Humphreys 'Introduction to Lie Algebras and Representation Theory'

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Ch. 1

3. Relative to $\{x, h, y\}$, $\operatorname{ad} x = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\operatorname{ad} h = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, $\operatorname{ad} y = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

$\therefore\ K = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{bmatrix}$

4. Relative to $\{x, y\}$, $\operatorname{ad} x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\operatorname{ad} y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$Z(L) = 0 \Rightarrow L \cong \operatorname{ad} L \cong \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}$

8. $\begin{bmatrix} m & n \\ p & q \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} + \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} m & n \\ p & q \end{bmatrix}^T = 0 \Leftrightarrow \begin{cases} m = -q^T \\ n = -n^T \\ p = -p^T \end{cases}$

$\therefore \{ e_{ii} - e_{i+l,i+l} : i = 1, \cdots, l \} \cup \{ e_{ij} - e_{j+l,i+l} : i,j = 1, \cdots, l,\ i \neq j \}$
$\cup \{ e_{i,j+l} - e_{j,i+l} : i,j = 1, \cdots, l,\ i < j \} \cup \{ e_{i+l,j} - e_{j+l,i} : i,j = 1, \cdots, l,\ i < j \}$ is a basis of D_l

In particular $\dim D_l = l + l(l-1) + \frac{l(l-1)}{2} + \frac{l(l-1)}{2} = 2l^2 - l$

9. For A_l: $e_{ii} - e_{i+1,i+1} = [e_{i,i+1}, e_{i+1,i}]$ $\quad \forall i = 1, \cdots, l$

$e_{ij} = \frac{1}{2} [e_{ii} - e_{jj}, e_{ij}]$ $\quad \forall i, j = 1, \cdots, l+1,\ i \neq j$

For B_l: $e_{ii} - e_{i+l,i+l} = [e_{i,1} - e_{1,i+l}, e_{1,i} - e_{i+l,1}]$ $\quad \forall i = 2, \cdots, l+1$

$e_{i1} - e_{1,i+l} = [e_{ii} - e_{i+l,i+l}, e_{i1} - e_{1,i+l}]$ $\quad \forall i = 2, \cdots, l+1$

$e_{1i} - e_{i+l,1} = [e_{ii} - e_{i+l,i+l}, e_{1i} - e_{i+l,1}]$ $\quad \forall i = 2, \cdots, l+1$

$e_{ij} - e_{j+l,i+l} = [e_{ii} - e_{i+l,i+l}, e_{ij} - e_{j+l,i+l}]$ $\quad \forall i,j = 2, \cdots, l+1,\ i \neq j$

$e_{i,j+l} - e_{j,i+l} = [e_{i1} - e_{1,i+l}, -e_{j1} + e_{1,j+l}]$ $\quad \forall i,j = 2, \cdots, l+1,\ i < j$

$e_{i+l,j} - e_{j+l,i} = [-e_{1i} + e_{i+l,1}, e_{1j} - e_{j+l,1}]$ $\quad \forall i,j = 2, \cdots, l+1,\ i < j$

For C_l: $e_{ii} - e_{i+l,i+l} = [e_{i,i+l}, e_{i+l,i}]$ $\quad \forall i = 1, \cdots, l$

$e_{ij} - e_{j+l,i+l} = [e_{i,j+l} + e_{j,i+l}, e_{j+l,j}]$ $\quad \forall i,j = 1, \cdots, l,\ i \neq j$

$e_{i,i+l} = \frac{1}{2} [e_{ii} - e_{i+l,i+l}, e_{i,i+l}]$ $\quad \forall i = 1, \cdots, l$

$e_{i,j+l} + e_{j,i+l} = [e_{ii} - e_{i+l,i+l}, e_{i,j+l} - e_{j,i+l}]$ $\quad \forall i,j = 1, \cdots, l,\ i \neq j$

$e_{i+l,i} = \frac{1}{2} [e_{i+l,i}, e_{ii} - e_{i+l,i+l}]$ $\quad \forall i = 1, \cdots, l$

$e_{i+l,j} + e_{j+l,i} = [e_{ii} - e_{i+l,i+l}, -e_{i+l,j} - e_{j+l,i}]$ $\quad \forall i,j = 1, \cdots, l,\ i \neq j$

For D_l $(l \geq 3)$: $e_{ii} - e_{i+l,i+l} = \frac{1}{3} \big([e_{i,j+l} - e_{j,i+l}, e_{j+l,i} - e_{i+l,j}]$
$+ [e_{i,k+l} - e_{k,i+l}, e_{k+l,i} - e_{i+l,k}]$
$- [e_{j,k+l} - e_{k,j+l}, e_{k+l,j} - e_{j+l,k}] \big)$ where i, j, k distinct
$\forall i = 1, \cdots, l$

$e_{ij} - e_{j+l,i+l} = [e_{ii} - e_{i+l,i+l}, e_{ij} - e_{j+l,i+l}]$ $\quad \forall i,j = 1, \cdots, l,\ i \neq j$

$e_{i,j+l} - e_{j,i+l} = [e_{ii} - e_{i+l,i+l}, e_{i,j+l} - e_{j,i+l}]$ $\quad \forall i,j = 1, \cdots, l,\ i < j$

$e_{i+l,j} - e_{j+l,i} = [e_{ii} - e_{i+l,i+l}, -e_{i+l,j} + e_{j+l,i}]$ $\quad \forall i,j = 1, \cdots, l,\ i < j$

10. (I think we need char $\mathbb{F} \neq 2$ for the isomorphisms to hold?)

