Averaging and Renormalization for the KdV-Burgers Equation

Alexandre J. Chorin
Department of Mathematics
University of California at Berkeley
Berkeley, CA 94720, USA

Abstract

We consider traveling wave solutions of the Korteveg-deVries-Burgers equation, and set up an analogy between the spatial averaging of these traveling waves and real space renormalization for Hamiltonian systems. The result is an effective equation which reproduces means of the unaveraged, highly oscillatory, solution. The averaging enhances the apparent diffusion, creating an "eddy" (or renormalized) diffusion coefficient; the relation between the eddy diffusion coefficient and the original diffusion coefficient is found numerically to be one of incomplete similarity, setting up an instance of Barenblatt's renormalization group. The results suggest a new relation between self-similar solutions of differential equations on one hand and renormalization groups and optimal prediction algorithms on the other. An analogy with hydrodynamics is pointed out.

1 Introduction

There are many equations of interest in science whose solution is too complicated to be computed with sufficient accuracy, and one may be interested in finding an effective equation, easier to work with, whose solution is an average of the real solution. Recently, there has been a major effort to find rational ways to derive such equations for nonlinear problems, see., e.g., [1],[2].

Optimal prediction methods [3] have been developed to find effective equations where the average is over a probability density for unresolved degrees of freedom. In the case of systems in thermal equilibrium the derivation of the reduced equations simplifies considerably, and turns out to be closely related to renormalization group methods [4].

It is of great interest to extend these simpler optimal prediction methods to non-Hamiltonian systems, and we attempt to do so here by analogy; a more ambitious extension of renormalization group ideas to systems far from equilibrium, including diffusive systems, has been presented in [5], [6],[7]. We work here with the specific example of the Korteveg-deVries-Burgers (KdVB) equation, with boundary conditions that give rise to traveling waves. The problem has a single dimensionless parameter that we call R (to highlight an analogy with the Reynolds number of fluid mechanics). For large values of R the traveling waves are highly oscillatory; we consider a spatial (not ensemble) average of these traveling waves and look for an equation whose solution approximates this average (see Figure 1). This problem was chosen because of its apparent simplicity, because previous work by Barenblatt et al. [8],[9] has suggested interesting conjectures about its solution, and also because of interesting connections with fluid mechanics.

In the following sections we describe the problem, review the equilibrium averaging theory for Hamiltonian systems, set up the analogy for the KdVB equation, and provide a numerical analysis of its validity. We also compare the numerical results to earlier analytical work and point out interesting scaling relations that emerge from the calculations. Note that the averaging/renormalization presented here is applied to stationary solutions; the extension to time dependent problems will be presented elsewhere. A key feature of the work is that the methods are applied to partial differential equations and thus require a careful consideration of scaling.

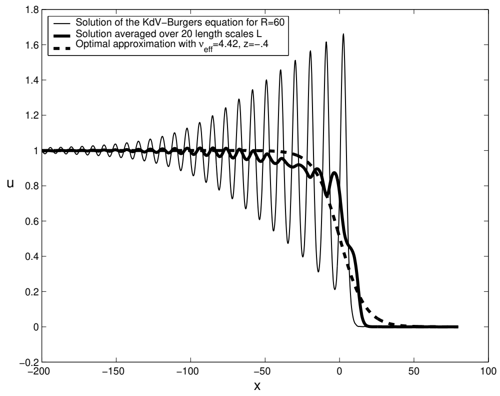

Figure 1: Solution of the KdV-Burgers equation for $R = 60$ (thin oscillatory line), its average for $\ell = 20$ (thick line), and its optimal approximation (broken line)

2 The KdV-Burgers equation, its numerical solution, and an asymptotic analysis

Consider the equation

$$u_t + uu_x = \alpha u_{xx} + \epsilon^2 u_{xxx}, \tag{1}$$

with boundary conditions

$$u(-\infty) = U, \quad u(+\infty) = 0, \quad u_x(-\infty) = 0, \tag{2}$$

where the subscripts denote differentiation, x is the spatial variable, t is time, and $\alpha \geq 0$, ϵ^2, $U \geq 0$ are given constants. The boundary conditions create a traveling wave solution moving to the right (towards $+\infty$) with velocity $U/2$. We pick as length scale $L = \epsilon/\sqrt{U}$, as time scale $T = \epsilon/U^{3/2}$, set $x = x'/L$, $t = t'/T$, $u' = u/(L/T)$, and drop the primes; the result is the equation

