# Qual transcript

A - Ian Agol, N - David Nadler, B - Richard Bamler, M - Holger Mueller, T - Chi Cheuk Tsang (Me!)

## 1 Algebraic Topology

A: Give an example of a space not homotopy equivalent to a CW complex.
T: Described $X$ = topologist sine curve, mentioned that $\pi_i(X) = 0$ for all $i$ and that there is a non-nullhomotopic map $X \to S^1$, which cannot be if $X$ is homotopy equivalent to a CW complex.
A: Fill out the details.
T: Proved that a CW complex is contractible if its homotopy groups are zero. Sketched a proof of $\pi_i(X) = 0$.
N: Compute the cohomology of $X$.
T: *stuck*
A: Hint: What would be a CW approximation for $X$?
T: A point is a CW approximation for $X$. *still stuck*. Guessed that $H^i(X) = 0$ but couldn't justify it, thought that $\pi_i(X) = 0 \implies H^i(X) = 0$ only for CW complexes (Note: it is actually true for any space)
N: An easier version of the question: consider the 1-cochain that assigns to each path the amount of times it passes through a given point, show that it is a coboundary.
T: Did that.
A: Let $X$ be a CW complex, of dimension $\leq 2n$, let $\xi_1, \xi_2$ be two $n$-dimensional complex bundles over $X$. Show that $\xi_1, \xi_2$ are stably isomorphic iff they are isomorphic.
T: Did that.
A: Let $X$ be a CW complex, of any dimension, let $\xi_1, \xi_2$ be two complex line bundles over $X$. Show that $\xi_1, \xi_2$ are isomorphic iff they have the same Chern class.
T: Did that.
A: Here is a question I don't know the answer to: Let $X$ be a CW complex, of dimension $\leq 2n$, let $\xi_1, \xi_2$ be two $n$-dimensional complex bundles over $X$. Is it true that $\xi_1, \xi_2$ are stably isomorphic iff they have the same Chern class.
T: Guessed that it was false, looked at $Vect_{\mathbb{C}^n}(S^{2n-1}) \cong \pi_{2n-2}(U(n))$, constructed a example for $S^5$ (that is actually incorrect!, pointed out after the exam by Ian)

B: There may be a way to construct counterexamples using interpretation of Chern class as curvature, try to carry that out.
T: Mentioned that flat vector bundles would have zero Chern class. Mentioned that flat vector bundles can be obtained by twisting the trivial bundle over the universal cover. But couldn't actually write down a counterexample.
(Note: The answer to Ian's question is actually 'no', see
https://math.stackexchange.com/questions/470464/when-characteristic-classes-determine-a-bundle)

## 2 Riemannian Geometry

M: Derive the geodesic equation using the action principle.
T: Defined geodesics to be curves $\gamma$ with $\nabla_{\gamma'}\gamma' = 0$. Proved that geodesics are exactly critical points of the energy functional.
N: Can you show how to express $\nabla_{\gamma'}\gamma' = 0$ in terms of Christoffel symbols?
T: Did that.
N: What are some variants of this computation?
T: Showed that a curve minimizes distance between 2 submanifolds iff $\nabla_{\gamma'}\gamma' = 0$ and $\gamma'$ orthogonal to the submanifolds at the endpoints.
B: Sometimes people define geodesics as critical points of the length functional instead of the energy functional, show that the 2 approaches are the same.
T: Didn't know how to show $\gamma$ is critical point of energy functional iff $\gamma$ is critical point of length functional and $|\gamma'|$ constant, could only show it for minimum points.
B: Let $\gamma$ be a length minimizing curve between 2 hypersurfaces $N_1, N_2$, conjecture an inequality involving $\int_\gamma Ric(\gamma'), H_{N_1}, H_{N_2}$
T: *stuck*
B: Hint: think about what happens when $N_1, N_2$ are horospheres in hyperbolic space.
T: Wrote down $\int_\gamma Ric(\gamma') \leq H_{N_1} + H_{N_2}$.
B: Prove this inequality.
T: Did that.

## 3 Lie Groups and Algebras

N: Consider $SO_+(3,1)$, what is another name for it?
T: $PSL(2,\mathbb{C})$.
N: Let's prove this from scratch. What is another name for $\mathfrak{so}(3,1) \otimes \mathbb{C}$?
T: $\mathfrak{so}(3,1) \otimes \mathbb{C} = \mathfrak{so}(3,1,\mathbb{C}) = \mathfrak{so}(4,\mathbb{C}) = D_2 = A_1 \times A_1 = \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$
N: Suppose we don't use the classification of simple Lie algebras using Dykin diagrams, how would you show that $\mathfrak{so}(4,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$?
T: (With help from David) Wrote down the action of $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$ on $End(\mathbb{C}^2)$ as $(X,Y) \cdot A = XA - AY$, showed that the action is skew symmetric w.r.t. the quadratic form det on $End(\mathbb{C}^2)$.

B: What is $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R})$ isomorphic to?
T: Mentioned that the same construction leads to $\mathfrak{sl}(2,\mathbb{R}) \times \mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{o}(2,2)$.
N: Continuing our discussion, deduce what $\mathfrak{so}(3,1)$ is from your calculations.
T: Mentioned that $\mathfrak{so}(3,1)$ are the real points of $\mathfrak{so}(3,1) \otimes \mathbb{C}$. Claimed that complex conjugation on $\mathfrak{so}(3,1) \otimes \mathbb{C}$ is swapping the two factors of $\mathfrak{sl}(2,\mathbb{C}) \times \mathfrak{sl}(2,\mathbb{C})$, so $\mathfrak{so}(3,1) = \mathfrak{sl}(2,\mathbb{C})$. But got stuck proving my claim.
A: I know you know that $SO_+(3,1) \cong PSL(2,\mathbb{C})$ from hyperbolic geometry, can you outline that argument?
T: Did that.