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Numerical study of slightly viscous flow

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A numerical method for solving the time-dependent Navier–Stokes equations in two space dimensions at high Reynolds number is presented. The crux of the method lies in the numerical simulation of the process of vorticity generation and dispersal, using computer-generated pseudo-random numbers. An application to flow past a circular cylinder is presented.

1. Introduction

The Navier–Stokes equations in two space dimensions can be written in the form

$$\partial_t \xi + (\mathbf{u}\,.\,\nabla)\,\xi = R^{-1}\Delta\xi, \tag{1a}$$

$$\Delta\psi = -\xi, \tag{1b}$$

$$u = -\partial_y \psi, \quad v = \partial_x \psi, \tag{1c}$$

where $\mathbf{u} = (u, v)$ is the velocity vector, $\mathbf{r} = (x, y)$ is the position vector, t is the time, ψ is the stream function, ξ is the vorticity, $\Delta \equiv \nabla^2$ is the Laplace operator and R is the Reynolds number. R is assumed to be so large that finite-difference methods are difficult to apply. Equations (1) are to be solved in a domain D, not necessarily finite, with boundary ∂D, and their solution must satisfy the boundary conditions

$$\mathbf{u} = 0 \quad \text{on} \quad \partial D \tag{2}$$

and the initial condition

$$\mathbf{u}(x, y, t = 0) \quad \text{given in } D. \tag{3}$$

Consider in particular the problem of flow past a cylinder of finite cross-section D'. In the vicinity of its boundary ∂D a boundary layer will form, whose thickness will be proportional to $R^{-\frac{1}{2}}$ (see, for example, Schlichting 1960, p. 109). Consider furthermore a finite-difference method whose grid is characterized near the boundary layer and in the wake by a mesh width δ. Since it is presumably necessary that a few mesh points fall within the boundary layer, we find that the condition

$$\delta^2 R = O(1) \tag{4}$$

must be satisfied. Analysis (Chorin 1969*a*) suggests the more stringent condition

$$\delta R = O(1). \tag{5}$$

Conditions similar to (4) and (5) were given by Keller & Takami (1966); they indicate that at Reynolds numbers of practical significance the number of mesh points as well as the amount of computational labour required to obtain a solution would be prohibitive. In practice, insuperable difficulties are encountered at

Reynolds numbers of a few hundred. It is therefore of interest to develop a grid-free numerical method in which the values of the velocity field near a boundary are not all computed but are merely sampled, with computational effort concentrated in regions of greatest interest. We shall now present such a method, which relies on a numerical simulation of the process of vorticity generation and dispersal, using computer-generated pseudo-random numbers. A summary of this method was presented in Chorin (1972).

2. Principle of the method

Consider first the flow of an inviscid fluid (i.e. $R = \infty$). Equations (1) reduce to

$$D\xi/Dt = 0, \quad \Delta\psi = -\xi, \tag{6}$$

where D/Dt denotes a total derivative. One could think of solving equations (6) in the absence of boundaries by partitioning the vorticity ξ into a sum of blobs, i.e. writing

$$\xi = \sum_{j=1}^{N} \xi_j, \tag{7}$$

where the functions ξ_j have small support, i.e. vanish outside a small region (or blob) around a point $\mathbf{r}_j$. ψ will then have the form

$$\psi = \sum_{j=1}^{N} \psi_j, \quad \text{with} \quad \Delta\psi_j = -\xi_j. \tag{8}$$

For $|\mathbf{r}-\mathbf{r}_j|$ large, ψ_j will tend to the form

$$\psi_j \simeq \frac{\bar{\xi}_j}{2\pi}\log|\mathbf{r}-\mathbf{r}_j|, \quad \bar{\xi}_j = \int \xi_j \, dx\, dy, \tag{9}$$

