A mathematical model for the scaling of turbulence

Grigory Isaakovich Barenblatt and Alexandre J. Chorin[†]

Department of Mathematics, University of California, Berkeley, CA 94720

Contributed by Alexandre J. Chorin, August 26, 2004

We present a simple physical model of turbulent wall-bounded shear flows that reveals exactly the scaling properties we had previously obtained by similarity considerations. The significance of our results for the understanding of turbulence is pointed out.

In a series of papers (1–13) we constructed a model of wall-bounded turbulent shear flow based on a hypothesis of incomplete similarity (14) and a vanishing viscosity principle (15), and then compared the model with the data and found an excellent agreement. In particular, we found that the mean flow in wall-bounded turbulence had a persistent dependence on the Reynolds number, contrary to often-used assumptions, and that the well known logarithmic law of the wall was invalid.

In the present paper we offer a simplified qualitative model of a turbulent boundary layer and, more generally, of wall-bounded turbulent flows. The derivation of the model is heuristic, but once the model is derived, we prove rigorously that it has the scaling properties that we had obtained for wall-bounded turbulence. This establishes the self-consistency and realizability of our assumptions, and also provides a mathematical explanation of their origin and meaning. Incomplete similarity leads to scaling laws with anomalous exponents and is closely related to the renormalization group (14, 16–18). We furthermore discuss the significance of the results for turbulence theory and in particular for scaling in the Kolmogorov range of scales. The striking conclusion from our analysis and the experimental data is that, as the Reynolds number R tends to infinity, the derivatives of the velocity field reach a limit and do so at an inverse logarithmic rate in R.

We briefly remind the reader of the fundamentals of similarity theory (14, 16). Suppose a variable a is a function of variables a_1, $a_2, \ldots, a_m, b_1, b_2, \ldots, b_k$, where $a_1, \ldots, a_m$ have independent units (for example, units of length and mass), while the units of $b_1, \ldots, b_k$ can be formed from the units of $a_1, a_2, \ldots, a_m$. Then there exist dimensionless variables $\Pi = (a/a_1^{\alpha_1} \cdots a_m^{\alpha_m})$, $\Pi_i = (b_i/a_1^{\alpha_{i1}} \cdots a_m^{\alpha_{im}})$, $i = 1, \ldots, k$, where the α_i, α_{ij} are simple fractions, such that Π is a function of the Π_i:

$$\Pi = \Phi(\Pi_1, \ldots, \Pi_k). \qquad [1]$$

This is just a consequence of the requirement that a physical relationship be independent of the size of the units of measurement. At this stage, nothing can be said about the function Φ. Now suppose the variables Π_i are either small or large, and assume that the function Φ has a nonzero finite limit as its arguments tend to either zero or to infinity; then $\Pi \sim$ constant, and one finds a power monomial relation between a and the a_i. This is a complete similarity relation. If the function Φ does not have the assumed limit, it may happen that for Π_1 small or large, $\Phi(\Pi_1) = \Pi_1^\alpha \Phi_1(\Pi_1) + \ldots$, where the dots denote lower-order terms, α is a constant, the other arguments of Φ have been omitted, and Φ_1 has a finite nonzero limit. One then obtains a scaling (power monomial) expression for a in terms of the a_i and b_i, with undetermined powers that must be found by means other than dimensional analysis. The resulting power relation is an *incomplete* similarity relation. Its existence is equivalent to the asymptotic invariance of the problem under the renormalization group (14, 16). Of course, one may well have functions Φ with neither kind of similarity.

Derivation of the Model

We first present a heuristic derivation of an equation for the mean flow in the intermediate layer in a wall-bounded turbulent flow. This derivation is heuristic, but the analysis of the equation will be rigorous, and the heuristic derivation includes several points needed for the interpretation of the rigorous results.

