Math 220, Spring 2012, homework 10, due Wednesday April 11

1. Consider the random variable $\sigma_n = \sum_1^n \xi_i^2$ for $n = 100$, where each ξ_i is a Gaussian variable with mean zero and variance 1. Calculate by Monte Carlo the mean and the variance of σ_n. (This illustrates the equivalence of ensembles.) (This is also the last computational problem).

2. Consider the partial differential equation $u_t = (u^2/2)_x$ (the subscripts denote differentiations) in $0 \le x \le 2\pi$, with the periodic boundary condition $u(x + 2\pi) = u(x)$ and with an initial condition such that $\int_0^{2\pi} u(x,0)dx = 0$. Assume the solution can be written as $u = \sum_{-N}^{N} u_k e^{ikx}$, where N is fixed (i.e., neglect all Fourier coefficients beyond the N-th). Check that at for all t, $u_0(t) = 0$. For $k > 0$, write $u_k = \alpha_k + i\beta_k$, and find the equations of motion for the α_k, β_k. Show that since u is real, u_{-k} is the complex conjugate of u_k. Check that the flow in the $2N$ dimensional space of the α_k, β_k is incompressible, and that the energy $E = \sum(\alpha_k^2 + \beta_k^2)$ is invariant. Given N and E, what is the microcanonical density for this system? Show that if $N \to \infty$ and equilbirium has been reached, then $E \to \infty$ unless $E = 0$. (This problem illustrates the difficulty in extending equilibrium statistical mechanics to problems described by partial differential equations).

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