where $|\mathbf{r}-\mathbf{r}_j|$ denotes the length of the vector $\mathbf{r}-\mathbf{r}_j$. The expression (9) is the stream function of a point vortex; we are thus assuming that distant blobs affect each other as if they were point vortices of appropriate strength $\bar{\xi}_j$. Neighbouring vortex blobs, however, affect each other's motion unlike neighbouring vortices, in particular, the velocity field should remain bounded, while the velocity field induced by a point vortex becomes unbounded near the vortex (Batchelor 1967, p. 95). If the blobs are small, one can assume that the velocity changes little over their area and, furthermore, that the amount of vorticity they contain, $\bar{\xi}_j$, is small, so that their effect on their immediate neighbours is small. These assumptions have now been justified by Dushane (1973). The gist of the analysis is as follows: Euler's equations are written in integral form, and it is then shown that the right-hand sides of equations (10) below are rectangle-rule approximations to the resulting integrals. From this fact, it is deduced that the error converges to zero with the area of the largest of the supports of the ξ_j. Thus we write

$$\psi = \sum_{j=1}^{N} \bar{\xi}_j \psi^0(\mathbf{r}-\mathbf{r}_j),$$

where $\psi^0(r)$ is a fixed function of r such that

$$\psi^0(r)\begin{cases} \sim (1/2\pi)\log r & \text{for} \quad r \text{ large}, \\ \to 0 & \text{as} \quad r \to 0, \end{cases}$$

and we have
$$\xi = \sum_{j=1}^{N} \bar{\xi}_j \xi_j^0, \quad \xi_j^0 = -\Delta\psi^0(\mathbf{r}-\mathbf{r}_j).$$
The motion of the vortex blobs is then described by
$$\frac{dx_i}{dt} = -\sum_{j\neq i} \bar{\xi}_j \frac{\partial\psi^0}{\partial y} \quad (\mathbf{r}-\mathbf{r}_j) \quad (i = 1, \ldots, N), \tag{10a}$$
$$\frac{dy_i}{dt} = \sum_{j\neq i} \bar{\xi}_j \frac{\partial\psi^0}{\partial x} (\mathbf{r}-\mathbf{r}_j) \quad (i = 1, \ldots, N), \tag{10b}$$
where (x_i, y_i) are the components of $\mathbf{r}_i$. This construction can be summarized as follows: if we consider a collection of vortices having a structure and density such that their density approximates the initial vorticity density, and if their motion is determined by equations (10), then their density will continue to approximate the vorticity density at later times. This statement indicates how a small viscosity can be taken into account through a judicious use of the relationship between diffusion and random walks (see, for example, Einstein 1956, p. 15; Wax 1954, p. 9). Consider the diffusion equation
$$\partial_t \xi = R^{-1}\Delta\xi, \quad \xi = \xi(x, y, t),$$
with initial data $\xi(0) = \xi(x, y, t = 0)$. A solution of this equation using random walks can be obtained as follows. Distribute over the x, y plane points of masses ξ_i and locations $\mathbf{r}_i = (x_i, y_i)$, $i = 1, \ldots, N$, N large, in such a way that the mass density approximates $\xi(0)$. Then move the points according to the laws
$$x_i^{n+1} = x_i^n + \eta_1, \quad y_i^{n+1} = y_i^n + \eta_2, \tag{11a,b}$$
where η_1 and η_2 are Gaussianly distributed random variables with zero mean and variance $2k/R$, k being the time step, and where $x_i^n \equiv x_i(nk)$ and $y_i^n \equiv y_i(nk)$. Then the mean density after n steps (11) will approximate $\xi^n \equiv \xi(nk)$. An algorithm for sampling η_1 and η_2 is readily designed (see, for example, Paley & Wiener 1934, p. 146). Boundaries on which ξ is prescribed are readily handled by maintaining a constant density across them and allowing points from both sides to cross at will. (For analyses, see Einstein (1956) and Wax (1954).)

Now approximate equations (10) by an algorithm of the form
$$x_i^{n+1} = x_i^n + ku^{n,\frac{1}{2}}, \quad y_i^{n+1} = y_i^n + kv^{n,\frac{1}{2}}, \tag{12a,b}$$
where $u^{n,\frac{1}{2}}$ and $v^{n,\frac{1}{2}}$ approximate the right-hand sides of equations (10), k is a time step and $x_i^n \equiv x(nk)$ and $y_i^n \equiv y(nk)$ are as above. Then the vorticity density generated by the motion of the vortices according to the laws
$$x_i^{n+1} = x_i^n + ku^{n,\frac{1}{2}} + \eta_1, \tag{13a}$$
$$y_i^{n+1} = y_i^n + kv^{n,\frac{1}{2}} + \eta_2 \tag{13b}$$
will approximate the solution of equations (1).

