DILATATIONS OF PSEUDO-ANOSOV MAPS AND STANDARDLY EMBEDDED TRAIN TRACKS

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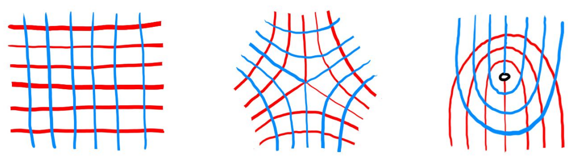
Let S be an orientable finite type surface. An orientation preserving homeomorphism $f : S \to S$ is *pseudo-Anosov* if there exists a transverse pair of measured, singular *stable* and *unstable* foliations ℓ^s, ℓ^u such that f stretches the measure of ℓ^u by $\lambda(f)$ and contracts the measure of ℓ^s by $\frac{1}{\lambda(f)}$. The number $\lambda(f) > 1$ is known as the *dilatation* of f.

Here a *foliation* is a partition of S into 1-manifolds. A *singular foliation* is a foliation except we allow finitely many *singularities*, where the foliation is locally conjugate to either

- the pull back of the foliation of $\mathbb{R}^2$ by vertical lines by the map $z \mapsto z^{\frac{n}{2}}$, for some $n \geq 3$, or
- the pull back of the foliations of $\mathbb{R}^2 \backslash \{(0,0)\}$ by vertical lines by the map $z \mapsto z^{\frac{n}{2}}$, for some $n \geq 1$.



A (singular) foliation is *measured* if it has a transverse measure, i.e. there is a way to measure the width of a band of leaves.

A pseudo-Anosov map is *fully-punctured* if all the singularities occur at the punctures of S, i.e. only the latter type of singularity occurs. Every pseudo-Anosov map can be made fully-punctured by puncturing at its singularities.

Intuition: The dilatation of a pseudo-Anosov map measures the complexity of its dynamics.

More precisely: Let $\mathcal{T}(S)$ be the *Teichmüller space* of S, defined as the space of conformal structures on S modulo diffeomorphisms isotopic to identity. The measured foliations ℓ^s and ℓ^u determine a conformal structure on S. Contracting and expanding the leaves of the two foliations deforms the conformal structure and determines a bi-infinite geodesic path α on $\mathcal{T}(S)$ (under the *Teichmüller metric*).

$\mathcal{T}(S)$ can be compactified with the space of projective measured (singular) foliations $\mathbb{P}\mathcal{MF}(S)$. α limits onto $[\ell^s]$ in the negative direction and onto $[\ell^u]$ in the positive direction. f acts as a translation of length $\log \lambda(f)$ along α.

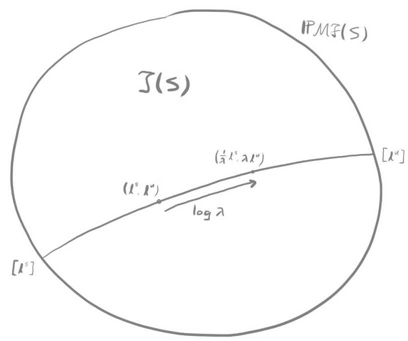
Now let $\mathcal{M}(S)$ be the *moduli space* of S, defined as the space of conformal structures on S modulo diffeomorphisms. α quotients down to a closed geodesic of length $\log \lambda(f)$ on $\mathcal{M}(S)$. Conversely, every closed geodesic on $\mathcal{M}(S)$ comes from a pseudo-Anosov map.

$\Rightarrow$ There is a minimum dilatation among all pseudo-Anosov maps on fixed S.

Minimum dilatation problem: What is this minimum dilatation?

Previous progress on the problem:

- The problem is solved for 'small' surfaces such as $S_{0,4}, ..., S_{0,9}$, and $S_{2,0}$ with computational aid.



- Penner showed that $\lambda(f)^{-\chi(S)} \geq 2^{\frac{1}{6}}$ but this is not sharp.
- The known and conjectured minimal dilatations have erratic patterns as the genus g and number of punctures s of S grows, but have nice asymptotic behavior at least when restricted to certain lines in the (g, s)-plane.

