Milnor & Stasheff 'Characteristic Classes'

Ch.2 Q.E
Ch.4 Q.D.E
Ch.5 Q.D
Ch.6 Q.E
Ch.7 Q. A,B,C
Ch.8 Q. A,B
Ch.9 Q.C
Ch.11 Q. D,E,F
Ch.12 Q. B
Ch.14 Q.B,C
Ch.15 Q.C,D

Ch. 2

E. Define $f : E(\xi) \to E(\xi)$ as follows:

Let $\{v_1, \cdots, v_n\}$ be a basis of ξ_x

Let $A = [\mu(v_i, v_j)]$, $B = [\mu'(v_i, v_j)]$

A, B are symmetric

Let $C = [c_{ij}] = \sqrt{B^{-1}}\sqrt{A}$, note that $C^T B C = A$

Let $f(v_i) = \Sigma c_{ij} v_j$, then $\mu'(f(u), f(v)) = \mu(u, v) \quad \forall u, v \in \xi_x$

If $\{v_1', \cdots, v_n'\}$ is another basis of ξ_x, say $v_i' = m_{ij} v_j$

Then $A' = M^T A M$, $B' = M^T B M$ where $M = [m_{ij}]$

$\Rightarrow C' = M^{-1} \sqrt{B^{-1}}\sqrt{A}\, M$

$\Rightarrow f$ is well-defined

Definition of f is smooth in $v_i, v_j \Rightarrow f$ is morphism of vector bundles

Ch. 4

D. Suppose $M^n \hookrightarrow \mathbb{R}^{n+1}$

Then $i^* w(T\mathbb{R}^{n+1}) = w(TM \oplus N)$

$$1 = w(TM)\, w(N) = (1 + w_1(TM) + \cdots + w_n(TM))(1 + w_1(N))$$

$$\Rightarrow w_i(TM) = w_1(N)^i$$
$$= w_1(TM)^i$$

If $\mathbb{R}P^n \hookrightarrow \mathbb{R}^{n+1}$, $w(\mathbb{R}P^n) = 1$ or $1 + a + \cdots + a^n$ by above

$(1+a)^{n+1} = 1$ or $1 + a + \cdots + a^n$

$(1+a)^{n+1} = 1$ or $(1+a)^{n+2} = 1$

$n+1 = 2^r$ or $n+2 = 2^r$

E. $w(T(\mathbb{R}P^2 \times \mathbb{R}P^2)) = (1+a_1)^3 (1+a_2)^3$

$$= 1 + (a_1 + a_2) + (a_1^2 + a_1 a_2 + a_2^2) + (a_1^2 a_2 + a_1 a_2^2) + a_1^2 a_2^2$$

where $H^*(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}/2) = \mathbb{Z}/2[a_1, a_2]/\langle a_1^3, a_2^3 \rangle$

In particular, $w_1^4(\mathbb{R}P^2 \times \mathbb{R}P^2) = \langle (a_1 + a_2)^4, [\mathbb{R}P^2 \times \mathbb{R}P^2] \rangle = 0$

$w_2^2(\mathbb{R}P^2 \times \mathbb{R}P^2) = \langle (a_1^2 + a_1 a_2 + a_2^2)^2, [\mathbb{R}P^2 \times \mathbb{R}P^2] \rangle = \langle a_1^2 a_2^2, [\mathbb{R}P^2 \times \mathbb{R}P^2] \rangle = 1$

$w(T\mathbb{R}P^4) = (1+a)^5$

$= 1 + a + a^4$ where $H^*(\mathbb{R}P^4; \mathbb{Z}/2) = \mathbb{Z}/2[a]/\langle a^5 \rangle$

In particular, $w_1^4(\mathbb{R}P^4) = \langle a^4, [\mathbb{R}P^4] \rangle = 1$

$w_2^2(\mathbb{R}P^4) = \langle 0^2, [\mathbb{R}P^4] \rangle = 0$

$\therefore$ ϕ, $\mathbb{R}P^2 \times \mathbb{R}P^2$, $\mathbb{R}P^4$, $\mathbb{R}P^2 \times \mathbb{R}P^2 \sqcup \mathbb{R}P^4$ have mutually distinct Stiefel-Whitney numbers $\Rightarrow$ they are pairwise not cobordant

