Math 128a, Chorin, Spring 2013, homework 9, due the week of April 8.

1. Consider the fourth-order Runge-Kutta scheme applied to the differential equation $y' = y$ with $y(0) = 1$. Verify in this special case that the scheme has a truncation error $O(h^4)$. (Hint: check first that $u_{i+1} = u_i e^h + O(h^5)$).

2. Consider again the equation $y' = y, y(0) = 1$, and the Euler scheme for solving it. In this special case you can solve the equation exactly and can therefore determine the errors exactly. (i) Using the mesh size $h = 0.2$, plot the local truncation error and the overall error for $0 \leq x = ih \leq 1$ as a function of i (for example for $i = 1$, when there is no previous error so that the two errors are equal, they are both equal to $e^h - (1 + h)$. (ii) Repeat this with $h = 0.1$, and note that the errors per time step are roughly 4 times smaller but the overall error at $x = 1$ is only about $1/2$ of what it was, and explain why this makes sense.

3. (This problem prepares the way for a computer problem to come). Suppose you want to find the best approximation of a function f by a polynomial of degree $\leq n$ on $[0, 1]$ in the $|| \cdot ||_2$ norm as follows (in the way you were told is NOT good): Define a function $G(a_0, a_1, \ldots, a_n)$ as follows:

$$G(a_0, \ldots, a_n) = ||f - (a_0 + a_1 x + \ldots + a_n x^n)||^2,$$

(=the squared distance between f and a function in the span of the appropriate powers of x). To minimize this distance, set up the $(n + 1)$ equations $\frac{\partial G}{\partial a_i} = 0$ for $i = 0, 1, \ldots, n$. Write the resulting linear system in the form $La = q$, where a is the vector whose components are the a_i, q is a known vector, and L is a matrix. Say clearly what L and q are. (In a computer problem to come, I will ask you to actually solve this system).