$$u_t + uu_x = \frac{1}{R} u_{xx} + u_{xxx}, \tag{3}$$

with $u_x(-\infty) = 0$, $u(+\infty) = 0$, $u(-\infty) = 1$; $R = \epsilon\sqrt{U}/\alpha$ is the "Reynolds number". For $R \leq 1$ the traveling wave has a monotonic profile, while for $R > 1$ the profile is oscillatory, with oscillations whose wave length is of order 1 [10]. At zero diffusion ($R = \infty$) the stationary asymptotic wave train extends to infinity on the left with a singularity near the location where there would be a shock in the absence of dispersion [11]. For finite R the wave train is damped as is the singularity, and the solution of (3) tends to 1 as x decreases even in the steady state.

The numerical solution of (3) requires some care. As usual we proceed by fractional steps corresponding to the several terms in equation (3). The fractional steps that correspond to the linear terms are performed via a Fourier expansion, with each coefficient advanced in time exactly (the earliest reference we could find for this procedure in this context is [12]. For the nonlinear term we use the fourth-order Lax-Wendroff-like method of Zalesak [13]; its accuracy is second order in time and fourth order in space, and so is the accuracy of the overall fractional step scheme, though in practice the error in time is negligible.

Equation (3) is solved numerically in a domain $-X+t/2 \leq x \leq X+t/2$, in which the traveling wave is stationary. X is picked large enough so that there are no significant oscillations of the solution outside this domain. The boundary conditions are imposed at the edges of this domain rather than at infinity.

The solution of equation (3), shifted by $-t/2$, tends to a steady state. For $R \sim 20$ the convergence is smooth and rapid (see Figure 2, where the residual variation is due to the translation); for greater values of R oscillations can persist for a long time and reaching a steady state at an acceptable cost may require a little averaging in time. At the steady state we average the solution at each point x over the region $(x-\ell/2, x+\ell/2)$ and call the result $\bar{u}$, the averaged solution at x; ℓ is a dimensionless multiple of the unit of length L specified above. We are looking for an effective equation $g(v, v_x, v_{xx}, \dots) = 0$ whose solution v approximates this average; this solution can be expected to be smoother than the solution of (3) and thus require fewer mesh points for an accurate numerical solution.

In [8],[9], Barenblatt et al. have given an asymptotic derivation of an effective equation for this problem. They assumed that R was much larger than 1, that the front was averaged over a length much larger than the wave length of the dispersive waves but much smaller than the width of the wave train (i.e., of the region where u is neither the constant 1 nor the constant 0). They also assumed that the dispersion in the effective equation was negligible and thus the effective equation had the form

$$\bar{u}\bar{u}_x = (\nu_{\text{eff}}\bar{u}_x)_x, \tag{4}$$

with an effective viscosity (not necessarily constant) $\nu_{\text{eff}} < 1$. If one writes $u = \bar{u} + u'$, and the spatial average of u' is zero, substitution into equation

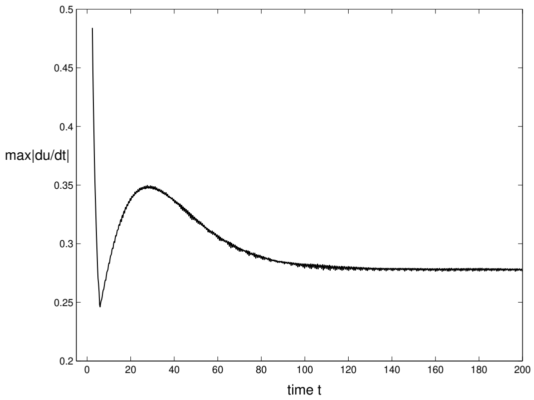

Figure 2: Decay of the time derivative of the solution of the KdVB equation to the constant that characterizes the traveling wave

(4) yields:

$$\overline{u'^2}/2 = \nu_{\text{eff}}\bar{u}_x. \tag{5}$$

An asymptotic analysis then suggested that ν_{eff} was approximately a constant, at least far from the edges of the oscillation region, and a numerical calculation supported this conclusion to some extent. We shall call equation (5) the hydrodynamic approximation and the value of ν_{eff} it yields ν_{hydro}.

