Math 128a, Spring 2013, sample final.

1. Consider the implicit scheme $u_{i+1} = u_i + hf(x_{i+1}, u_{i+1})$ for solving the equation $y' = f(x, y)$, $y(0) = a$, in the special case $f(x, y) = 2 + \cos(y)$. This is a nonlinear equation for u_{i+1}. Set up an iteration scheme for finding u_{i+1} and find under what conditions your iteration converges.

2. Suppose you are trying to evaluate $I = \int_{-1}^{1} f(x)dx$. Find x_1, x_2, w_1, w_2 so that the equation $I = w_1 f(x_1) + w_2 f(x_2)$ is exact whenever f is a polynomial of degree ≤ 3.

3. Consider the quadrature rule $\int_a^b f(x)dx = \sum hf(a+ih)+e$, where i takes the values $0, 1, \ldots, n-1$ with $n = (b-a)/h$ and e is the error. Show that $e = Ch + O(h^2)$, where C does not depend on h.

4. Suppose you are solving the equation $y' = f(x, y)$, $y(0) = a$ by a second order Runge-Kutta method, and you go from x_0 to $x_{i+1} = x_i + h$ by a step of length $h = .1$ and get $u_{i+1} = 1.764$. Then you go back to x_i, advance the solution by two steps of length $h/2$, and get at $x_i + h$ the value $u = .7615$. Use these facts (i) to estimate the local truncation error in the step of length h, and (ii) to estimate the length h of the step needed to get a local truncation error approximately equal to 10^{-5}.

5. Estimate the condition number of the matrix with rows (1, .99), (.99, .98).

6. Suppose you are solving the equation $y' = y$, $y(0) = 1$ by the second order Runge-Kutta scheme $u_{i+1} = u_i + hf(x_i + h/2, u_i + (h/2)f(x_i, u_i))$. (i)Explain why the error per step at the $(i+1)$th step is $u_i e^h - u_{i+1}$; (ii) Find an expression for u_{i+1} using the scheme and the special form of the equation; and (iii) use this expression to find the order of of the local truncation error for this special equation (i.e., the exponent ℓ in the expression $\tau = O(h^\ell)$) when $f(x, y) = y$.)

7. Find coefficients a_{-1}, a_0, a_1, a_2 such that, for a function f with enough derivatives, one has
$$a_{-1}f(x-h) + a_0 f(x) + a_1 f(x+h) + a_2 f(x+2h) = f''(x) + O(h^3).$$

8. Suppose the function $f(x) = \sin \pi x$ is interpolated on $[0, 1]$ at 10 equidistant points (which include 0 and 1) by a polynomial P of degree 9. Find the smallest bound B you can such that $||f - P||_\infty \leq B$. (The question asks only for the bound; you don't have to find the polynomial).

9. Find the solution of the difference equation $u_{i+1} = 4u_i - 4u_{i-1}$ subject to the conditions $u_0 = 1, u_1 = 2$.