Place in the flow an obstacle with boundary ∂D. In the case of an inviscid flow only the normal component of $\mathbf{u}$ can be required to vanish on the boundary. This requirement can be satisfied by adding to the flow induced by the vortices a potential flow with velocity on the boundary so designed that it cancels the normal velocity due to the vortices. This potential flow can be found by solving an integral equation on ∂D (see Kellogg 1929, p. 311) and does not require the imposition of a grid on D. For details, see below. When R is finite, the tangential component of $\mathbf{u}$ has to vanish on ∂D as well. Suppose that at some time t the flow we have so far, which is the sum of the flow due to the vortices and of a potential flow, fails to satisfy this second boundary condition. The effect of viscosity will be to create a thin boundary layer which will ensure a smooth transition from the boundary to the flow inside D. The vorticity in that boundary is readily evaluated; it can then be partitioned among vortex blobs and the latter can be allowed to diffuse according to the laws (13). Once this has occurred, $u^{n,\frac{1}{2}}$ and $v^{n,\frac{1}{2}}$ will be small, and in the neighbourhood of a boundary the random component of equations (13) will be dominant. When a vortex, new or old, crosses ∂D it disappears. This process imitates the physical process of vorticity generation (see the discussion in Batchelor (1967, p. 277)).

It is clear that our method can be applied to flows in finite domains as well as to flows in exterior regions. The example of flow past an obstacle does, however, indicate an advantage of our method: no asymptotic expansion of the solution far from the body need be known in advance.

3. Implementation of the method

We shall now give details of the algorithm just outlined by presenting an explicit form for the blob stream function $\psi^0(r)$ and a construction of $u^{n,\frac{1}{2}}$ and $v^{n,\frac{1}{2}}$ to be used in (13). The method of calculating the potential component of the flow will be presented in the next section.

Consider blob stream functions of the form

$$\psi^0(r) = \begin{cases} (2\pi)^{-1}\log r & (r \geqslant \sigma), \\ (2\pi)^{-1} r/\sigma & (r < \sigma), \end{cases} \tag{14}$$

where $r = |\mathbf{r}|$ and σ is a cut-off length, to be determined later. The reason for considering this particular form will appear below. The total circulation around a vortex of this form is 1, and the associated velocity field is continuous and bounded. Assume that at time $t = nk$ we have a velocity field $\mathbf{u}^n$ with vorticity approximated by

$$\xi^n = -\sum_{j=1}^{N} \xi_j \Delta\psi^0(\mathbf{r}-\mathbf{r}_j) \quad (\Delta \equiv \nabla^2).$$

We now present a sequence of steps which will yield ξ^{n+1}.

Divide the boundary ∂D into M segments of equal length h, with centres Q_i, $i = 1, \ldots, M$; let the co-ordinates of Q_i be (X_i, Y_i). Let $\mathbf{u}_\xi = (u_\xi, v_\xi)$ be the velocity induced by the vortices present at time $t = nk$; we have at $\mathbf{r} = (x, y)$

$$u_\xi(\mathbf{r}) = \frac{1}{2\pi}\sum_j{}_1 \frac{y_j - y}{r_j^2}\xi_j + \frac{1}{2\pi}\sum_i{}_2 \frac{y_i - y}{\sigma r_i}\xi_i, \tag{15a}$$

$$v_\xi(\mathbf{r}) = -\frac{1}{2\pi}\sum_j{}_1 \frac{x_j - x}{r_j^2}\xi_j - \frac{1}{2\pi}\sum_i{}_2 \frac{x_i - x}{\sigma r_i}\xi_i, \tag{15b}$$

