Z_{(0,\infty)}$, where $\pi : \mathbb{R}^1 \to S^1$ is the universal cover, so a computation of the self-Hom can look like

$$
\begin{align*}
\text{Hom}(\pi_! Z_{(0,\infty)}, \pi_! Z_{(0,\infty)}) &= \text{Hom}(Z_{(0,\infty)}, \pi^* \pi_! Z_{(0,\infty)}) \\
&= \text{Hom}(Z_{(0,\infty)}, \pi^* \pi_! Z_{(0,\infty)}) \\
&= \Gamma((0, \infty), \pi^* \pi_! Z_{(0,\infty)}) \\
&= (\pi^* \pi_! Z_{(0,\infty)})^c \\
&= \Gamma_c(\{\epsilon + n | n \in \mathbb{Z}_{\geq 0}\}; \mathbb{Z}) = \mathbb{Z}^{\mathbb{Z}_{\geq 0}}.
\end{align*}
$$

4 Wrapped sheaves

Let $X, Y$ be topological spaces. A sheaf $K \in \text{Sh}(X \times Y)$ produces a functor

$$
\text{Sh}(X) \to \text{Sh}(Y) \\
F \mapsto K \circ F := p_2!(K \otimes p_1^*F)
$$

which is usually referred as convolution with the sheaf kernel $K$.

Now let $\varphi : S^*M \times I \to S^*M$ be a contact isotopy where $I$ is an open interval containing $0$. A theorem of Guillermou, Kashiwara, and Schapira is that there exists a unique sheaf $K = K(\varphi) \in \text{Sh}(M \times M \times I)$ such that $K|_0 = Z_\Delta$ and $SS^\infty(K)$ is contained in the movie of $\varphi$. By convolution, we get, for a fixed sheaf $F \in \text{Sh}(M)$, a family $\{F_t\}_{t \in I}$ such that $F_0 = F$ and $SS^\infty(F_t) = \varphi_t SS^\infty(F)$, and we can think it as isotope $F$ by $\varphi$. Furthermore, when $\varphi$ is positive, then there is a continuation map $K_s \to K_t$ when $t \geq s$.

As an example, consider the isotopy on $S^*\mathbb{R}^1 \cong \mathbb{R}^1 \times \pm 1$ which is given by the formula $\varphi(x, \pm 1) = (x \pm t, \pm 1)$. Its GKS sheaf quantization has $\mathbb{Z}_{\{t \geq 0\}}$ as the slice at $t > 0$:
It sends \( F = \mathbb{Z}_\{0\} \), by convolution, to \( \mathbb{Z}_{(\epsilon, \zeta)}[1] \):

Now, we define the category of wrapped sheaves. First, take \( \tilde{w}_{\Lambda}(M) \) to be the collection of sheaves \( F \) such that \( SS^\infty(F) \) is a subanalytic Legendrian away from \( \Lambda \), \( \text{supp}(F) \) is compact, and \( F_x \) to be perfect for all \( x \in M \).

**Definition 4.1.** The category of wrapped sheaves \( w_{\Lambda}(M) \) is defined to be

\[
w_{\Lambda}(M) := \frac{\tilde{w}_{\Lambda}(M)}{\text{isotopies}}.
\]

Similarly to the wrapped Fukaya category, morphisms in \( w_{\Lambda}(M) \) is computed by the colimit

\[
\text{Hom}_w(G, F) = \text{colim}_{F \rightarrow F^w} \text{Hom}(G, F^w).
\]

As one can guess, we consider yet again the case \( M = S^1 \) and \( \Lambda = S^*_0 \leq S^1 \). It’s now not hard to convince oneself that the object corresponds to \( L = T^*_\epsilon S^1 \) is \( \mathbb{Z}_{\{0\}} \). So by the picture above, we consider \( \text{Hom}_w(\mathbb{Z}_J, \mathbb{Z}_J) = \text{colim}_w \text{Hom}(\mathbb{Z}_J, (\mathbb{Z}_J)_w) \) where \( J \hookrightarrow S^1 \) is some small open interval. Depending on how large the wrapping \( w \) is, the situation in the universal cover is given by the following picture:
So \( \text{Hom}(Z_J, (Z_J)^w) = \mathbb{Z}^\oplus n \) where \( n \) is the number of times when the lift of \( J \) passes over itself after extended by \( w \). Thus, we conclude again that \( \text{Hom}_w(Z_{\{0\}}, Z_{\{0\}}) = \mathbb{Z}^{\geq 0} \).

Finally, we mention that the comparison map is given by

\[
\mathcal{M}_A^+ : w\text{sh}_A(M) \rightarrow \text{Sh}_A(M) \\
F \mapsto \colim_{F \rightarrow F^w} F^w,
\]

That is, we take the colimit directly on the objects. One can conclude, from the last picture that, in the case \( M = S^1 \) and \( \Lambda = S_{0, \leq}^* S^1 \), \( \mathcal{M}_A^+ Z_{\{0\}} = \pi_1 Z_{(0, \infty)} \), which is the reason for our guess earlier.

References