Consider a turbulent flow bounded by a wall, and a point P a distance y above that wall (Fig. 1). We wish to determine the mean velocity profile $u = u(y)$ parallel to the wall as a function of the distance y from the wall. We consider the flow in the region $\delta \leq y \leq d$, i.e., outside a viscous sublayer of thickness δ near the wall where viscous effects are dominant; d is a length typical of the height of the intermediate layer. A prime example of a wall-bounded flow is flow in a pipe, where d is the diameter of the pipe.

It is widely believed that the mechanics of a turbulent flow are determined by a mixing length ℓ, a turbulence analog of a mean-free path in kinetic theory (19, 20). Following von Kármán (21) we assume $\ell = -u'/u''$. The simplest assumption about ℓ is $\ell = y$ (an irrelevant proportionality constant is omitted), and thus $yu'' + u' = 0$, $yu' = C_1$, and

$$u = C_1 \ln y + C_2. \qquad [2]$$

(We denote by C_i constants whose value is immaterial and, whenever it leads to no confusion, set $C_i = 1$.) From this assumption we obtain $u(d) = \ln d + C_2$, $u(\delta) = \ln \delta + C_2$, and $u(d) - u(\delta) = \ln(d/\delta)$. Redefine the velocity u as $u = u - u(d)$; then $u(\delta) = -\ln(d/\delta)$. Clearly, $u'(\delta) = 1/\delta$.

We now ask how these conclusions are modified when one takes into account the direct effect of viscosity on the flow. The analog of a Reynolds number here is d/δ, the ratio of the scale of the domain to a viscous length. The boundary conditions,

$$u(\delta) = -\ln(d/\delta), \quad u'(\delta) = 1/\delta, \qquad [3]$$

should remain unchanged; they describe the mean velocity on the outer edge of the viscous sublayer $0 < y < \delta$ whose formation is due to viscous effects and which is not modified by the inclusion of viscous effects in the description of the layer in which we are interested. The importance of coherent structures in turbulence has been shown in numerical computations (22) and in experiments (23), they are responsible for most of the turbulent shear (24), and their scale increases with the Reynolds number (22, 25). Thus, ℓ should increase; assume the increase is proportional to the small parameter $1/\ln(d/\delta)$; this is plausible since the quantity $\ln(d/\delta)$ has already appeared in the analysis as the major large parameter (also see the discussion below). Thus,

$$\ell = \left(1 + \frac{1}{\ln(d/\delta)}\right) y. \qquad [4]$$

Then $u'' = -(u'/y)(1 + (1/\ln(d/\delta))^{-1}$ and neglecting terms quadratic in the small parameter $1/\ln(d/\delta)$ we find

[†]To whom correspondence should be addressed. E-mail: chorin@math.berkeley.edu.

APPLIED MATHEMATICS

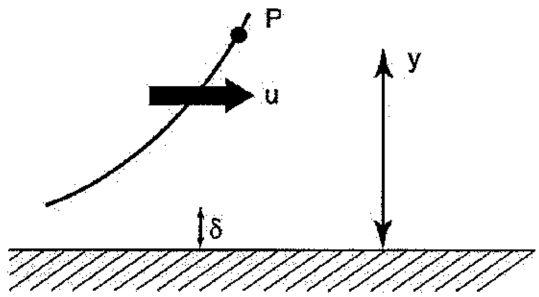

Fig. 1. The geometry of the problem.

$$u'' = -\left(1 - \frac{1}{\ln(d/\delta)}\right)u'/y. \qquad [5]$$

This is our model equation. The main assumption in this heuristic derivation is Eq. **4**, which quantifies the impact of a viscous length scale on the mixing length.

Properties of the Model Equation

The derivation we have just given is heuristic, but there is nothing heuristic about the analysis we now give.

