DIMENSIONAL ANALYSIS AND SCALING

Every physical quantity is expreseed in terms of units, or “dimensions”. For example, the statement “ the distance between Berkeley and San Francisco is 17” is meaningless- the obvious question is “17 what ?”. 17 kilometers is different from 17 miles. “mile” and ”kilometer” are examples of units. To provide units adequate for conveying quantitative information in a given field of science one has to pick some basic variables and assign them units, for example, one can choose to assign units to variables that describe lengths, for example inches or kilometers, and/or to variables that describe time, for example hours or milliseconds. Once one has units for length and time, one automatically has units for velocity, which is the ratio of distance covered divided by the time used, and its units are the ratios of units of distance divided by units of time, for examples, inches divided by hours, or ”inches per hour”. We shall denote the units of a variable x by $[x]$, and denote by L the unit of length and by T the unit of time (not to be confused in later chapters with temperature), so that if u is a velocity, its units are $[u] = L/T$. How many basic variables are needed depends on the subject; usually for mechanics one needs three, for example, L, T, and a unit of mass M. The other variables then have units which can be derived from these units, as was the case for velocity above. One says that the units of these other variables depend on the units of length, time, and mass, while the units of length, time, and mass are independent (of each other). If one wishes also to consider electric forces one may need to add a unit of charge, and if the result of one’s efforts is to be sold, one needs a unit of money. It is obvious that if the size of the unit in which a quantity is expressed is increase by a factor α, the numerical value of the size of the quantity is divided by α, so that 20 kilometers equal 20000 meters. A quantity is dimensionless if it does not change when units are changed, for example, the Mach number in fluid mechanics is the ratio of two velocities and does not change when L, T, change. From now on we assume that the variables to which basic units have been assigned do not change, and discuss what happens when the sizes of these basic units change.

An equation relating physical quantities makes sense only if the units on both sides are equal, or else the equation becomes false when the size of the units changes. For example, the heat equation, if written as $u_t = u_{xx}$, makes no sense at first sight, because the units of the left-hand-side are $[u_t] = [u]/T$ while on the right-hand-side are $[u_{xx}] = [u]/L^2$, so that if the equation is true when time is measured in seconds and distance is measured in centimeters, it will be false if time is measured in minutes. (“x is measured in seconds” means “the units of x are seconds”). The equation $u_t = u_{xx}$ must be understood as the equation $u_t = \nu u_{xx}$, where the coefficient ν has units $[\nu] = L^2/T$, and where L, T have been chosen so that the numerical value of κ is one. This last equation will then remain true when the units are changed provided the numerical value of ν changes in the appropriate manner.

Suppose that $a_1, a_2, \ldots, a_n$ are variables with independent units, and that the sizes of the units have been decided, so that the variables have numerical values $a'_1, a'_2, \ldots, a'_n$. We now show that it is possible to pick sizes for the units which will assign to a_1 a new numerical value larger than the previous one by a factor A, while keeping the numerical values of all the other variables unchanged. For simplicity, assume that there are three basic units, say L, T and M. The number of variables with independent units can then be $1, 2$ or 3 (see the exercises). We write things out in the case of two independent variables; the case of three variables is similar. Let the units of a_1 be $L^{\alpha_1}T^{\beta_1}M^{\gamma_1}$, those of a_2 be $L^{\alpha_2}T^{\beta_2}M^{\gamma_2}$. If L is divided by a factor λ then a'_1 is multiplied by λ^{α_1}, if T is divided by a factor τ then a'_1 is multiplied by τ^{β_1}, and if M is divided by μ then a'_1 is multiplied by μ^{γ_1}, with analogous statements for a'_2. We claim that it is possible to pick α, τ, μ so that the new numerical value of a_1 is Au'_1, where A is an arbitrary non-negative constant, while the other numerical values are unchanged, i.e.,

$$\lambda^{\alpha_1}\tau^{\beta_1}\mu^{\gamma_1} = A,$$

and

$$\lambda^{\alpha_2}\tau^{\beta_2}\mu^{\gamma_2} = 1.$$

Taking logarithms, one finds a pair of linear equations for the variables $log(\lambda), log(\tau), log(\tau)$:

$$\alpha_1 log(\lambda) + \beta_1 log(\tau) + \gamma_1 log(\mu) = log(A),$$
$$\alpha_2 log(\lambda) + \beta_2 log(\tau) + \gamma_2 log(\mu) = 0.$$