3 Renormalization and averaging for Hamiltonian systems

To explain what will be done below for the KdVB equation, we quickly review optimal prediction at equilibrium [3], which for a Hamiltonian system is a special case of real-space renormalization [4]. Consider a system of ordinary differential equation

$$\frac{d\phi}{dt} = R(\phi), \quad \phi(0) = x, \tag{6}$$

where ϕ, x, and R are n-dimensional vectors with components $\phi = (\phi_1, \ldots, \phi_n)$, etc. and t is the time. Assume that the system is Hamiltonian: there exists a function $H = H(\phi)$ such that $R_i = \frac{\partial H}{\partial \phi_{i+1}}$ for i odd and $R_i = -\frac{\partial H}{\partial \phi_{i-1}}$ for i even. The system (6) then preserves the canonical probability density $Z^{-1}\exp(-H/T)$, where Z is the partition function.

Suppose we want to calculate the average value of a function $A = A(\phi)$ with respect to the canonical density. One can do so by Markov-Chain Monte-Carlo based on the Hamiltonian H, or one can solve equations (6) in time, average the values of A and hope for ergodicity.

Divide the component of ϕ into groups: $\hat{\phi} = (\phi_1, \ldots, \phi_m)$, $\tilde{\phi} = (\phi_{m+1}, \ldots, \phi_n)$, so that $\phi = (\hat{\phi}, \tilde{\phi})$, and similarly $x = (\hat{x}, \tilde{x})$, $R = (\hat{R}, \tilde{R})$. Note that $\hat{R} = \hat{R}(\phi)$, i.e., a subset of components of R is in general a function of all the components of ϕ. Suppose the function A depends only on the values of the reduced set of m variables $\hat{\phi}$; then there is no need to calculate the full statistics of ϕ. Indeed, one can approximate the system (6) by

$$\frac{d\hat{\phi}}{dt} = E\left[\hat{R}(\phi)|\hat{\phi}\right], \quad \hat{\phi}(0) = \hat{x}, \tag{7}$$

where $E\left[\hat{R}(\phi)|\hat{\phi}\right]$ is the conditional expectation of $\hat{R}(\phi)$ given $\hat{\phi}$. $E[\hat{R}|\hat{\phi}]$ is a function of $\hat{\phi}$ only [3] and is the best approximation of $\hat{R}(\phi)$ by a function of $\hat{\phi}$ in the mean square sense. Hald's theorem [3] states that the reduced system (7) is Hamiltonian as well, with a new Hamiltonian $\hat{H} = -\log \int e^{-H} d\tilde{\phi}$, where $d\tilde{\phi} = d\phi_{m+1} d\phi_{m+2} \ldots d\phi_n$; we have folded the temperature T into the Hamiltonian and assume the reader can insert the appropriate factors of T into the equations of motion. The partition function of the reduced system equals that of the original system, $\hat{Z} = \int e^{-\hat{H}} d\hat{\phi} = Z$, and if the initial data for the reduced system are drawn from the new canonical density $Z^{-1} e^{-\hat{H}}$ then the joint probability density of the variables $\hat{\phi}$ equals their marginal density in the full system. The average of the function $A = \hat{A}(\hat{\phi})$ can now be computed by Monte-Carlo based on the Hamiltonian $\hat{H}$ or by solving equations (7) and relying on ergodicity. However, in general, the reduced Hamiltonian system does not produce accurate time evolutions for the for the components $\hat{\phi} = \hat{\phi}(t)$, see [3]; it only reproduces the correct equilibrium statistics.

This construction is closely related to renormalization, in the present context, to real-space renormalization [14]. Associate the variables ϕ_i, $i = 1, \ldots,$ with points on a regular lattice. Divide the points into groups, each containing the same number q of variables, each group having the same shape and the same number of lattice points (for example, one could divide a one-dimensional lattice into groups of two points). Pick as components of $\hat{\phi}$ one of the variables in each group (for example, in the case of a one-dimensional lattice divided into groups of two, take the leftmost of each pair). Pick as components of $\tilde{\phi}$ the variables which are not in $\hat{\phi}$. Then write $H^{(0)} = H$, $\phi^{(0)} = \phi$, $H^{(1)} = \hat{H}$, $\phi^{(1)} = \hat{\phi}$, and relabel the components of

$\phi^{(1)}$ so that they are attached to the points in the original lattice. This transformation: $H^{(0)} \rightarrow H^{(1)}$, $\phi^{(0)} \rightarrow \phi^{(1)}$, followed by relabeling, is a renormalization group (RNG) transformation as well as a spatial averaging. Note that the Hamiltonian $H^{(1)}$ was obtained from $H^{(0)}$ by averaging the right-hand-sides of the equations of motion (see equation (7)).