where $r_j = [(x_j - x)^2 + (y_j - y)^2]^{\frac{1}{2}}$, Σ_1 is a sum over all vortices such that $r_j > \sigma$ and Σ_2 is a sum over all vortices such that $r_i \leqslant \sigma$. Let $\mathbf{n} = (n_1, n_2)$ be the outward normal to ∂D. We find a potential flow $\mathbf{u}_p$ such that $\mathbf{u}_p.\mathbf{n} = -\mathbf{u}_\xi.\mathbf{n}$ (at Q_i, $i = 1, \ldots, M$). The details of the evaluation of $\mathbf{u}_p$ will be presented in the next section. $\mathbf{u}_p + \mathbf{u}_\xi$ satisfies the normal boundary condition on ∂D. We write

$$\mathbf{u}^{n,\frac{1}{2}} = (u^{n,\frac{1}{2}}, v^{n,\frac{1}{2}}) = \mathbf{u}_p + \mathbf{u}_\xi,$$

and use this velocity field in equations (13) to advance the position of the existing vortices; those vortices which cross ∂D are eliminated.

Let $\mathbf{s}$ be a unit vector tangent to ∂D. The total vorticity in the boundary layer which appears when the condition $\mathbf{u}.\mathbf{s} = 0$ is applied is $(\mathbf{u}_p + \mathbf{u}_\xi).\mathbf{s}$ per unit length of ∂D. We now partition the resulting vortex sheet into M blobs, centred at the Q_i. We evaluate $(\mathbf{u}_p + \mathbf{u}_\xi).\mathbf{s}$ at Q_i, and assign to the newly created vortices the vorticity $\bar{\xi} = (\mathbf{u}_p + \mathbf{u}_\xi).\mathbf{s}h$. The newly created vortices cannot be point vortices, since the flow field in the neighbourhood of a point vortex is very different from that near a vortex sheet; in particular, it is not bounded, while in the neighbourhood of a vortex sheet, the velocity does remain bounded, with its tangential components suffering a jump as the sheet is crossed. It is clear that an array of vortices with the structure (14) will approximate these features, since if one draws a line through the centre of such a vortex the velocity field where $r < \sigma$ has a constant magnitude and changes sign abruptly at the centre. Furthermore, as a vortex of this structure leaves the surface, its induced velocity field must exactly annihilate the tangential velocity at the boundary. This condition can be satisfied if

$$\sigma = h/2\pi, \tag{16}$$

and thus the cut-off σ is determined. The newly created vortices then move according to the laws (13); those which leave the fluid disappear; the evaluation of ξ^{n+1} is complete.

The use of equations (15) amounts to an apparently cumbersome method of solution of Poisson's equation. However, the method is intended for use in problems where intense vorticity is confined to small regions, which makes (15) usable, and the alternative methods of solution employ a grid, which would destroy the principle of our method.

4. The evaluation of the potential component of the flow

To complete the description of our algorithm, we now describe how the potential component $\mathbf{u}_p$ is evaluated (see Smith 1970). The equation to be solved is

$$\Delta\psi = -\xi = -\nabla \times \mathbf{u} = 0, \tag{17}$$

subject to the boundary condition

$$\mathbf{u}.\mathbf{n} = -\mathbf{u}_\xi.\mathbf{n} \quad \text{on} \quad \partial D.$$

Equation (17) can be satisfied by a flow of the form

$$\mathbf{u} = \nabla\phi, \tag{18}$$

where ϕ has the form

$$\phi(\mathbf{r}) = \frac{1}{2\pi}\int_{\partial D} \alpha(q) \log R(q)\, dq, \tag{19}$$

where q is a point on ∂D, with co-ordinates (x_q, y_q), and

$$R(q) = [(x-x_q)^2+(y-y_q)^2]^{\frac{1}{2}}.$$

$\alpha(q)$ is a single-layer source (see Kellogg 1929, p. 311), and satisfies the integral equation

$$\alpha(q) - \frac{1}{\pi}\int_{\partial D} \alpha(q')\, \partial_n(\log R(q'))\, dq' = -2\mathbf{u}_\xi . \mathbf{n}, \tag{20}$$

where ∂_n denotes a derivative in the direction of $\mathbf{n}$. We approximate (20) by a system of linear equations. A source of intensity 1 at Q_i induces at the point $Q_j, i \neq j$, a velocity field with components

