Math 128a, Chorin, Spring 2013, homework 3, due the week of February 18.

1. Construct orthonormal polynomials Q_0, Q_1, Q_2 of degrees $0, 1, 2$ on the interval $[0, 1]$, and use them to find the best approximation of $f(x) = e^x$ in the $2-$norm on that interval. Plot both this approximation and the sum of the first 3 terms in the Taylor series around the origin, (i.e. up to and including the quadratic term), and comment on what you see.

2. Check that the functions $1, \sin(x), \cos(x), \sin(2x), \cos(2x)$, etc., are orthogonal on $[0, 2\pi]$. Find the corresponding orthormal functions. Use then to to find the best approximation of the function $f(x)$ by a linear combination of the functions $1, \sin(x), \cos(x), \ldots, \sin(5x), \cos(5x)$, when $f(x) = 1$ when $0 \leq x \leq \pi$ and $f(x) = -1$ when $\pi < x \leq 2\pi$. Plot $f(x)$ and its approximation.

3. Show that if a vector v is orthogonal to each of the vectors $e_0, e_1, \ldots, e_n$, then it is orthogonal to every vector in the span of $e_0, e_1, \ldots, e_n$.

4. Show that if the polynomial P_1 of degree 1 is orthogonal to the function $f(x) = 1$ on an interval $[a, b]$, then P_1 must have a zero in this interval.

5. Consider the function $f(x) = e^x$ on the interval $[0, 1]$, and suppose you interpolate it by a polynomial P_n at the $n + 1$ points i/n, with $i = 0, 1, 2, \ldots, n$ where n is a positive integer.

   (i) Let $z_1 = 1/(2n)$. Show that

   $$|e^{z_1} - P_n(z_1)| \leq A_n = e((2n-1)!!)/(2^{n+1}(n+1)!),$$

   where $m!! = 1 \cdot 3 \cdot 5 \cdots m$ if m is odd.

   (ii) Let $z_2 = 1/2$, and assume that n is odd, so that z_2 is half-way between two interpolation points. Show that

   $$|e^{z_2} - P_n(z_2)| \leq B_n = 2e(n!!)/(2^{n+1}(n+1)!).$$

   (iii) Show that $A_n/B_n \to \infty$ as n increases.