1. Consider the fourth-order Runge-Kutta scheme applied to the differential equation $y'=y$ with $y(0)=1$. Verify in this special case that the scheme has a truncation error $O(h^4)$. (Hint: check first that $u_{i+1} = u_i e^h + O(h^5)$).

2. Consider again the equation $y'=y$, $y(0)=1$, and the Euler scheme for solving it. In this special case you can solve the equation exactly and can therefore determine the errors exactly. (i) Using the mesh size $h=0.2$, plot the local truncation error and the overall error for $0 \leq x = ih \leq 1$ as a function of $i$ (for example for $i=1$, when there is no previous error so that the two errors are equal, they are both equal to $e^h - (1+h)$). (ii) Repeat this with $h=0.1$, and note that the errors per time step are roughly 4 times smaller but the overall error at $x=1$ is only about $1/2$ of what it was, and explain why this makes sense.

3. (This problem prepares the way for a computer problem to come). Suppose you want to find the best approximation of a function $f$ by a polynomial of degree $\leq n$ on $[0, 1]$ in the $\|\cdot\|_2$ norm as follows (in the way you were told is NOT good): Define a function $G(a_0, a_1, \ldots, a_n)$ as follows:

$$G(a_0, \ldots, a_n) = \|f - (a_0 + a_1 x + \ldots + a_n x^n)\|^2,$$

($=$ the squared distance between $f$ and a function in the span of the appropriate powers of $x$). To minimize this distance, set up the $(n+1)$ equations $\frac{\partial G}{\partial a_i} = 0$ for $i = 0, 1, \ldots, n$. Write the resulting linear system in the form $La = q$, where $a$ is the vector whose components are the $a_i$, $q$ is a known vector, and $L$ is a matrix. Say clearly what $L$ and $q$ are. (In a computer problem to come, I will ask you to actually solve this system).