$$U_1(ij) = -\frac{1}{2\pi}\frac{X_j - X_i}{R_{ij}^2}, \quad U_2(ij) = -\frac{1}{2\pi}\frac{Y_j - Y_i}{R_{ij}^2},$$

$$R_{ij}^2 = (X_j - X_i)^2 + (Y_j - Y_i)^2.$$

We approximate $\alpha(q)$ by the M component vector $\boldsymbol{\alpha} = (\alpha(Q_1), \ldots, \alpha(Q_M))$, which must thus satisfy the matrix equation

$$\mathbf{A}\boldsymbol{\alpha} = \mathbf{b},$$

where the components of $\mathbf{b}$ are the values of $-\mathbf{u}_\xi . \mathbf{n}$ evaluated at the points Q_i, and the matrix $\mathbf{A}$ has the components

$$\begin{cases} a_{ij} = U_1(ij)\, n_1 + U_2(ij)\, n_2 & (i \neq j) \\ a_{ii} = \frac{1}{2}h^{-1} & (i = 1, \ldots, N). \end{cases}$$

The discrete form of (18) and (19) is then

$$\mathbf{u}_p(\mathbf{r}) = \sum_{i=1}^{M} \mathbf{u}_p(i),$$

where

$$\mathbf{u}_p(i) = \begin{cases} \dfrac{1}{2\pi}\alpha(Q_i)\dfrac{\mathbf{r}(Q_i)}{r^2(Q_i)} & \text{if} \quad r(Q_i) \geqslant \frac{1}{2}h, \\ \frac{1}{2}h^{-1}\alpha(Q_i)\,\mathbf{n}(Q_i) & \text{if} \quad r(Q_i) < \frac{1}{2}h, \end{cases}$$

where $\mathbf{r}(Q_i)$ is the vector joining Q_i to $\mathbf{r}$, and $r(Q_i) = |\mathbf{r}(Q_i)|$.

5. Heuristic considerations

The crux of our method is the representation of the flow by a randomly placed set of vortices of similar finite structure. This representation was suggested by the author's work on turbulence theory (Chorin 1969*b*, 1970, 1973), and it may be of interest to summarize the relevant considerations.

Much of the theory of turbulence is concerned with the behaviour of the spectrum of the flow at large frequencies (see Batchelor 1960, p. 103). The reason is that this behaviour seems to be independent of the particular flow under consideration. The hope is that an understanding of this behaviour would suggest a way to incorporate these frequencies into a numerical method; finite-difference methods in particular can handle only a bounded range of frequencies.

The high-frequency range of the spectrum is associated with the less smooth part of the flow. The crucial assumption in the author's work is that the loss of smoothness in incompressible flow does not occur uniformly in each flow but is localized in certain regions, much in the same way as the high-frequency components of one-dimensional compressible flow are occasioned by the appearance of shocks. An argument was given to the effect that these rough regions consist of circular vortices; in order to match the observed spectra these circular vortices must have a core of universal structure. Their locations may be thought of as random. By constructing the flow from such elements, one ensures that the high-frequency range is taken into account. It is interesting to note that similar considerations can be applied to Glimm's (1965) solution of nonlinear hyperbolic systems.

The question now arises as to what is the order of magnitude of the errors induced by our approximation. In the case of infinite R (inviscid flow), we have a fully deterministic method of solving Euler's equations, and as long as k is of order h, we expect the overall error to be of $O(k)$. In the case of finite R, the velocity at any one point or at any one instant is a random variable, and convergence can be expected only in the mean, i.e. as one averages over large regions, or over long times or over ensembles. In the mean, diffusion is represented without error; the crucial problem is to assess the effect of the interaction between the random and deterministic parts of equations (13). As far as the determination of inertial effects is concerned, the random variables η_1 and η_2 can be viewed as harmful perturbations. The standard deviation of η_1 and η_2 is $[2k/R]^{\frac{1}{2}}$. After n steps, the total effect of the random perturbations will be to induce a displacement of order

$$[n.2k/R]^{\frac{1}{2}} = O(R^{-\frac{1}{2}})$$

in the location of the vortices. If this can be identified with an error in the evaluation of the nonlinear terms, its magnitude will be of $O(R^{-\frac{1}{2}})$. We thus conjecture that the mean error in our calculation is of $O(k)+O(R^{-\frac{1}{2}})$. The second term in this estimate may appear shocking, since it does not depend on k. However, the relations (4) and (5) indicate that, if $R^{-\frac{1}{2}}$ is not small, a difference method may be used and our algorithm becomes unnecessary. We therefore do not expect valid solutions at low R.