Theorem 1 (Hironaka-T.)**.** *Let $f : S \to S$ be a fully-punctured pseudo-Anosov map. Suppose f has at least two puncture orbits, then $\lambda(f)^{-\chi(S)} \geq \mu^4$, where $\mu = \frac{1+\sqrt{5}}{2}$ is the golden ratio.*

Moreover, this lower bound is asymptotically sharp, i.e. there exists a sequence of fully-punctured pseudo-Anosov maps $f_i : S_i \to S_i$ such that $\lambda(f_i)^{-\chi(S_i)} \searrow \mu^4$.

Remark.

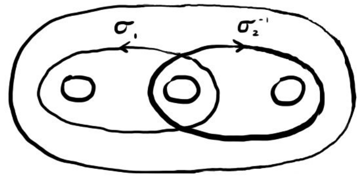
- f has at least two puncture orbits $\Leftrightarrow$ the mapping torus of f has at least two boundary components.
- The assumption that f has at least two puncture orbits is necessary; there are examples of f with only one puncture orbit for which the inequality fails.
- The quantity $\lambda(f)^{-\chi(S)}$ is known as the *normalized dilatation* of f and comes up naturally in Thurston-Fried fibered face theory.

One sentence summary of proof: Use standardly embedded train tracks to encapsulate the dynamics of f by a reciprocal Perron-Frobenius matrix, then apply a theorem of McMullen.

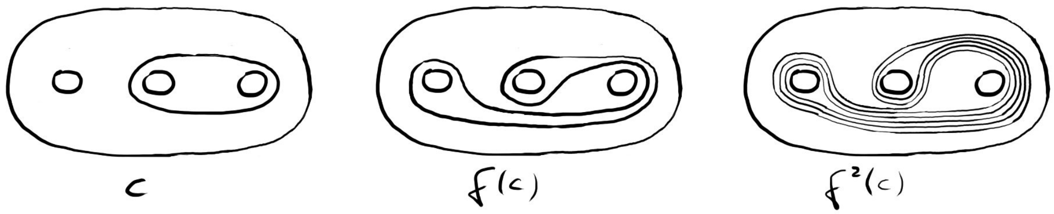
We expand on some of these ideas for the rest of the talk.

Let c be a simple closed curve on S. $f^n(c) \to [\ell^u]$ as $n \to \infty$ due to the dynamics on $\mathbb{P}\mathcal{MF}(S)$.

Example. Consider the map on the 4-punctured sphere defined as the composition of two half-twists, $f = \sigma_2^{-1}\sigma_1$.



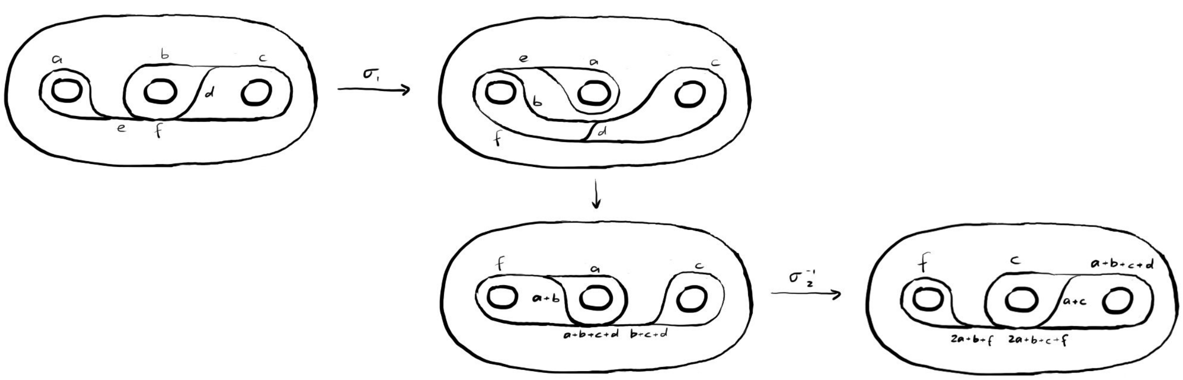
We draw out the iterates of a simple closed curve c under f.