With a suitable representation for the Hamiltonians $H^{(0)}$, $H^{(1)}$, this operation can be repeated, and one obtains a sequence of Hamiltonians $H^{(0)}, H^{(1)}, H^{(2)}, \ldots$. The fixed points of the transformation $H^{(n)} \rightarrow H^{(n+1)}$ are the critical points of the system if the system is infinite. The sequence of Hamiltonians can be expected to converge to one of the stable fixed points. If the Hamiltonians have a representation of the form $H^{(n)} = \sum a_i^{(n)} \psi_i$, where the ψ_i's are a suitable basis and the expansion has a finite number of terms, then the matrix A with elements $\frac{\partial a_i^{(n+1)}}{\partial a_j^{(n)}}$ is readily computed [14], the eigenvalues of this matrix for $H^{(n)}$'s near critical points determine the critical exponents of the system (see any textbook on critical phenomena, e.g., [14],[15]). The corresponding eigenvectors are the "scaling fields" at the critical points, i.e., fields that scale simply near those points.

4 An analogy between the RNG and the averaging of the KdV-Burgers equation

We draw an analogy between the conditional expectations which define the renormalized variables in the equilibrium Hamiltonian case and an averaging in space which defines "renormalized" variables for solutions of the KdVB equations that are stationary in a moving frame. Averaging over an increasing length scale corresponds either to more renormalization steps or, equivalently, to renormalization with a greater number of variables grouped together. We pick a class of equations in which to seek the "effective" equation, the one whose solutions best approximate the averages of the true solution in the mean square sense; the choice of mean-square approximation in the KdVB case corresponds to the use of L_2 norms implied by the use of conditional expectations in the Hamiltonian case, and the choice of a class of equations in which to look for the effective equation is analogous to the choice of a basis for the representation of the Hamiltonian; the calculation of the best coefficients in the chosen class of "effective" equations corresponds to the evaluation of the coefficients in the series for the renormalized Hamiltonians. Note that in the Hamiltonian case we average the right-hand-sides of the equations and in the analogous KdVB case we attempt to average the solutions; this must be so because in the KdVB case we do not have the Hald theorems which guarantee that averaging the right-hand-sides produces the correct statistics for the solutions. One expects that the effective equation would have smoother solutions than the original equation and would require fewer mesh points to be properly approximated; in this sense the number of variables is decreased, though we choose not to change the scale of the averaged solution and thus have no analogue of the relabeling of the variables in the Hamiltonian case.

We initially looked for an effective equation in the class of equations of the form

$$vv_x = \nu v_{xx} + v_{xxx} + \beta |v_x|^\alpha v_{xx}, \tag{8}$$

where $\nu \geq 0, \alpha \geq 0, \beta \geq 0$ were constants. This form was suggested by the work of Barenblatt et al. [8],[9]. The average solution is

$$\bar{u}(x) = \frac{1}{\ell} \int_{x-\ell/2}^{x+\ell/2} u(s)ds,$$

where u is a solution of equation (2) es, and ℓ is dimensionless. The problem is to find the value of the parameters in the effective equation which minimizes

$$I = \min_z \int_{-\infty}^{+\infty} |\bar{u}(x) - v(x+z)|^2 dx.$$

The shift z is needed because the problem is translation invariant, and the numerical procedures can produce a shift that has no intrinsic significance. We found that the last term in equation (8) had little effect on the minimum of I and could safely be omitted. The effective equation thus has exactly the form of the original equation but a different value of the coefficient ν (or alternately, a different Reynolds number). The dispersive term can also be omitted (as was done by Barenblatt [8]), see also below, but we do not do so mostly for esthetic reasons.