At the other extreme, some difficulty may be expected at very high Reynolds numbers. This is so because the boundary layers formed by the algorithm are made up of a few bouncing vortices and are thus noisy; turbulence effects should therefore appear at too small a value of R, as they do, for example, in noisy wind tunnels or around rough bodies.

6. Application: flow past a circular cylinder

Consider a circular cylinder of radius 1, immersed in a fluid of density 1, to which is imparted at $t = 0$ a constant velocity of magnitude 1. The Reynolds number based on cylinder radius is $R = \nu^{-1}$, where ν is the viscosity. (In the literature one encounters a Reynolds number $R' = 2R$ based on cylinder diameter.) Let the origin O be fixed in the centre of the cylinder's base, with the negative-x axis pointing in the direction of the motion. In the resulting frame of reference the velocity at infinity is $(1, 0)$, and the cylinder is at rest. ∂D is the circumference of the base.

Divide ∂D into M pieces of length $h = 2\pi/M$. The cut-off length is $\sigma = h/2\pi = 1/M$. One of the important functionals of the flow is the drag coefficient C_D, which in our units is simply the force per unit length of the cylinder. We have

$$C_D = C_\nu + C_p,$$

where C_ν is the skin drag, given by

$$C_\nu = -\frac{1}{R}\int_{\partial D} \xi_\partial \sin\theta \, d\theta, \tag{21}$$

and C_p is the form drag, given by

$$C_p = \int_{\partial D} p_\partial \cos\theta \, d\theta, \tag{22}$$

where $r\cos\theta = x$, $r\sin\theta = y$, ξ_∂ is the vorticity on ∂D and p_∂ is the pressure on ∂D. p_∂ can be evaluated using the formula

$$p_\partial(\theta) = -\frac{1}{R}\int_{\theta_0}^{\theta} \partial_n \xi \, ds + \text{constant}, \tag{23}$$

where $\partial_n \xi$ is the normal derivative of ξ and the integration is carried out along ∂D. The problem at hand is to evaluate ξ and $\partial_n \xi$ given our random array of vortices. Introduce the regions A_j and A_j^+ defined by

$$A_j = \{x, y \,|\, 1 \leqslant r < 1+\mu, (j-\tfrac{1}{2})\,2\pi/M \leqslant \theta < (j+\tfrac{1}{2})\,2\pi/M\},$$
$$A_j^+ = \{x, y \,|\, 1+\mu \leqslant r < 1+2\mu, (j-\tfrac{1}{2})\,2\pi/M \leqslant \theta < (j+\tfrac{1}{2})\,2\pi/M\},$$

where $\mu = (2k/R)^{\frac{1}{2}}$ is the standard deviation of η_1 and η_2. $\xi(A_j)$ and $\xi(A_j^+)$ are defined as the sums of the vorticities ξ_i associated with vortices whose centres fall within A_j and A_j^+ divided by the areas of A_j and A_j^+ respectively. We now identify $\xi(A_j)$ with $\xi(Q_j)$, and $(\xi(A_j^+) - \xi(A_j))/\mu$ with $\partial_n \xi(Q_j)$. It is worth emphasizing that the grid just introduced is used not to advance the calculation, but only to diagnose its outcome. $\xi(A_j)$ and $\xi(A_j^+)$ are random variables, and can be expected to have substantial variance; we therefore introduce the averaged drag $C_D(t, T)$, defined by

$$C_D(t, T) = \int_{t-T}^{t} C_D(t)\, dt, \tag{24}$$

where $C_D(t)$ is the drag C_D at time t. The integrals (21)–(24), can be evaluated through the use of the trapezoidal rule.