$A_1 = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\rangle = C_1$

$A_1 \cong B_1$ by sending $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 2 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$D_1 = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle \cong \mathbb{F}$

$C_2 \cong B_2$ by sending $\begin{bmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix} \longmapsto \begin{bmatrix} 0 & & & & \\ & 1 & & & 0 \\ & & 1 & & \\ & 0 & & -1 & \\ & & & & -1 \end{bmatrix}$, $\begin{bmatrix} \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix} \end{bmatrix} \longmapsto \begin{bmatrix} 0 & & & & \\ & 1 & & & 0 \\ & & -1 & & \\ & 0 & & -1 & \\ & & & & 1 \end{bmatrix}$

$\begin{bmatrix} \begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix} \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & 0 & 0 & 2 \\ 0 & & & & \\ -2 & & 0 & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix}$, $\begin{bmatrix} \begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & -1 \\ 0 & 0 \end{smallmatrix} \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & & & & \\ 0 & & 0 & & \\ 0 & & & & \\ 1 & & & & \end{bmatrix}$, $\begin{bmatrix} 0 & \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \\ 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} & & & 0 & 0 \\ & 0 & & 0 & -2 \\ & & & 2 & 0 \\ & & & & \\ & & & 0 & \end{bmatrix}$

$\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \\ 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} & 0 & 0 & 0 & 0 \\ & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \\ & & & 0 & 0 \\ & 0 & & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} \\ 0 & 0 \end{bmatrix} \longmapsto \begin{bmatrix} & 0 & 0 & 0 & 0 \\ & 0 & 1 & & \\ 0 & 0 & 0 & 0 & \\ & & & 0 & 0 \\ & 0 & & -1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & 0 \end{bmatrix} \longmapsto \begin{bmatrix} & & 0 & & \\ & & & & \\ 0 & 0 & 1 & & 0 \\ 0 & -1 & 0 & & \end{bmatrix}$,

$\begin{bmatrix} 0 & 0 \\ \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & 0 \end{bmatrix} \longmapsto \begin{bmatrix} & & 0 & & \\ 0 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} & 0 \end{bmatrix} \longmapsto \begin{bmatrix} & & 0 & & \\ 0 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & \\ 0 & & 0 & 0 & -1 \\ 0 & & & 0 & 0 \end{bmatrix}$

$A_3 \cong D_3$ by sending $\begin{bmatrix} 1 & & & \\ & -1 & & 0 \\ & & 0 & \\ 0 & & & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \\ & & & 0 & & \\ & 0 & & & -1 & \\ & & & & & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & & & \\ & 1 & & 0 \\ & & -1 & \\ 0 & & & 0 \end{bmatrix} \longmapsto \begin{bmatrix} 1 & & & & & \\ & -1 & & & 0 & \\ & & 0 & & & \\ & 0 & & -1 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix}$,

$\begin{bmatrix} 0 & & & \\ & 0 & & 0 \\ & & 1 & \\ 0 & & & -1 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & & & & & \\ & 1 & & & 0 & \\ & & -1 & & & \\ & & & 0 & & \\ & 0 & & & -1 & \\ & & & & & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ & 0 & & \end{bmatrix}^{(T)} \longmapsto \begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{smallmatrix} \\ 0 & 0 \end{bmatrix}^{(T)}$, $\begin{bmatrix} 0 & 0 & 1 & 0 \\ & 0 & & \end{bmatrix}^{(T)} \longmapsto \begin{bmatrix} 0 & \begin{smallmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix} \\ 0 & 0 \end{bmatrix}^{(T)}$,

$\begin{bmatrix} 0 & 0 & 0 & 1 \\ & 0 & & \end{bmatrix}^{(T)} \longmapsto \begin{bmatrix} 0 & \begin{smallmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \\ 0 & 0 \end{bmatrix}^{(T)}$, $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & 0 & & \end{bmatrix}^{(T)} \longmapsto \begin{bmatrix} \begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \end{bmatrix}^{(T)}$, $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ & 0 & & \end{bmatrix}^{(T)} \longmapsto \begin{bmatrix} \begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{smallmatrix} \end{bmatrix}^{(T)}$,

$\begin{bmatrix} & 0 & & \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^{(T)} \longmapsto \begin{bmatrix} \begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \end{bmatrix}^{(T)}$