5 Dimensional analysis and similarity

We briefly remind the reader of the fundamentals of similarity theory [5],[6]. Suppose a variable a is a function of variables $a_1, a_2, \ldots, a_m$, $b_1, b_2, \ldots, b_k$, where in some system of units $a_1, \ldots, a_m$ have independent units while the units of $b_1, \ldots, b_k$, can be formed from the units of $a_1, a_2, \ldots, a_m$. Then there exist dimensionless variables $\Pi = \frac{a}{a_1^{\alpha_1} \cdots a_m^{\alpha_m}}$, $\Pi_i = \frac{b_i}{a_1^{\alpha_{i1}} \cdots a_m^{\alpha_{im}}}$,

$i = 1, \ldots, k$, where the α_i, α_{ik} are integers, such that Π is a function of the Π_i:

$$\Pi = \Phi(\Pi_1, \ldots, \Pi_k). \tag{9}$$

This is just a consequence of the requirement that a physical relationship be independent of the units of measurement. Suppose the variables Π_i are small, and suppose one assumes that the function Φ (about which we know nothing at this stage) has a non-zero finite limit as its arguments tend to zero; then $\Pi \sim$ constant, and one finds a power monomial relation between a and the a_i. The resulting relation is invariant under a group of scaling transformations generated by changes in the units. A similar argument works if the Π_i's are very large. If the function Φ does not have the assumed limit, it may happen that for Π_1 small, $\Phi(\Pi_1) = \Pi_1^q \Phi'(\Pi_1) + \ldots$, where the dots denote lower order terms, q is a constant, the other arguments of Φ have been omitted and Φ' has a finite non-zero limit. One can then obtain a power monomial expression for a in terms of the a_i and b_1, with undetermined powers which must be found by other means. The resulting power relation is an incomplete similarity relation, and the corresponding group of transformations under which the relationship is invariant is Barenblatt's renormalization group. A relation between Barenblatt's RNG and more standard definitions of the RNG is discussed in [7].

A logarithmic change of variables can transform the problem of determining exponents in incomplete similarity into a wave propagation problem where the wave velocity is unknown, see [5],[6], and applications in [2]. In our current problem the wave velocity is determined in advance.

6 Numerical results

In summary, we consider the equation

$$uu_x = \frac{1}{R} u_{xx} + u_{xxx}, \quad u_x(-\infty) = 0, \quad u(-\infty) = 1, \quad u(+\infty) = 0, \tag{10}$$

approximated numerically as described above, define $\bar{u}$, consider an effective equation of the form

$$vv_x = \nu v_{xx} + v_{xxx},$$

where $v = v(\nu, x)$ satisfies the same boundary conditions as u, and look for $\nu = \nu_{\text{eff}}$ which minimizes

$$I(\nu) = \min_z \int_{-\infty}^{+\infty} \left(v(x+z) - \bar{u}(x)\right)^2 dx.$$

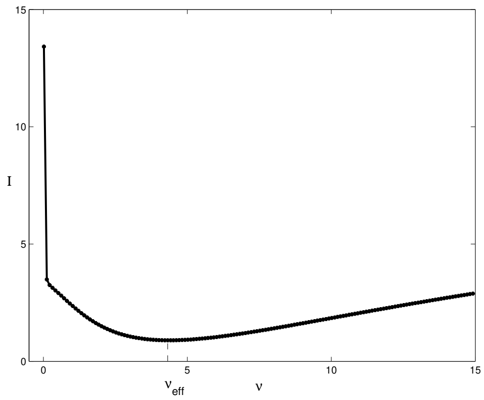

Figure 3: The mean-square difference I as a function of the viscosity ν

The general situation is shown in Figure 1, where we have plotted u, $\bar{u}$, for $R = 60$, and v for $\nu = \nu_{\text{eff}} = 4.42$; $z = -.4$.

In Figure 3 we plot $I(\nu)$ as a function of ν for $R = 60$, $\ell = 12$. $I(\nu)$ first decreases fast, then slowly, and then increases. For smaller values of R and ℓ, there is a local minimum at the point where the rapid decrease becomes a slow decrease; since this local minimum disappears for larger values of ℓ we pay no attention to it here. The value ν_{eff} is ν at the minimum of $I(\nu)$; the flatness of the curve at that point, especially for large values of R, can make the minimization of I rather painful, and rather than use sophisticated minimization routines we found the minima by tabulation. Note the resemblance of the procedure to system identification methods.