$\therefore |\mathcal{N}_4| \geq 4$

Ch.5

D. Let $\{x_i\}, \{y_i\}$ be orthonormal basis of X, Y respectively

Let $A = [\langle x_i, y_j \rangle]$

$A = PU_1$ for some symmetric P and orthogonal U_1

$P = U_2^t D U_2$ for some diagonal D and orthogonal U_2

Let $x_i' = \sum u_{ij} x_j$ where $U_2 = [u_{ij}]$, $y_i' = \sum v_{ij} y_j$ where $U_2 U_1 = [v_{ij}]$

Then $\{x_i'\}, \{y_i'\}$ are still orthonormal basis of X, Y and

$\langle x_i', y_j' \rangle = 0$ if $i \neq j$

Let $W_i = \operatorname{span}\{x_i', y_i'\}$, W_i are mutually orthogonal

$\dim W_i \leq 2 \Rightarrow \exists \sigma_i \in O(W_i)$ s.t $\sigma_i x_i' = y_i'$, $\sigma_i y_i' = x_i'$

Let $\mathbb{R}^{n+k} = (\oplus W_i) \oplus Z$, let $\sigma = (\oplus \sigma_i) \oplus id \in O(\mathbb{R}^{n+k})$

Then $\sigma x_i' = y_i'$, $\sigma y_i' = x_i' \Rightarrow \sigma X = Y$, $\sigma Y = X$

Note that $\alpha(X, Y) = \sum \angle(x_i', y_i')$ in the above notation

$\Rightarrow \alpha(X, Y) \geq 0$ equality holds iff $X = Y$

and $\alpha(X, Y) = \alpha(Y, X)$

Also note that $\alpha(X, Y) = \operatorname{dist}(X \cap S^{n+k-1}, Y \cap S^{n+k-1})$

$\Rightarrow \alpha(X, Z) \leq \alpha(X, Y) + \alpha(Y, Z)$

$\therefore \alpha$ is a metric

$X^\perp = \oplus x_i'^{\perp} \oplus Z$, $Y^\perp = \oplus y_i'^{\perp} \oplus Z$ in the above notation

$\therefore \alpha(X^\perp, Y^\perp) = \sum \angle(x_i'^{\perp}, y_i'^{\perp}) = \sum \angle(x_i, y_i) = \alpha(X, Y)$

E. Consider the map $\phi : G_n(\mathbb{R}^{n+k}) \longrightarrow G_k(\mathbb{R}^{n+k})$

$$V \longmapsto V^\perp$$

Near $E_n \in G_n(\mathbb{R}^{n+k})$, ϕ has coordinate representation

$\operatorname{span}\left\{\begin{bmatrix} I \\ A \end{bmatrix}\right\} \longmapsto \operatorname{span}\left\{\begin{bmatrix} -A^T \\ I \end{bmatrix}\right\}$ hence is smooth

ϕ is a bijection $\Rightarrow$ ϕ is a diffeomorphism

In fact we claim that ϕ respects the CW structures defined on $G_n(\mathbb{R}^{n+k})$ via the flag $E_0 \subseteq \cdots \subseteq E_{n+k}$ and on $G_k(\mathbb{R}^{n+k})$ via the flag $E_{n+k}^\perp \subseteq \cdots \subseteq E_0^\perp$, hence ϕ is an isomorphism of CW complexes

Given $V \in e_\sigma \subseteq G_n(\mathbb{R}^{n+k})$ where $\sigma = (\sigma_1, \cdots, \sigma_n)$

$$\dim(V \cap E_\mu) - \dim(V \cap E_{\mu-1}) = \begin{cases} 1 & \text{if } \mu = \sigma_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

But $\dim(V \cap E_\mu) = \dim V + \dim E_\mu - \dim V \oplus E_\mu$

$$= \dim V + \mu - \dim(V^\perp \cap E_\mu^\perp)^\perp$$

$$\therefore \dim(V^\perp \cap E_{\mu-1}^\perp)^\perp + 1 - \dim(V^\perp \cap E_\mu^\perp)^\perp = \begin{cases} 1 & \text{if } \mu = \sigma_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

$$\dim(V^\perp \cap E_\mu^\perp) - \dim(V^\perp \cap E_{\mu-1}^\perp) = \begin{cases} 0 & \text{if } \mu = \sigma_i \text{ for some } i \\ 1 & \text{otherwise} \end{cases}$$

$\therefore V \in e_{\bar\sigma} \subseteq G_k(\mathbb{R}^{n+k})$ where $\bar\sigma = (1, \cdots, \widehat{n+k+1-\sigma_i}, \cdots, n+k)$