$D_2 = \left\langle \begin{bmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} -1 & 0 \\ 0 & 0 \end{smallmatrix} \end{bmatrix}, \begin{bmatrix} \begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix} & 0 \\ 0 & \begin{smallmatrix} 0 & 0 \\ 0 & -1 \end{smallmatrix} \end{bmatrix} \right\rangle \cong \mathbb{F}^2$

12. For A_l: $x \in A_l \iff \operatorname{tr} x = \operatorname{tr} x^T = 0 \iff x^T \in A_l$

For B_l: $x \in B_l \iff x \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} x^T \iff \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} x = -x^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I \\ 0 & I & 0 \end{bmatrix} \iff x^T \in B_l$

For C_l: $x \in C_l \iff x \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = -\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} x^T \iff \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} x = -x^T \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \iff x^T \in C_l$

For D_l: $x \in D_l \iff x \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = -\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} x^T \iff \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} x = -x^T \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \iff x^T \in D_l$

Ch. 2

3. Let $[a_{ij}] \in Z(gl(n,\mathbb{F}))$

$$0 = [[a_{ij}], e_{kl}] = \sum_i a_{ik} e_{il} - \sum_j a_{lj} e_{kj}$$

$$\therefore \begin{cases} a_{kk} = a_{ll} \\ a_{ik} = 0 & \text{if } i \neq k \\ a_{lj} = 0 & \text{if } j \neq l \end{cases}$$

$\therefore [a_{ij}] = a_{11} \cdot I \in s(n,\mathbb{F}) \Rightarrow Z(gl(n,\mathbb{F})) \subseteq s(n,\mathbb{F})$

$I \in Z(gl(n,\mathbb{F})) \Rightarrow s(n,\mathbb{F}) \subseteq Z(gl(n,\mathbb{F}))$

The same reasoning shows that $Z(sl(n,\mathbb{F})) \subseteq s(n,\mathbb{F}) \cap sl(n,\mathbb{F}) = \begin{cases} s(n,\mathbb{F}) & \text{if char } \mathbb{F} \mid n \\ 0 & \text{otherwise} \end{cases}$

If char $\mathbb{F} \mid n$, $I \in Z(sl(n,\mathbb{F})) \Rightarrow s(n,\mathbb{F}) \subseteq Z(sl(n,\mathbb{F}))$

$\therefore Z(sl(n,\mathbb{F})) = \begin{cases} s(n,\mathbb{F}) & \text{if char } \mathbb{F} \mid n \\ 0 & \text{otherwise} \end{cases}$

4. Suppose $\dim L = 3$, $\dim [L,L] = 1$, $[L,L] \subseteq Z(L)$

Let $[L,L] = \langle x \rangle$, $L = \langle x, y, z \rangle$

Then $[x,y] = 0$, $[x,z] = 0$, $[y,z] = kx$ for some $k \in \mathbb{F}^*$

Conversely let L_k be the Lie algebra defined by these relations, it remains to show that $L_{k_1} \cong L_{k_2}$

$$L_{k_1} = \langle x_1, y_1, z_1 \rangle \cong L_{k_2} = \langle x_2, y_2, z_2 \rangle$$

$$x_1 \mapsto x_2$$
$$y_1 \mapsto y_2$$
$$z_1 \mapsto \frac{k_1}{k_2} z_2$$

5. Note that for every ideal $I \trianglelefteq L$, $[L/I, L/I] = [L,L]/I = L/I$

If $\dim I = 2$, $\dim(L/I) = 1$ so $[L/I, L/I] = 0 \neq L/I$

If $\dim I = 1$, $\dim(L/I) = 2$

$\dim [L/I, L/I] = 0$ or $1 \Rightarrow [L/I, L/I] \neq L/I$

$\therefore$ The only ideals in L are 0 and L

In particular, $\dim sl(2,\mathbb{F}) = 3$ & $[sl(2,\mathbb{F}), sl(2,\mathbb{F})] = sl(2,\mathbb{F})$ if char $\mathbb{F} \neq 2$

by calculation in ch.1 Q9 $\Rightarrow sl(2,\mathbb{F})$ simple

6. If char $\mathbb{F} \neq 3$, suppose $I \trianglelefteq sl(3,\mathbb{F})$

Let $\sum a_{ij} e_{ij} \in I$, $[[[\sum a_{ij} e_{ij}, e_{12}], e_{13}], e_{23}] = -a_{31} e_{13} \in I$

Similarly, $-a_{ij} e_{ij} \in I \quad \forall i \neq j$

$\therefore a_{ij} = 0 \quad \forall i \neq j$ or some $e_{ij} \in I$, $i \neq j$

Case 1: $a_{ij} = 0 \quad \forall i \neq j$

$[\sum a_{ii} e_{ii}, e_{12}] = (a_{11} - a_{22}) e_{12} \in I$