Thurston observed that $f^n(c)$ can be approximated by a train track τ for n large, which in turn approximates the underlying foliation of ℓ^u. Note that f folds up the edges of τ. Consider weights on the edges of τ, thought of as widths of the edges. Then the folding moves add up the weights and determines a transition matrix $A : \mathbb{R}^{\mathcal{E}(\tau)} \to \mathbb{R}^{\mathcal{E}(\tau)}$.

One can use the Perron-Frobenius theorem to show that A has a unique (up to scaling) eigenvector w with positive eigenvalue λ. Hence w must determine the measure on ℓ^u and λ must be equal to the dilatation $\lambda(f)$. In other words, the spectral radius of A equals $\lambda(f)$.

Example. For the above example, $f^n(c)$ can be approximated by the following train track τ. We demonstrate foldings from τ to $f(\tau)$.



By following through the weights, we compute the transition matrix A to be $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$, which has spectral radius μ^2.

Theorem 2 (McMullen)**.** *Let A be a n-by-n reciprocal Perron-Frobenius matrix with spectral radius $\rho(A)$, where $n \geq 2$. Then $\rho(A)^n \geq \mu^4$.*

Here *reciprocal* means that the characteristic polynomial of A is palindromic up to sign. We will explain later why the transition matrix is always reciprocal.

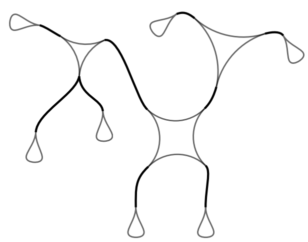
However Theorem 1 does not follow from Theorem 2 just yet. A may not be Perron-Frobenius and the dimension of A is usually way bigger than $-\chi(S)$!

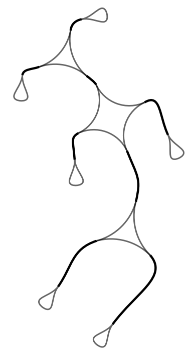
The new idea is to use standardly embedded train tracks.

Definition. A train track τ is said to be *standardly embedded* if its set of edges $\mathcal{E}$ can be partitioned into a set of *infinitesimal edges* $\mathcal{E}_{\text{inf}}$ and a set of *real edges* $\mathcal{E}_{\text{real}}$, such that:

- The union of infinitesimal edges is a disjoint union of cycles, which we call the *infinitesimal polygons.*
- The real edges connect up the vertices of the infinitesimal polygons.
- The smoothing at each vertex v is defined by separating the infinitesimal edges and the real edges.

Here are two examples.





Theorem 3 (Hironaka-T.). *If f has at least two puncture orbits, then the approximating train track τ can be chosen to be standardly embedded.*

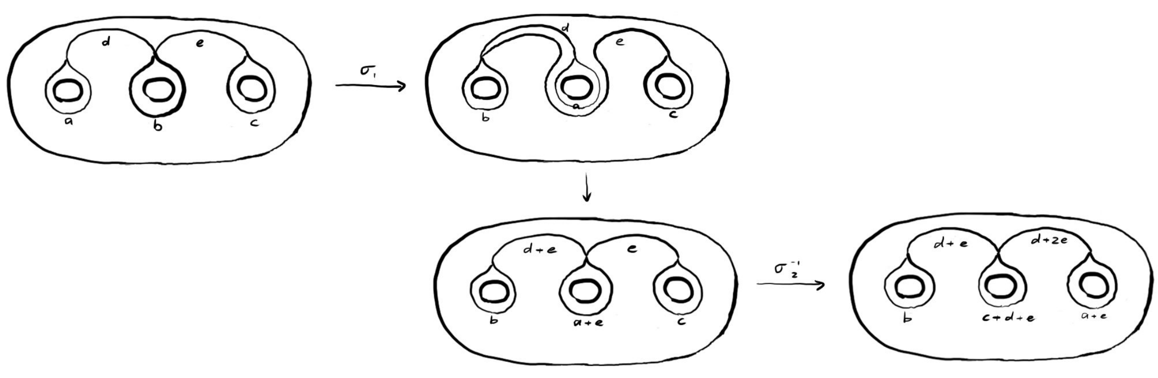
Notice that infinitesimal edges can never get folded onto other edges. So by separating the infinitesimal and real edges, the transition matrix can be split into