The time step k and the number M of vortices created per time step remain to be chosen. As k decreases M must increase; this is so because the deterministic component of the right-hand sides in equations (13) is proportional to k, while the random component has standard deviation proportional to $\sqrt{k}$. Thus as k is decreased, each vortex has an increasing number of opportunities to cross ∂D and disappear; since a minimum number of vortices must be maintained in the fluid, more and more must be created. For a given k, M is chosen so large that a further increase does not affect the solution. k must be chosen so that a decrease in k will not affect the flow; the solution is rather insensitive to k. After some experimentation the value $k = 0{\cdot}2$ was picked. The required M is 20. All the calculations below were made with these parameters.

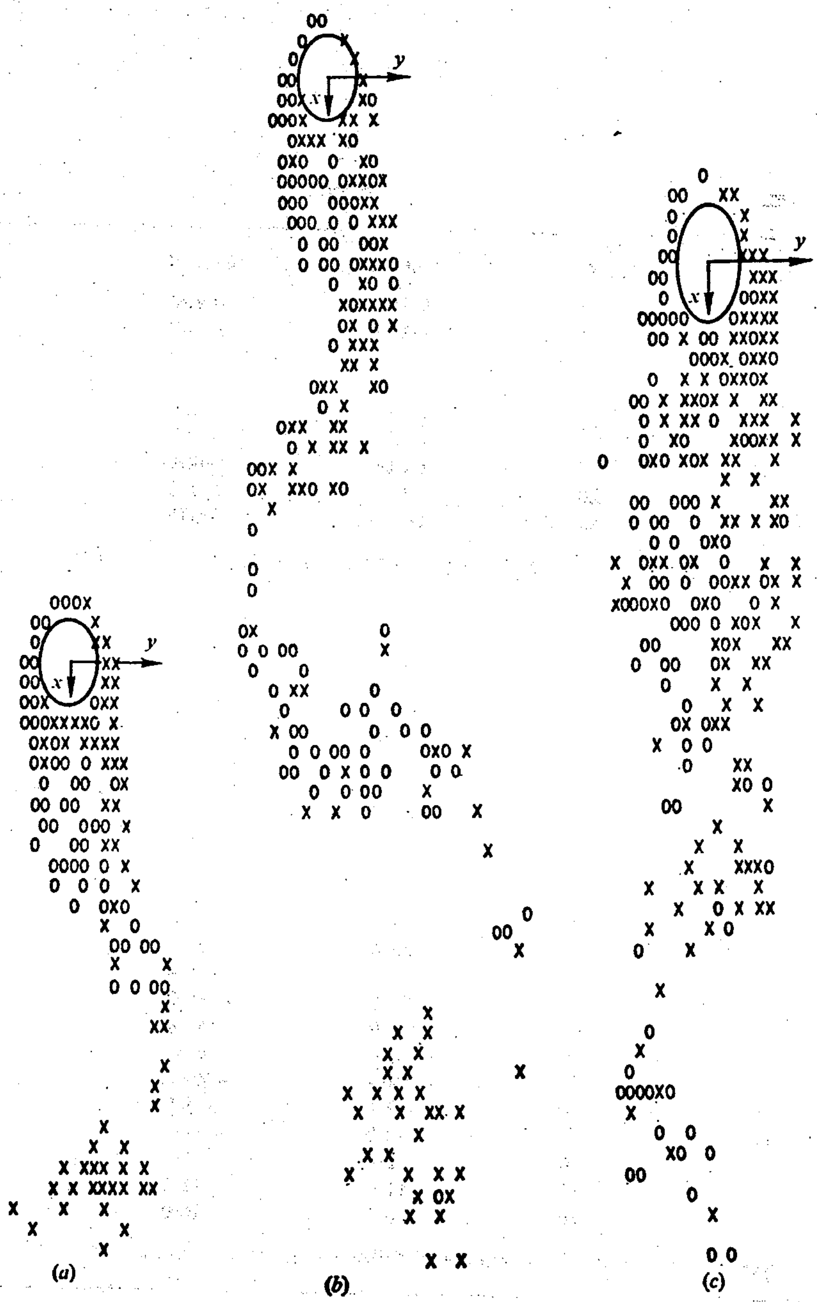

FIGURE 1. Flow at (*a*) $R = 1000$, $t = 12$; (*b*) $R = 1000$, $t = 24$; (*c*) $R = 100$, $t = 16$.