The solution of Eq. **5** with the boundary conditions of Eq. **3** is

$$u = \ln\left(\frac{d}{\delta}\right)\left(\frac{y}{\delta}\right)^{1/\ln(d/\delta)} - 2\ln\left(\frac{d}{\delta}\right).$$

The quantity $yu'(y) = (d/d \ln y)u$ can be calculated,

$$yu' = (y/\delta)^{1/\ln(d/\delta)} = \exp\left(\frac{\ln(y/d) + \ln(d/\delta)}{\ln(d/\delta)}\right),$$

so that for each fixed y, its limit as $\delta \to 0$ is $e = 2.71\ldots$. As $\delta \to 0$, u converges pointwise to the curve (or, in logarithmic coordinates, straight line) $u = e \cdot \ln y$. There is a separate function u for each value of δ/d; the envelope of these curves is $u = \ln y$; thus, in the $(\ln y, u)$ plane the curves have this straight line as envelope and have a common asymptotic slope e (see Fig. 2).

Consider these facts from the point of view of the similarity theory presented above. The solution u is a function of y, δ, and d. The derivative du/dy must satisfy the relation

$$\frac{du}{dy} = \frac{1}{y}\Phi\left(\frac{y}{\delta}, \frac{d}{\delta}\right), \qquad [6]$$

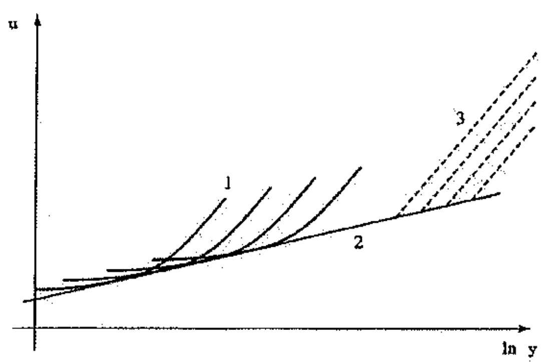

Fig. 2. The solutions of Eq. **5**. Continuous straight line, envelope; broken lines, asymptotic slopes.

where u is dimensionless, though we have omitted a division by a characteristic velocity scale, y/δ, d/δ are large and Φ is unknown. If one makes an assumption of complete similarity, $\Phi \sim C$, where C is a constant, one finds $u' = C/y$, and from the boundary conditions $u = \ln y$, which is the equation of the envelope. The same wrong result can be obtained by simply neglecting the small term $1/\ln(d/\delta)$ in Eq. **5**. A substitution of the true expression for u yields relation

$$\Phi = \left(\frac{y}{\delta}\right)^{1/\ln(d/\delta)}, \qquad [7]$$

which represents incomplete similarity in y/δ with anomalous exponent $1/\ln(d/\delta)$ and no similarity in d/δ.

The relation between the coefficient of the right hand side of Eq. **5** and the parameter δ is a critical relation in the following sense: Consider instead of Eq. **5** the more general equation

$$u'' = -(1 - \psi(\delta/d))u'/y, \qquad [8]$$

where $\psi = \psi(\delta/d)$ is a monotonic function of δ/d, with $d\psi/d\delta > 0$, and with the boundary conditions of Eq. **3**. The solution of Eq. **8** is

$$u_\psi = (\psi)^{-1}(y/\delta)^\psi - \ln(y/\delta) - \psi^{-1}. \qquad [9]$$

One easily finds $ydu_\psi/dy = (y/\delta)^\psi$ and hence that the limit of ydu_ψ/dy as $\delta \to 0$ is

$$\begin{array}{ll} 1 & \text{if } \psi \ln(d/\delta) \to 0; \\ e & \text{if } \psi \ln(d/\delta) \to 1; \\ \infty & \text{if } \psi \ln(d/\delta) \to \infty. \end{array}$$

In the first case the solution is asymptotically given by a complete similarity relation; the second case is the one we discussed; in the third case there is no similarity result. This conclusion affirms once more the close connection between criticality and incomplete similarity (see refs. 17 and 18). Note that in the first two cases the limit $\delta \to 0$ is approached at the a rate proportional to ψ, i.e., logarithmically in the critical case and faster in the subcritical ($\psi \ln(d/\delta) \to 0$) case, as is consistent with the assumed form of the coefficient ℓ.

