THE PHYSICISTS’ PATH INTEGRALS.

In physics books the Wiener integral is often presented in a different way. Consider the integral
\[ \int F dW, \]
where
\[ F = \exp(\int_0^1 F(w(s)ds)\phi(w(1))), \]
as in the solution of the heat equation with potential. Discretize the integral in the exponent
as
\[ \sum kU_i \]
for
\[ i = 1, 2, \ldots, n, \]
where
\[ k = 1/n. \]
The discrete integral is now an average over a set of dependent Gaussian variables. As in
the example at the end of the previous section, define the independent variables
\[ \tau_i = w(ik) - w((i - 1)k), i = 1, 2, \ldots, n, \]
so that
\[ w_i = \sum_{i=1}^n \tau_i. \]
We then find
\[ \int F dW = \int d\tau_1 \ldots \int d\tau_n \phi(x_n) \exp(\sum_{i=1}^n kU(w_i)) \frac{\exp(-\sum_{i=1}^n \tau_i^2/(2k))}{(2\pi k)^{n/2}} d\tau_1 \ldots d\tau_n, \]
or, rearranging the sums in the exponentials and using the definition of the \( \tau_i \),
\[ \int F dW = \int d\tau_1 \ldots \int d\tau_n \phi(x_n) \exp\left(\sum_{i=1}^n (-\sum_{i=1}^n \tau_i^2/(2k) - kU(w_i))\right) \frac{1}{(2\pi k)^{n/2}} d\tau_1 \ldots d\tau_n. \]

Physicists often define the Wiener integral as the limit, as \( n \to \infty \), of this last expression, and often write the limit as
\[ \frac{1}{Z} \int e^{-\int_0^1 \left[ \frac{1}{2} (\frac{dw}{ds})^2 - U(x + w(s)) \right] ds} \phi(x + w(t))[dw]. \]

The ratio \( dw/dt \) appears because \( \tau_i/k = (w(ik) - w((i - 1)k))/k \), which looks like
an approximation to \( dw/ds \), and \( \int_0^1 (dw/ds)^2 ds \) can be thought of as \( \sum k(dw_i/ds)^2 \).
This way of writing seems to suggest that \( Z \) is the limit of the product of the factors \( \sqrt{2\pi k} \), \([dw]\) is the limit of the product of the \( d\tau_i \), and \( dw/dt \) is the limit of the ratio \( (w_{i+1} - w_i)/k \) (i.e., a derivative of BM); none of these limits makes
sense by itself, but the limit of the whole expression does make sense, as we
already know. This limit is often called a “path integral”.

This procedure has some advantages: it is possibly more intuitive; it extends to
“Feynman integrals”, another kind of sum over paths that appears in physics,
where factors such as \( \exp(-x^2/2) \) are replaced by \( \exp(-ix^2/2) \) \( i \) is the unit imaginary number), and which cannot be interpreted as an expected value over
a probability measure; finally, the expression in brackets in equation (??) has
an interesting physical interpretation, as will be discussed below.