For figures 1(*a*) and (*b*) the flow at $R = 1000$ at two times is visualized. The domain is divided into squares of side $\epsilon = 0{\cdot}3$; if a square contains no vortices nothing is printed; if the sum of the vortices in the square is positive a cross is printed; if the sum is negative a circle is printed. Note the deformation of the circle by the computer printer. This visualization may be crude, but it is in

t	$C_D(t, t)$	$C_D(t, 2)$
2	0·993	0·993
4	1·118	1·232
6	1·020	0·833
8	1·056	1·158
10	1·068	1·118
12	1·034	0·863
16	1·014	0·894
20	1·049	1·216
24	1·060	1·073

TABLE 1. Average drag as a function of time at $R = 1000$

j	$\xi(A_j)$	$\xi(A_j^+)$
1	10·8	14·4
2	6·0	5·9
3	−21·3	−20·9
4	8·1	7·9
5	−25·4	−24·9
6	7·2	7·0
7	52·1	51·1
8	22·7	49·3
9	35·7	35·0
10	20·6	20·2
11	−9·5	−6·4
12	−30·9	−30·3
13	−52·6	−51·6
14	−24·8	−41·9
15	−19·4	−49·3
16	−9·1	−8·9
17	−8·3	−11·4
18	2·3	2·2
19	11·6	11·4
20	5·1	10·0

TABLE 2. Vorticity distribution at $R = 1000$, $t = 10$

keeping with the spirit of our method, in which no location is certain; the visualization is of course most inadequate at boundaries. In table 1 the values of $\xi(A_j)$ and $\xi(A_j^+)$ for $t = 10$ and $R = 1000$ are tabulated. Separation can be detected when $\xi(A_j)$ and $\xi(A_j^+)$ differ appreciably. It can be seen that the separation of the boundary layers occurs (asymmetrically) around $\theta = 126°$ and $\theta = 288°$. The values of $C_D(t, t)$ and $C_D(t, 2)$ at $R = 1000$ are tabulated in table 2. The numbers printed yield a mean drag of 1·04, in excellent agreement with experiment (Schlichting 1960, p. 16).

We now decrease R. At $R = 500$ we obtain a mean drag $C_D = 1{\cdot}15$. In figure 1 (c) we visualize the flow at $R = 100$, which is at the lower limit of applicability of the method; $\epsilon = 0{\cdot}25$. $C_D(t, 2)$ climbs to a maximum of 2·80, and then oscillates between 1·30 and 1·18. (The experimental values lie between 1·20 and 1·25.) The skin drag is $0{\cdot}26 \pm 0{\cdot}02$; the experimental value is 0·28.

As we increase R we find that $C_D = 1{\cdot}09$ at $R = 5000$; this rise is of course experimentally observed. At $R = 10\,000$, C_D is approximately $0{\cdot}87$, about $\frac{2}{3}$ of the experimental value. We can conjecture that the rough representation of the boundary layer triggers a premature onset of the drag crisis, analogous to the effect of a rough boundary or a noisy flow. This conjecture is apparently confirmed by the fact that, at $R = 100\,000$, $C_D = 0{\cdot}29$, in good agreement with the experimental value beyond the drag crisis. However, more thought is required before we are ready to claim that the method is able to follow a transition to turbulence. Beyond $R = 10\,000$ the vortex street behind the cylinder becomes disorderly at about 10 units of length behind the cylinder. In all our calculations, the number of vortices in the fluid at $t = 30$ is approximately 300, and it takes about 12 minutes of CDC 6400 time to follow the evolution from $t = 0$ to $t = 30$.†

7. Conclusion and further work

We have presented a numerical method containing a random element which makes possible the analysis of flow at high Reynolds number with comparatively little effort. The price paid for this achievement is the loss of pointwise convergence in either space or time. The method will be applied to other problems besides the one presented here, but the most fascinating subject for further research, both theoretical and numerical, is the possibility that this method is able to simulate the transition to turbulence.

Another problem under investigation is the development of a similar method for three-dimensional flow problems, in which vortices will be replaced by vortex lines.

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† The computer program used to obtain the results above is available from the author.

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