In summary, one of the following possibilities must hold for solutions of Eq. **8**: (*i*) A faster than inverse logarithmic rate of convergence (in δ) of the profiles gradients to a limit, asymptotic complete similarity, and a logarithmic dependence of u on y; (*ii*) an inverse logarithmic rate of convergence of the profile gradients to a limit, incomplete similarity, and a power law profile; or (*iii*) no well defined limiting velocity profile.

The Scaling of Turbulence

Consider first wall-bounded turbulent flows, for example, flow in a pipe. In Fig. 3 we display mean velocity profiles obtained experimentally by Zagarola in a Princeton experiment (26). The flaws in the experimental procedure (see ref. 11) do not invalidate the qualitative shape of the curves.

It is usually stated that the mean velocity profile in most of the pipe, at large Reynolds numbers R and away from the center and from the near vicinity of the wall, is logarithmic and R-independent; this is usually derived by a complete similarity argument (see ref. 19): the mean velocity u should depend on the viscosity ν, the fluid density ρ, the pipe diameter d, the shear stress at the wall τ, and the distance to the wall y. Dimensional analysis allows the relationship between these quantities to be written in the form

$$du/dy = (u_*/y)\Phi(y/\delta, d/\delta),$$

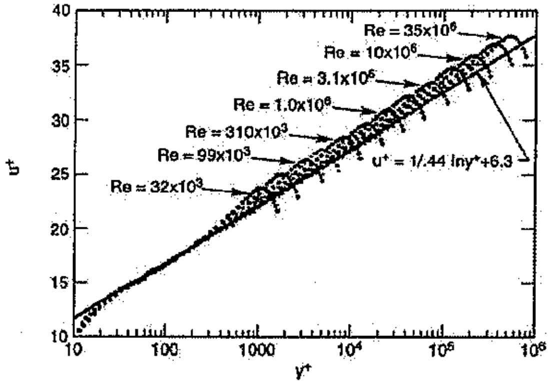

Fig. 3. The experimental graphs of velocity vs. distance from the wall [reproduced with permission from ref. 26 (Copyright 1996, Mark V. Zagarola)].

where $u_* = \sqrt{(\tau\rho)}$, $\delta = \nu/u_*$, and $d/\delta = u_* d/\nu$ is a function of the conventional Reynolds number $R = \bar{u}d/\nu$, with $\bar{u}$ equal to the flux divided by the cross section of the pipe (compare with Eq. **6**). This can be rewritten as

$$du/dy = (u_*/y)\Phi(y/\delta, R),$$

where both arguments of the unknown function Φ are large. If one then makes a complete similarity assumption that the function Φ tends to a nonzero constant C as its arguments become large, one obtains $(du/dy)/u_* = C/y$; one usually defines κ = "Kármán's constant" $= 1/C$ so that this becomes $(du/dy)/u_* = (1/\kappa)/y$. An integration, and an additional assumption about the integration constant, yields the "logarithmic law of the wall" $u/u_* = (1/\kappa)\ln y + B$, where B is implicitly claimed to be R-independent. This conclusion is consistent with Eq. **2**.

However, though this derivation is often reproduced, the conclusion is untenable. The "constants" κ, B vary widely from experiment to experiment, and the data (see Fig. 3) do not show a single R-independent line in the ($\ln y$, u) plane but rather a series of curves, one per Reynolds number, with a straight envelope and a common asymptotic slope, much like Fig. 2. It is therefore more sensible to assume (see Eq. **7**)

$$\Phi = A(R)(y/\delta)^{\alpha(R)} \qquad [10]$$

(incomplete similarity in y/δ and no similarity in R); $\alpha(R)$ and $A(R)$ are functions of R. We further expand $A(R)$, $\alpha(R)$ in powers of $1/\ln R$, consistent with Eq. **4**. One can show directly that this dependence of A, α on R is critical. This yields

$$A(R) = A_0 + A_1/\ln R + \ \ldots, \ \alpha(R) = \alpha_0 + \alpha_1/\ln R + \ \ldots.$$