In Figures 4, 5 we plot ν_{eff} as a function of R with $\ell = 20$, in Figure 4 in regular coordinates and in Figure 5 in log-log coordinates. We see that ν_{eff}, the renormalized or eddy viscosity, increases as the "bare" viscosity $1/R$ decreases. This is plausible because as $1/R$ decreases the fluctuations in u increase and the portion of the x-axis in which there are oscillations increases. From Figure 5 we conjecture that ν_{eff} is proportional to R^{α} with $\alpha = 3/4$ (a least square fit gives $\alpha = .7506$). If indeed this is the case, one can obtain an effective equation from this scaling relation even when R is too large for an affordable solution of the original equation.

As we have shown in the preceding section, one has

$$\nu_{\text{eff}} = \Phi(R, \ell)$$

where Φ is an unknown dimensionless function, and we are interested in the case where both R and ℓ are large. The suggestion here is that $\Phi(R, \ell) =$

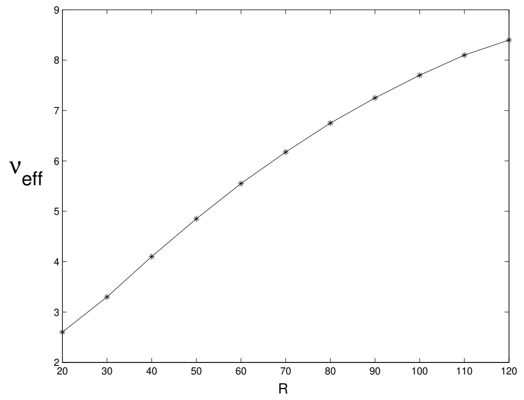

Figure 4: $\nu_{\rm eff}$ as a function of R

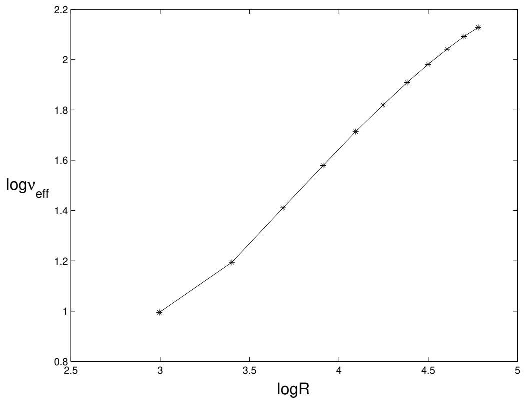

Figure 5: Log-log plot of $\nu_{\rm eff}$ as a function of R

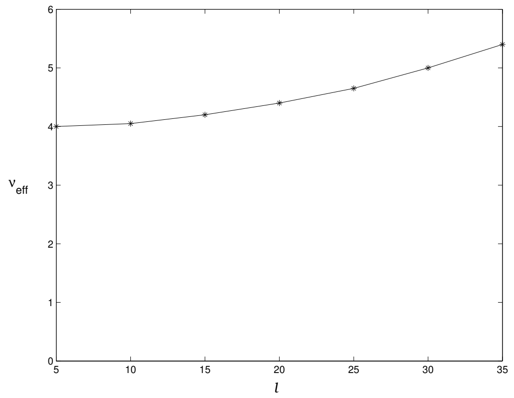

Figure 6: ν_{eff} as a function of ℓ for $R = 60$

$R^{3/4}\Phi(\ell)$, an incomplete similarity relation.

Note that in fluid mechanics one also expects ν_{eff}, the eddy viscosity, to increase with the Reynolds number R defined by the "bare" (i.e. real) viscosity. For example, one can use the scaling laws for the intermediate region of wall-bounded flow [16] to derive an eddy viscosity of the form $\nu_{\text{eff}} = C_1 - C_2\frac{1}{\ln R}$, with $C_1 \geq 0$, $C_2 \geq 0$. However, the increase is much slower. This qualitative difference is due to the geometric facts already mentioned, and also to the lack of any stochastic structure in the oscillatory wave train (for the importance of this last element, see e.g. [17]). In either case, the effect of the bare viscosity propagates from one scale of description to the next and is never forgotten. Note that in the fluid dynamics problem incomplete similarity also plays a key role, as must be the case if the effect of the bare viscosity propagates to averaged descriptions of the phenomena.