This system of equations fails to have a solution only if the left-hand-side of the first equation is a multiple of the left-hand-side of the second equation, in which case the units of a_1 are the units of a_2 raised to some power, i.e., the units of a_1, a_2 are not independent.

Suppose a variable a is a function of variables $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_k$, where $a_1, \ldots$, where the $a_i, i = 1, \ldots, m$ have independent units (for example a_1 could be a length and a_2 could be a time), while the units of $b_1, \ldots, b_k$, can be formed from the units of $a_1, a_2, \ldots, a_m$; (in the example just used, b_1 could be a velocity)‘. Then one can find dimensionless variables $\Pi, \Pi_i, i = 1, m$

$$\Pi = \frac{a}{a_1^{\alpha_1}\cdots a_m^{\alpha_m}}, \ \ \Pi_i = \frac{b_i}{a_1^{\alpha_{i1}}\cdots a_m^{\alpha_{im}}}, \ \ i = 1, \ldots, k, \tag{1}$$

where the α_i, α_{ij} are fractions, and the relation between a and the a_i, b_i becomes

$$\Pi = \Phi(a_1, \ldots, a_m, \Pi_1, \ldots, \Pi_k), \tag{2}$$

where Φ is some unknown function to be determined. Now change the units of measurement. The dimensionless quantities are unchanged, but the quantities $a_1, \ldots, a_m$ can take any value one wishes, as we have just shown. This means that Φ cannot be a function of a_1 etc. and we have:

$$\Pi = \Phi(\Pi_1, \ldots, \Pi_k), \tag{3}$$

and the number of variables has been decreased. This device can be a useful way to simplify problems by reducing the number of variables, see the example in the exercises.

Equation (**??**) can be simplified further when one of the variables (say Π_1) is very small (The case where Π_1 is very large can be reduced to the case where Π_1 is very small by making Φ a function of $1/\Pi_1$). Assume that the function Φ has a nonzero finite limit as Π_1 tends to zero. Then one can approximately set $\Pi_1 = 0$ and the number of variables has been reduced by one. If Π_1 is the only argument of Φ then we have found a monomial relation between a and the a_i: $a = Ca_1^{\alpha_1} \cdots a_m^{\alpha_m}$, where C is a constant, and we have found a complete solution of the problem up to a constant. This is a "complete similarity" relation. It embodies the usual intuition that very small variables can be omitted (and note that it makes sense to speak of small variables only when these variables are dimensionless).

If the function Φ does not have the assumed limit, it may happen that for Π_1 small, $\Phi(\Pi_1) = \Pi_1^{\alpha}\Phi_1(\Pi_1) + \cdots$, where the dots denote lower-order terms, α is a constant, the other arguments of Φ have been omitted and, Φ_1 has a finite nonzero limit. One can then obtain a monomial expression for a in terms of the a_i and b_i, with undetermined powers that must be found by means other than dimensional analysis. The resulting power relation is an 'incomplete" similarity relation. The exponent α is known in the physics literature as an anomalous scaling exponent. Note that if the anomalous exponent is negative, we have the counterintuitive result that the impact of the small variable increases when its magnitude decreases. In physics, incomplete similarity is usually discussed in the context of the renormalization group; for instances of incomplete simlarity, see chapter **??** and chapter **??** below. Of course, one may well have functions Φ with neither kind of similarity.