We then keep only the two leading terms in the series. To have a finite limit in the subcritical case one must have $\alpha_0 = 0$; this leaves three constants, α_1, A_0, and A_1, to determine. We determined them by comparison with experimental data (9, 4, 6, 12). The results agree extremely well with experiment, and even more important, the constants do not move as one processes all pipe and boundary layer flows for which data are available (12). The logarithmic law of the wall corresponds to the wrong logarithmic solution in the analysis of the previous section.

A significant conclusion one can now draw is that as the Reynolds number increases, the first moments of the velocity tend to a limit, and do so at an inverse logarithmic rate.

One may ask next how higher moments of the velocity field behave as the Reynolds number increases. If the first moments tend to a limit at an inverse logarithmic rate, one expects higher moments to converge to a limit at the same rate or at a lower rate; they can also fail to have a limit. Consider for example the structure functions for the velocity field in homogeneous isotropic turbulence (the objects of Kolmogorov's theory). Consider a homogeneous isotropic flow and the velocity $\mathbf{u}$ at two points $\mathbf{x}$, $\mathbf{x} + \mathbf{r}$, and in particular the velocity component u_D in the direction of the line that joins these points. Then, drop the subscript D. The pth order structure function is $s_p = \langle(u(\mathbf{x} + \mathbf{r}) - u(\mathbf{x}))^p\rangle$, where the brackets denote an average. The function s_p depends on the Reynolds number R (for a suitable definition of R, see refs. 3 and 14), a bulk length scale L, the variable distance $r = |\mathbf{r}|$, and the mean rate of energy dissipation ε. Dimensional analysis yields $s_p = (\varepsilon r)^{2/3}\Phi_p(L/r, R)$, where Φ_p is an unknown dimensionless function of two large arguments. If one makes the complete similarity assumptions $\Phi_p \sim C$ for large arguments, one finds the classical Kolmogorov scaling $s_p \sim (\varepsilon r)^{p/3}$, and in particular $s_2 \sim (\varepsilon R)^{2/3}$, from which it follows that in the inertial range the energy spectrum $E(k)$ is proportional to $k^{-5/3}$, the famous Kolmogorov–Obukhov spectrum.

But it is not necessarily so. One should expect the Kolmogorov–Obukhov spectrum to be a manifestation of criticality (see ref. 15), and one should assume incomplete similarity (also see ref. 8). In the plethora of possible assumptions one may as well start with the form of Eqs. **7** and **10** so that for $p = 2$ one finds

$$s_2 = A(R)(\varepsilon r)^{2/3} r^{\alpha_1/\ln R + \ldots},$$

with $A(R) = A_0 + A_1/\ln R + \ldots$, assuming, of course, in addition that the critical dependence on R is inverse logarithmic as above. This yields a spectrum

$$E(k) \sim k^{-5/3 + \alpha_1/\ln R},$$

i.e., a viscous correction to the Kolmogorov spectrum that changes slowly with R. Such a correction could easily be mistaken for a constant correction if one is not too careful with the data, and it is perfectly consistent with at least some of the data (see ref. 11). However, the critical dependence on R may well be slower than inverse logarithmic, in which case it would be even easier to overlook it in the processing of the data, especially when one is unaware of the possibility of such dependence. It has long been speculated that the exponent in the spectrum of the inertial range was in some sense a "critical" exponent, but the possibility that it should be R-dependent had not been properly considered.

The same argument applies to the case $p = 3$, where it is believed that the Kolmogorov scaling is exact; this would simply mean $\alpha = 0$ for $p = 3$. As one goes up in p, one should expect the critical dependence on R to slow down and even change sign, so that eventually the moments do not have an inviscid limit and the critical dependence becomes a dependence on a growing function of R. It is compatible with the data, and with some numerical simulations (7, 13), to assume that this turnover happens at $p = 3$. If this is so, it explains the divergence of the measured higher structure functions from the Kolmogorov predictions without in any way invalidating the Kolmogorov theory.