In Figure 6 we display the dependence of ν_{eff} on ℓ for $R = 60$. What is surprising at first sight is that ν_{eff} is not approximately constant as ℓ changes, but of course it cannot be because of the boundary conditions. This is the kind of phenomenon that one encounters in wave propagation analyses of time-dependent scaling, and the relationship between the present problem and those other studies, if any, remains to be elucidated. The relationship between ν_{eff} and ℓ thus displays an incomplete similarity as well.

In Figure 7 we show the mean square difference I at $\nu = \nu_{\text{eff}}$ between the mean solution $\bar{u}$ and the solution v of the effective equation, for $R = 60$ and various values of ℓ. As expected, this difference goes down monotonically. If the dispersive term in our effective solution is dropped the curve is changed very little and we do not display the result. Barenblatt's conjecture that

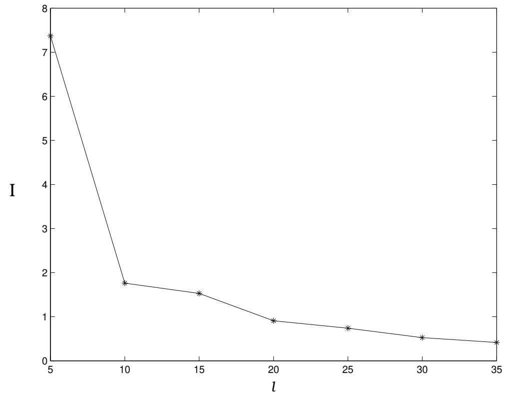

Figure 7: Mean-square difference between the mean solution and the best approximation as a function of ℓ for $R = 60$

the dispersive term is not significant after averaging is thus justified.

Note that when $I = I(\nu_{\text{eff}})$ decreases, the mean solution is increasingly well approximated by a self-similar solution (the solutions of $vv_x = \nu v_{xx}$ with our boundary conditions, for different values of ν, can be mapped on each other by a change of the length scale). A self-similar solution is approached as ℓ increases and one approaches a fixed point of the RNG — indeed, we conjecture that self-similar solutions are the scaling fields of RNG fixed points (i.e., solutions that scale simply near those fixed points), and thus Barenblatt's RNG is a restriction of more general RNG's to the eigenspace of the relevant eigenvalues in the neighborhood of the fixed points. We shall discuss this issue in detail in a separate paper.

Finally, we consider the validity of the asymptotic analysis of Barenblatt et al. (see equation (5)). Barenblatt et al. looked for an equation satisfied exactly by a mean solution, while we are content to look for an equation that produces an approximation to that mean in an L_2 sense; as we have just seen, the difference decreases when ℓ increases. Clearly, equation (5) does not hold in the major part of the wave system where $v_x \sim 0$ when v is identified with $\bar{u}$ for large ℓ, and thus ν_{hydro} is very large. However, one can check whether it holds in the transition region where $\bar{u} \sim v$ varies rapidly. The width of the transition region can be defined as $x_2 - x_1$, where x_1, x_2 satisfy $v(x_1) = 0.75$ and $v(x_2) = 0.25$; ν_{hydro} is then

$$\nu_{\text{hydro}} = (x_2 - x_1) \int_{x_1}^{x_2} (v - u)^2 dx.$$

The ratio $\nu_{\text{hydro}}/\nu_{\text{eff}}$ is plotted in Figure 8 for $\ell = 20$ and several values of

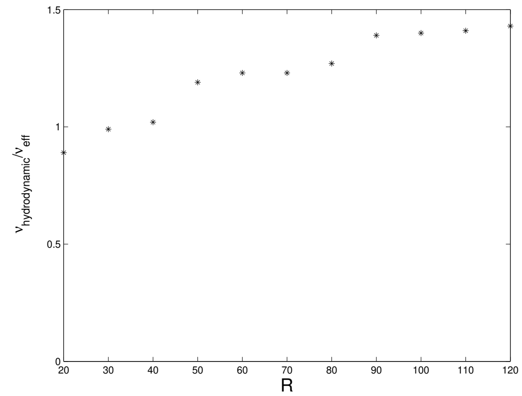

Figure 8: Validation of the Barenblatt asymptotic solution

R. The error in the ratio is substantial because $x_2 - x_1$ is not large and the number of mesh points in the transition zone is rather small (of order 10). Note that one of the assumptions in Barenblatt's theory is not satisfied — with our parameter values the width of the transition zone is comparable to the wave length of the dispersive waves. Under these conditions, the closeness of the ratio to one is a surprising confirmation of Barenblatt's theory.

