Math 128a, Chorin, Spring 2013, homework 2, due the week of February 11.

1. Consider the equation $tanh(\alpha x) = x$, where α is a parameter. Note that $x = 0$ is a solution for every value of α. ($tanh(x)$ is the function $tanh(x) = (e^x - e^{-x})/(e^x + e^{-x})$). (i) Use the Taylor series for $e^{\alpha x}$ to show that for α positive and small the equation has no roots other than zero, (ii) show that for α large it has two other roots, one positive and one negative. (These facts will be used in a computer assignment to come).

2. Find the polynomial of degree 1 that interpolates a function $f(x)$ at the two distinct points x_1, x_2 (i) using the equation of a line through two given points, and (ii), by using the Lagrange interpolation polynomials, and verify that the two are exactly the same (as they must be because they agree at 2 points).

3. Consider again the function $f(x) = e^x$ on the interval $[0, 1]$, and interpolate it at the point$x = x_1$ in the interval by a polynomial P_0 of degree 0, i.e., by the constant e^{x_1}. Show that the function $f - P_0$ is an increasing function of x in the given interval. Conclude that the function $|f - P_0|$ in the interval is largest at either 0 or 1. Plot the maximum of this function over the interval as a function of x_1 (i.e., for every value of x_1, find the largest value that $|f - e^{x_1}|$ can have in the interval). Observe that the maximum value of $|f - e^{x_1}|$ on the interval is smallest when x_1 is such that the values of $|f - P_0|$ at 0 and 1 are equal, and explain why this must be so.

4. Consider again the function $f(x) = e^x$ in $[0, 1]$, and interpolate it at the points x_1, x_2 by a polynomial $P_1(x)$ of degree 1. Show that the function $f - P_1$ is convex up (i.e., its second derivative is positive). Show that the function $|f - P_1|$ has 3 maxima, one smaller than both x_1, x_2, one larger than both, and one between x_1, x_2. (I will explain the significance of the these problems in class).

5. Suppose that a long time ago, before the computer era, a scientist was trying to evaluate the function $sin(x)$ for $x = 0.05$, and had at her disposal a table of values of $sin(x)$ for $x = 0.1, 0.2, 0.3$. Suppose she did it (i) by linear interpolation for the value at $x = 0.1$ and $x = 0.2$, and (ii) by quadratic interpolation from the values at $x = 0.1, 0.2, 0.3$. In each case, give a bound for the error.