We thank Ms. O.-L. Loubière for help with graphics. This work was supported in part by National Science Foundation Grant DMS 97-32710 and in part by the Office of Science, Office of Advanced Scientific Computing Research, Mathematical, Information, and Computational Sciences Division, Applied Mathematical Sciences Subprogram of the U.S. Department of Energy, under Contract DE-AC03-76SF00098.

1. Barenblatt, G. I. (1991) *C. R. Acad. Sci. Ser. II* **313**, 107–112.
2. Barenblatt, G. I. (1993) *J. Fluid Mech.* **248**, 513–520.
3. Barenblatt, G. I. & Chorin, A. J. (1996) *Proc. Natl. Acad. Sci. USA* **93**, 6749–6752.
4. Barenblatt, G. I. & Chorin, A. J. (1997) *Comm. Pure Appl. Math.* **50**, 381–398.
5. Barenblatt, G. I. & Chorin, A. J. (1998) *SIAM Rev.* **40**, 265–291.
6. Barenblatt, G. I. & Chorin, A. J. (1998) *Phys. Fluids* **10**, 1043–1044.
7. Barenblatt, G. I. & Chorin, A. J. (1999) *ESAIM Proc.* **7**, 22–35.
8. Barenblatt, G. I. & Goldenfeld, N. (1995) *Phys. Fluids A*, 3078–3082.
9. Barenblatt, G. I. & Prostokishin, V. M. (1993) *J. Fluid Mech.* **248**, 521–529.
10. Barenblatt, G. I., Chorin, A. J. & Prostokishin, V. M. (1997) *Appl. Mech. Rev.* **50**, 413–429.
11. Barenblatt, G. I., Chorin, A. J. & Prostokishin, V. M. (1999) *Physica D* **127**, 105–110.
12. Barenblatt, G. I., Chorin, A. J. & Prostokishin, V. M. (2000) *J. Fluid Mech.* **410**, 263–283.
13. Chorin, A. J. (1998) *Q. Appl. Math.* **56**, 767–786.
14. Barenblatt, G. I. (1996) *Scaling, Self-Similarity, and Intermediate Asymptotics* (Cambridge Univ. Press, Cambridge, U.K.).
15. Chorin, A. J. (1994) *Vorticity and Turbulence* (Springer, Boston).
16. Barenblatt, G. I. (2002) *Scaling* (Cambridge Univ. Press, Cambridge, U.K.).
17. Kadanoff, L. (2000) *Statistical Physics: Statics, Dynamics, and Renormalization* (World Scientific, Singapore).
18. Goldenfeld, N. (1992) *Lectures on Phase Transitions and the Renormalization Group* (Perseus, Reading, MA).
19. Monin, A. & Yaglom, A. (1971) *Statistical Fluid Mechanics* (MIT Press, Cambridge, MA).
20. Bernard, P. & Wallace, J. (2002) *Turbulent Flow: Analysis, Measurement, and Prediction* (Wiley, Hoboken, NJ).
21. Von Kármán, T. (1930) *Proc. Int. Congr. Appl. Mech. 3rd*, 85–93.
22. Chorin, A. J. (1973) *J. Fluid Mech.* **57**, 785–796.
23. Brown, G. & Roshko, A. (1974) *J. Fluid Mech.* **64**, 775–816.
24. Bernard, P., Thomas, J. & Handler, R. (1993) *J. Fluid Mech.* **253**, 385–419.
25. Yaglom, A. (2002) in *New Trends in Turbulence*, eds. Lesieur, M., Yaglom, A. & David, F. (Springer, Berlin), pp. 3–52.
26. Zagarola, M. (1996) Ph.D. thesis (Princeton Univ., Princeton).