

3. Assuming that all the integrals are meaningful, derive the following equalities and inequalities:

(i)  $\int x^2 |f(x)|^2 dx \int k^2 |\hat{f}(k)|^2 dk = \int |xf(x)|^2 dx \int |f'(x)|^2 dx$ , where  $\hat{f}$  is the Fourier transform of  $f$ ,  $f'$  is the derivative of  $f$ , and all integrals are from  $-\infty$  to  $+\infty$ .

(ii)  $\int |xf(x)|^2 dx \int |f'(x)|^2 dx \geq [\int |xf'f^*| dx]^2$ , where the  $*$  denotes a complex conjugate. (hint:  $|(f, g)| \leq \|f\| \cdot \|g\|$ )

(iii)  $[\int |xf'f^*| dx]^2 = (1/4) [\int x(|f|^2)' dx]^2$ .

(iv)  $(1/4) [\int x(|f|^2)' dx]^2 = (1/4) [\int |f(x)|^2 dx]^2 = (1/4) \|f\|^4$ .

Putting all these statements together you get the Heisenberg inequality of quantum mechanics,

$$\int x^2 |f(x)|^2 dx \int k^2 |\hat{f}(k)|^2 dk \geq (1/4) \|f\|^4,$$

which asserts that there is a lower bound on the error in the simultaneous measurement of the position and the momentum of a particle.

4. Consider a "turbulent" flow  $u = u(x)$  on the x-axis ( $x$  has dimension  $L$ ),  $[u] = L/T$ , where  $L$  is a unit of length and  $T$  is a unit of time. Let  $\hat{u} = \hat{u}(k)$  be the Fourier transform of  $u$ . Find the dimensions of  $\hat{u}(k)$ . Check that  $[k] = L^{-1}$ , so that the exponential in the Fourier transform formula is well-defined. Define an "energy spectrum"  $E(k)$  as  $E(k) = |\hat{u}^2(k)|$ . Find the dimensions of  $E$ . Define a rate of energy dissipation  $\epsilon$  as  $\frac{d}{dt}U$ , where  $U = \int u^2(x) dx$  and  $t$  is the time. Find the dimensions of  $\epsilon$ . Assume that  $E(k)$  is a function only of  $\epsilon$  and  $k$  (we will eventually see why this makes sense). Deduce that  $E(k)$  is proportional to  $\epsilon^{2/3} k^{-2}$ .