do Carmo 'Riemannian Geometry'

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Ch.0

5. a) On $\{z \neq 0\}$, $(x,y) \in B_1(0) \mapsto [x : y : \sqrt{1-x^2-y^2}]$ are local coordinates

on which $\tilde{\varphi}(x,y) = (x^2-y^2,\ xy,\ x\sqrt{1-x^2-y^2},\ y\sqrt{1-x^2-y^2})$

$$D\tilde{\varphi}(x,y) = \begin{bmatrix} 2x & -2y \\ y & x \\ \frac{1-2x^2-y^2}{\sqrt{1-x^2-y^2}} & \frac{-xy}{\sqrt{1-x^2-y^2}} \\ \frac{-xy}{\sqrt{1-x^2-y^2}} & \frac{1-x^2-2y^2}{\sqrt{1-x^2-y^2}} \end{bmatrix} \quad \text{is injective}$$

On $\{y \neq 0\}$, $(x,z) \in B_1(0) \mapsto [x : \sqrt{1-x^2-z^2} : z]$ are local coordinates

on which $\tilde{\varphi}(x,z) = (2x^2+z^2-1,\ x\sqrt{1-x^2-z^2},\ xz,\ z\sqrt{1-x^2-z^2})$

$$D\tilde{\varphi}(x,z) = \begin{bmatrix} 4x & 2z \\ \frac{1-2x^2-z^2}{\sqrt{1-x^2-z^2}} & \frac{-xz}{\sqrt{1-x^2-z^2}} \\ z & x \\ \frac{-xz}{\sqrt{1-x^2-z^2}} & \frac{1-x^2-2z^2}{\sqrt{1-x^2-z^2}} \end{bmatrix} \quad \text{is injective}$$

On $\{x \neq 0\}$, $(y,z) \in B_1(0) \mapsto [\sqrt{1-y^2-z^2} : y : z]$ are local coordinates

on which $\tilde{\varphi}(y,z) = (1-2y^2-z^2,\ y\sqrt{1-y^2-z^2},\ z\sqrt{1-y^2-z^2},\ yz)$

$$D\tilde{\varphi}(y,z) = \begin{bmatrix} -4y & -2z \\ \frac{1-2y^2-z^2}{\sqrt{1-y^2-z^2}} & \frac{-yz}{\sqrt{1-y^2-z^2}} \\ \frac{-yz}{\sqrt{1-y^2-z^2}} & \frac{1-y^2-2z^2}{\sqrt{1-y^2-z^2}} \\ z & y \end{bmatrix} \quad \text{is injective}$$

b) Suppose $\varphi(x,y,z) = \varphi(x',y',z')$

If $x = 0$, $x'y' = x'z' = 0$, so $x' = 0$ or $y' = z' = 0$

$$\begin{cases} -y^2 = -y'^2 \\ yz = y'z' \end{cases} \quad \text{or} \quad -y^2 = x'^2$$

$y = y',\ z = z'$ or $x' = y = 0$ which is the last case

or $y = -y',\ z = -z'$

$\therefore (x,y,z) = \pm(x',y',z')$

Similarly, if $y = 0$, $(x,y,z) = \pm(x',y',z')$

If $x \neq 0,\ y \neq 0$, $\frac{z}{x} = \frac{yz}{xy} = \frac{y'z'}{x'y'} = \frac{z'}{x'}$, $\frac{z}{y} = \frac{z'}{y'}$

If $z = 0$, $z' = 0$, $2x^2 = x^2 - y^2 + 1 = x'^2 - y'^2 + 1 = 2x'^2 \Rightarrow x = \pm x'$

Similarly $y = \pm y'$

$xy = x'y' \neq 0 \Rightarrow x = x',\ y = y'$ or $x = -x',\ y = -y'$

If $z \neq 0$, $z' \neq 0$, $\frac{x}{y} = \frac{x}{z}\frac{z}{y} = \frac{x'}{z'}\frac{z'}{y'} = \frac{x'}{y'}$

$\frac{1}{y^2} = 1 + \left(\frac{x}{y}\right)^2 + \left(\frac{z}{y}\right)^2 = 1 + \left(\frac{x'}{y'}\right)^2 + \left(\frac{z'}{y'}\right)^2 = \frac{1}{y'^2} \Rightarrow y = \pm y'$

$\therefore (x,y,z) = \pm(x',y',z')$

$\therefore \tilde{\varphi}$ injective $\Rightarrow \tilde{\varphi}$ embedding

6. $$DG(x,y)=\begin{bmatrix} -(r\cos y+a)\sin x & -r\sin y\cos x \\ (r\cos y+a)\cos x & -r\sin y\sin x \\ -\frac{1}{2}r\sin y\sin\frac{x}{2} & r\cos y\cos\frac{x}{2} \\ \frac{1}{2}r\sin y\cos\frac{x}{2} & r\cos y\sin\frac{x}{2}\end{bmatrix}$$ is injective if $r\neq 0$ and $a\neq\pm r$

Let $\alpha,\beta: \mathbb{R}^2\to\mathbb{R}^2$ be defined by $\alpha(x,y)=(x,y+2\pi)$, $\beta(x,y)=(x+2\pi,-y)$

$\pi=\langle\alpha,\beta\rangle$ acts properly discontinuously on $\mathbb{R}^2$ with $\mathbb{R}^2/\pi\cong K$

$G\alpha=G$, $G\beta=G$

$\therefore G$ induces $\bar{G}: K\to\mathbb{R}^4$, G immersion $\Rightarrow\bar{G}$ immersion

