Math 128a, Chorin, Spring 2013, homework 11, due the week of April 22.

1. Write the differential equation $y''' + x^2 y'' - (y')^2 - 5y = 0$, with data $y(0) = a, y'(0) = b, y''(0) = c$, as a first order system.

2. Show that the vector function $f = (f_1, f_2)$, with $y = (y_1, y_2)$ and $f_1 = y_1^2 - y_2^2, f_2 = x^2 y_1 + \cos(y_2)$, is a Lipshitz continuous function of y in the region $-1 \le x, y_1, y_2 \le 1$ in the $||\cdot||_\infty$ vector norm.

3. Suppose you are solving systems of equations of the form $y' = f(x, y), y(0) = a$, where y, f, a are vectors. We saw that it is desirable to have schemes with the following property: if one applies them to the solution of the scalar equation $y' = \lambda y$, where $\lambda < 0$ may be large in absolute value, then all the solutions of the resulting difference scheme remain bounded as $i \to \infty$ even when h, the mesh size, is too large for the scheme to be accurate. For each of the following schemes, determine whether this property holds.
(i) $u_{i+1} = u_i + hf_i$ (as usual, $f_i = f(x_i, u_i)$);
(ii) $u_{i+1} = u_i + hf_{i+1}$;
(iii)$u_{i+1} = u_i + (h/2)(f_i + f_{i+1})$;
(iv)$u_{i+1} = u_{i-1} + 2hf_i$;
(v)$u_{i+1} = u_{i-1} + (h/3)(f_{i-1} + 4f_i + f_{i+1})$.
(Note that the answer is "yes" for the lower-order implicit schemes).

4. Consider the scheme $u_{i+1} = u_i + h(\alpha f_i + (1 - \alpha)f_{i+1})$, where α is a constant in $[0, 1]$. (i) Show that it is convergent. (ii) Find the order of the local truncation error. (iii) Determine the values of α for which it has the property described in the previous problem. (Be careful: $|x| \le 1$ means $-1 \le x \le 1$).

5. Consider the differential equation $y'' + p(x)y' + q(x)y = f$, where p, q, f are functions of x in $[0, 1]$. Suppose you approximate y' by $(u_{i+1} - u_{i-1})/(2h)$ and y'' by $(u_{i+1} + u_{i-1} - 2u_i)/h^2$, where the u_i are values of u at the mesh points ih, $i = 0, 1, \dots, N$ and $h = 1/N$. The differential equation is approximated by equations of the form $L_i u = f_i$, where $L_i u$ is a linear combination of u_{i-1}, u_i, u_{i+1} and $f_i = f(ih)$, for $i = 1, 2, \dots, N - 1$ (at the end points $i = 0$ and $i = N$ $L_i u$ is not defined).
(i)Find $L_i u$.
(ii) Show that the equation $L_i u = f_i$ is consistent (i.e., show that if you replace u_i by y_i in this equation, the remainder is $O(h^2)$.
(iii) Prescribe values $y(0) = a$ and $y(1) = b$ for the solution of the differential equation, so that you no longer have an initial value problem. Set $u_0 = a$ and $u_N = b$. Write the equations $L_i u = f_i$ with these values of u_0, u_N as a linear system of $N - 1$ algebraic equations.