$$A = \begin{bmatrix} P & * \\ 0 & A^{\text{real}} \end{bmatrix}$$

where P is a permutation matrix and $A^{\text{real}} : \mathbb{R}^{\mathcal{E}_{\text{real}}(\tau)} \to \mathbb{R}^{\mathcal{E}_{\text{real}}(\tau)}$ records how weights on the real edges get added up under the folding moves. We have $\rho(A^{\text{real}}) = \rho(A) = \lambda(f)$.

Another crucial property of standardly embedded train tracks is that the number of real edges $|\mathcal{E}_{\text{real}}|$ is given by $-\chi(S)$.

Example. For the example above, we can choose the standardly embedded train track



By following through the weights, we compute the transition matrix A to be $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$. So the real transition matrix A^{real} is $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ which again has spectral radius μ^2.

The remaining task is to show that A^{real} is reciprocal, for then Theorem 1 would follow from Theorem 3 and Theorem 2.

We first note the following properties of reciprocal matrices.

Proposition 4.

(1) A symplectic matrix is reciprocal.

(2) A permutation matrix is reciprocal.

(3) Suppose matrix A admits a block decomposition $\begin{bmatrix} B & * \\ 0 & C \end{bmatrix}$. Then two of A, B, or C being reciprocal implies that the remaining one is reciprocal as well.

In particular, showing that A^{real} is reciprocal is equivalent to showing that A is reciprocal.

Definition. The Thurston symplectic form is an antisymmetric bilinear form ω defined on $\mathbb{R}^{\mathcal{E}(\tau)}$ by

$$\omega(w_1, w_2) = \sum_{v \in \mathcal{V}(\tau)} \sum_{e_1 \text{ left of } e_2} (w_1(e_1)w_2(e_2) - w_1(e_2)w_2(e_1))$$

where the second summation is taken over all pairs of half-edges e_1, e_2 for which e_1 is on the left of e_2.

The Thurston symplectic form generalizes the algebraic intersection number: If w_1 and w_2 determine simple closed curves c_1 and c_2 carried by τ, then $\omega(w_1, w_2) = \langle c_1, c_2 \rangle$.

It can be checked that each folding move preserves ω, hence the transition matrix A preserves ω.

But ω is not a symplectic form in general! So we cannot just apply Proposition 4(1). Instead, we analyze the radical $\text{rad}(\omega)$ of ω and show that f acts as a reciprocal matrix on $\text{rad}(\omega)$. Then combining reciprocity on $\text{rad}(\omega)$ and on $\mathbb{R}^{\mathcal{E}(\tau)}/\text{rad}(\omega)$ (by Proposition 4(1)), we get reciprocity on $\mathbb{R}^{\mathcal{E}(\tau)}$ (by Proposition 4(3)).

Future directions:

- Can the lower bound be improved if 'at least 2 puncture orbits' is replaced with 'at least q puncture orbits' for $q \geq 3$?
- Can the lower bound be improved if it is known that S has many punctures (that belong to the same orbit)?

The idea is that topological information should translate to dynamic information about the transition matrix. By further developing McMullen's theory of Perron-Frobenius directed graphs, one might hope to obtain better bounds in the presence of these dynamic conditions.

Remark. Upcoming work will show that if we replace the hypothesis in Theorem 1 by 'at least 1 puncture orbit', then the sharpest lower bound is μ^2.

A natural guess for the sharpest lower bound if we replace the hypothesis in Theorem 1 by 'at least 3 puncture orbit' is $(2+\sqrt{3})^2$.

It is interesting to note that μ^2, μ^4, $(2+\sqrt{3})^2$ are all metallic ratios, i.e. numbers of the form $\frac{n+\sqrt{n^2-4}}{2}$.

$$\mu^2 = \frac{3+\sqrt{5}}{2}, \mu^4 = \frac{7+\sqrt{45}}{2}, (2+\sqrt{3})^2 = \frac{14+\sqrt{192}}{2}$$