Suppose $G(x,y)=G(x',y')$

$$\begin{cases} r\sin y\cos\frac{x}{2}=r\sin y'\cos\frac{x'}{2} \\ r\sin y\sin\frac{x}{2}=r\sin y'\sin\frac{x'}{2}\end{cases}$$

$$\Rightarrow\begin{cases}\sin y=\sin y' \\ x=x'+4k\pi\end{cases}\text{ or }\begin{cases}\sin y=-\sin y' \\ x=x'+(4k+2)\pi\end{cases}\text{ or }\sin y=\sin y'=0,\text{ if } r\neq 0$$

$$\Rightarrow\begin{cases}y=y'+2m\pi \\ x=x'+4k\pi\end{cases}\text{ or }\begin{cases}y=-y'+(2m+1)\pi \\ x=x'+4k\pi\end{cases}\text{ or }\begin{cases}y=y'+(2m+1)\pi \\ x=x'+(4k+2)\pi\end{cases}\text{ or }\begin{cases}y=-y'+2m\pi \\ x=x'+(4k+2)\pi\end{cases}$$

$$\text{or }\begin{cases}y=m\pi \\ y'=n\pi\end{cases}$$

Also $$\begin{cases}(r\cos y+a)\cos x=(r\cos y'+a)\cos x' \\ (r\cos y+a)\sin x=(r\cos y'+a)\sin x'\end{cases}$$

$$\therefore\begin{cases}y=-y'+(2m+1)\pi \\ x=x'+4k\pi\end{cases}\Rightarrow r\cos y+a=-r\cos y+a\Leftrightarrow\cos y=0\Leftrightarrow y=(n+\tfrac{1}{2})\pi$$

$$\Rightarrow y=y'+2(n-m)\pi$$

$$\begin{cases}y=y'+(2m+1)\pi \\ x=x'+(4k+2)\pi\end{cases}\Rightarrow r\cos y+a=-r\cos y+a\Leftrightarrow\cos y=0\Leftrightarrow y=(n+\tfrac{1}{2})\pi$$

$$\Rightarrow y=-y'+2(n-m)\pi$$

$$\begin{cases}y=m\pi \\ y'=n\pi\end{cases}\Rightarrow\begin{cases}y=y'+2k\pi \\ \cos x=\cos x' \\ \sin x=\sin x'\end{cases},\text{ if } a, r, a\pm r\neq 0$$

$$\Rightarrow\begin{cases}y=y'+2k\pi \\ x=x'+4l\pi\end{cases}\text{ or }\begin{cases}y=-y'+2(n+k)\pi \\ x=x'+(4l+2)\pi\end{cases}$$

$\therefore$ There are only 2 cases: $\begin{cases}y=y'+2k\pi \\ x=x'+4l\pi\end{cases}$ or $\begin{cases}y=-y'+2k\pi \\ x=x'+(4l+2)\pi\end{cases}$

$$\Leftrightarrow(x,y)=\alpha^k\beta^{2l}(x',y')\text{ or }(x,y)=\alpha^k\beta^{2l+1}(x',y')$$

$\therefore\bar{G}$ is injective $\Rightarrow\bar{G}$ is embedding

Ch 1

4. a) $L_{(x,y)}(u,v) = (yu+x, yv)$

$\therefore DL_{(x,y)} = \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}$

$$g_{(x,y)} = L_{(x,y)}^{*}\, g_{(0,1)} = L_{(x,y)}^{*}(dx^2+dy^2) = \frac{1}{y^2}(dx^2+dy^2)$$

b) The metric can be rewritten as $-\frac{4}{(z-\bar{z})^2}\, dz\, d\bar{z}$

Pullback of this by $\varphi(z) = \frac{az+b}{cz+d}$ is

$$-\frac{4}{\left(\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}\right)^2} \frac{a(cz+d)-c(az+b)}{(cz+d)^2} dz \; \frac{a(c\bar{z}+d)-c(a\bar{z}+b)}{(c\bar{z}+d)^2} d\bar{z}$$

$$= -\frac{4(ad-bc)^2}{[(az+b)(c\bar{z}+d)-(a\bar{z}+b)(cz+d)]^2}\, dz\, d\bar{z}$$

$$= -\frac{4}{(z-\bar{z})^2}\, dz\, d\bar{z}$$

Meanwhile $\varphi^{-1}(z) = \frac{dz-b}{-cz+a}$, so φ is an isometry

5. We do this by induction

For $n=0$, this is clear

Suppose $\varphi: S^n \to S^n$ is an isometry

φ must send a pair of points realizing the diameter to another such pair

In particular $\varphi(0,\cdots,0,1) = -\varphi(0,\cdots,0,-1)$

$O(n+1)$ acts transitively on $S^n \Rightarrow$ W.L.O.G. assume $\varphi(0,\cdots,0,\pm 1) = (0,\cdots,0,\pm 1)$

Also φ sends a path realizing the distance between $(0,\cdots,0,\pm 1)$ to another such path, in a length-preserving way, and such paths are great circle arcs

Meanwhile φ preserves $E := S^n \cap \{x_{n+1}=0\} = \{x \in S^n : d(x,(0,\cdots,0,1)) = \frac{\pi}{2}\}$

$\therefore \varphi$ restricts to an isometry on E

But $E \cong S^{n-1}$, so $\varphi|_E \in O(n-1)$ by induction

By reasoning above, $\varphi = \varphi|_E \times id_{\mathbb{R}} \in O(n)$