These calculations were run in a computational domain of width $2X$ between 200 and 400 (depending on the length of the wave train), generally with 1024 mesh points, and with a time step k equal to the smallest of: $0.3h$ (to keep the advection term stable), $2h^2/R$, $2h^3$ (to keep overall accuracy), where h is the mesh size. The calculations were run up to a time $T = 200$.

7 Conclusions

We have set up an analogy between real-space renormalization (and thus optimal prediction algorithms) and the spatial averaging of the stationary KdV-Burgers equation. The analogy leads to incomplete similarity relations between "bare" and "renormalized" or "eddy" diffusion, and allows us to check and extend earlier averaging calculations; it also leads to interesting conjectures regarding the relation between renormalization and self-similarity. A procedurs similar in principle can be constructed for time-dependent problems, but requires substantial further technicalities, as will be explained in subsequent work. Note that the most interesting parts of our analysis result from the fact that we are dealing with partial differential equations rather than with systems of ordinary differential equations.

The specific relation between viscosity on different scales is quantitatively different here from what is seen in hydrodynamics, as it should be in this very special model, but the model, like the Navier-Stokes equations, does exhibit a persistent effect of the bare viscosity as the scale of the model increases, expressed through an incomplete similarity. One of the reasons for picking the KdV-Burgers model for study is the fact, explained in [18], that vortex-stretching in the Navier-Stokes equations induces dispersive behavior and is the main source of small-scale oscillations, like the dispersion here; we shall attempt to extend the method of the present paper to that interesting situation.

8 Acknowledgments

I would like to thank Professors P. Colella and M. Wright for helpful numerical advice, and Professor G.I. Barenblatt for suggesting the problem and for many helpful discussions. This work was supported in part by the National Science Foundation under Grant DMS 97-32710, and in part by the Office of Science, Office of Advanced Scientific Computing Research, Mathematical, Information, and Computational Sciences Division, Applied Mathematical Sciences Subprogram, of the U.S. Department of Energy, under Contract No. DE-AC03-76SF00098.

References

[1] W. E and B. Engquist (2003), Comm. Math. Sci. 1, 87.

[2] C. Rowley, I. Kevrikides, J. Marsden, and K. Lust (2002), Nonlinearity 16, 1257.

[3] A.J. Chorin, O. Hald and R. Kupferman (2002), Physica D 166, 239.

[4] A.J. Chorin (2003), Multiscale Modeling Simulation 1, 105.

[5] G.I. Barenblatt (1996), Scaling, Self-similarity and Intermediate Asymptotics. Cambridge University Press, Cambridge.

[6] G.I. Barenblatt (2003), Scaling. Cambridge University Press, Cambridge.

[7] N. Goldenfeld (1992), Lectures on Phase Transitions and the Renormalization Group. Perseus Books, Reading, Mass.

[8] G.I. Barenblatt and G.I. Shapiro (1984). Izvestya Atmospheric Oceanic Physics 20 (1984), 210.

[9] G.I. Barenblatt, M. Ivanov, and G.I. Shapiro (1985). Arch. Rat. Mech. Anal. 87, 293.

[10] J. Bona and M. Schonbek (1985), Proc. Roy. Soc. Edinburgh 101A (1985), 207.

[11] A. Gurevich and L. Pitaevskii (1974), Sov. Phys.–JETP 38, 291.

[12] R. Fornberg and G. Whitham (1978), Proc. Roy. Soc. London A 289 , 373.

[13] S. Zalesak (1984), in "Advances in Computer Methods for Partial Differential Equations V", R. Vichnevetsky and R. Stepleman (eds.), 1984, 491.

[14] T. Burkhardt and J. van Leeuwen (1982), Real-Space Renormalization. Springer, Berlin.

[15] L. Kadanoff (2000), Statistical Physics: Statics, Dynamics, and Renormalization. World Scientific, Singapore.

[16] G.I. Barenblatt, A.J. Chorin, and V.M. Prostokishin (1997), Appl. Mech. Rev. 50, 413.

[17] P. Lochak and C. Meunier (1988), Multiphase Averaging for Classical Systems, Springer, New York.

[18] A.J. Chorin (1993), J. Comput. Phys. 107 , 1.