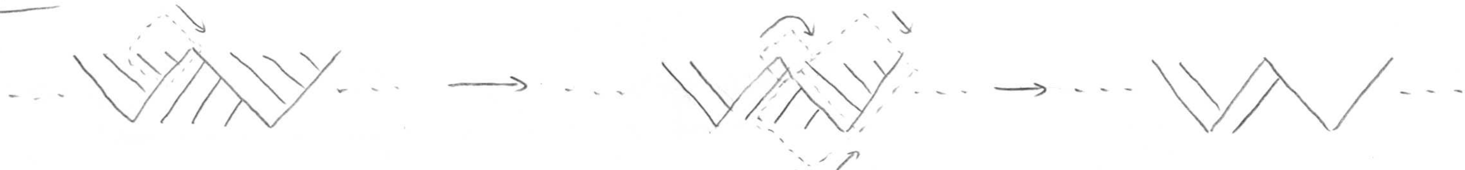
Hatcher 'Algebraic Topology'

Ch.0 Q6, 7, 11, 13, 14, 23, 27
Ch. 1.1 Q10, 13
Ch. 1.2 Q12–18, 20
Ch. 1.3 Q3, 5, 6, 10–15, 19–21, 24, 25, 29, 32, 33
Ch 1.A Q7, 10–13
Ch 1.B Q6, 9
Ch. 2.1. Q1, 2, 7, 10, 14, 19, 20, 26, 28
Ch. 2.2. Q1–3, 8, 13, 16, 18, 25, 26, 33, 36, 42, 43
Ch. 2.3. Q1–4
Ch. 2.B. Q3–5, 7, 11
Ch. 2.C. Q1, 4, 5, 8
Ch. 3.1. Q4, 10, 13
Ch. 3.2 Q2, 3, 5, 10, 12, 14, 16
Ch. 3.3 Q6, 10–12, 14, 21–24, 30, 31, 35
Ch. 3.A Q2, 6
Ch. 3.B Q2–5
Ch. 4.1 Q2, 9, 10, 13, 14, 18–23
Ch. 4.2 Q3, 4, 7–11, 13, 17–19, 22–26, 28, 30, 31, 34, 38
Ch. 4.3 Q3, 8–10, 15, 16, 21, 23
Ch. 4.D. Q1, 4–6, 9

Ch 0

6.



Repeat on other positions to get deformation retract in weak sense of Y onto Z

Z deformation retracts onto a point $\overset{Q4}{\Rightarrow} Y$ contractible

$Q5 \Rightarrow \exists$ deformation retract of Y onto a point

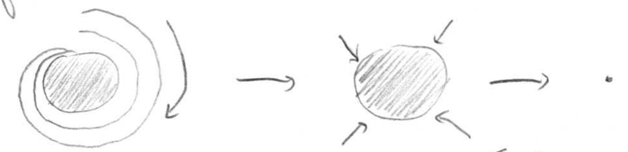
$\Rightarrow \exists$ deformation retract of Y onto Z

7.

Repeat on other positions to get deformation retract in weak sense of X to $[0, \infty)$

$\times \mathbb{R}$ & one point compactify $\Rightarrow$ deformation retract in weak sense of  to  $= D^2$

$\Rightarrow$ deformation retract in weak sense of Y to 

But  $\overset{Q4}{\Rightarrow} Y$ contractible

$Q5 \Rightarrow \exists$ deformation retract of Y onto a point

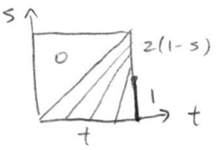
11. Let $kfg \simeq id_X$, $fgk \simeq id_X$, $lhf \simeq id_Y$, $hfl \simeq id_Y$

$lh \simeq lh(fgk) = (lhf)gk \simeq gk$

$\therefore (gk)f \simeq id_Y$

$\therefore f$ is homotopy equivalence with homotopy inverse gk

13. Define $\alpha(s,t)$ by



Define $r_t^s = r^0_{\alpha(s,t)} \, r^1_{\alpha(1-s,t)}$

When $s=0$, $r^0_{\alpha(0,t)} \, r^1_{\alpha(1,t)} = r^0_t \, r^1_0 = r^0_t \, id_X = r^0_t$

$s=1$, $r^0_{\alpha(1,t)} \, r^1_{\alpha(0,t)} = r^0_0 \, r^1_t = id_X \, r^1_t = r^1_t$

$t=0$, $r^0_{\alpha(s,0)} \, r^1_{\alpha(1-s,0)} = id_X \, id_X = id_X$

$$t=1, \quad r^0_{\alpha(s,1)} \, r^1_{\alpha(1-s,1)} = \begin{cases} r^0_1 \, r^1_{2s} & \text{if } 0 \le s \le \frac{1}{2} \\ r^0_{2(1-s)} \, r^1_1 & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

$$= \begin{cases} r^0_1 \, r^1_{2s} & \text{if } 0 \le s \le \frac{1}{2} \\ r^0_{2(1-s)} & \text{if } \frac{1}{2} \le s \le 1 \end{cases} \subseteq A$$

13. $r_t^0|_A,\ r_t^1|_A = id_A \Rightarrow r_t^s|_A = id_A$

$\therefore r_t^s$ is deformation retract of X onto A

14. We show this for any compact orientable surface Σ_g

Note that $\begin{bmatrix} v \\ e \\ f \end{bmatrix} = \begin{bmatrix} 1 \\ 2g \\ 1 \end{bmatrix} + (v-1)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (f-1)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Start with the standard CW structure with $\begin{bmatrix} v \\ e \\ f \end{bmatrix} = \begin{bmatrix} 1 \\ 2g \\ 1 \end{bmatrix}$ 

For each $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, modify a face 

For each $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, modify a face 

This works for non-orientable surfaces too

23. Let $X = A \cup B$ where A, B, and $C = A \cap B$ are contractible

C & A contractible $\Rightarrow C \hookrightarrow A$ is homotopy equivalence

$\therefore \exists$ deformation retract of A onto C

Similarly $\exists$ deformation retract of B onto C

These combine to give deformation retract of X onto C

$\therefore X \simeq C \simeq pt.$

27. $A \hookrightarrow M_f$ is homotopy equivalence $\Rightarrow \exists$ deformation retract of M_f onto A

$\Rightarrow \exists$ deformation retract of $M_f \cup_A X$ onto X

Define $f_t : A \to M_f$, $f_t(a) = (a, t)$

$M_f \cup_A X$ and $M_f \cup_f X$ are subcomplexes of $M_f \times [0,1] \cup_{f_t} X \times [0,1]$

(X, A) has homotopy extension property

$\overset{Q26}{\Rightarrow} X \times [0,1]$ deformation retracts onto $X \times \{0\} \cup A \times [0,1]$

$\Rightarrow M_f \times [0,1] \cup_{f_t} X \times [0,1]$ deformation retracts onto $M_f \cup_A X$ and onto $M_f \cup_f X$

Also $M_f \cup_f X$ deformation retracts onto $B \cup_f X$

Finally,

$$\begin{array}{ccccc} X & \overset{\simeq}{\hookrightarrow} & M_f \cup_A X & \overset{\simeq}{\searrow} & \\ \downarrow & & & & M_f \times [0,1] \cup_{f_t} X \times [0,1] \\ B \cup_f X & \overset{\simeq}{\hookrightarrow} & M_f \cup_f X & \overset{\simeq}{\nearrow} & \end{array}$$

$\therefore X \to B \cup_f X$ is a homotopy equivalence

Ch. 1.1

10. Let $[\alpha] \in \pi_1(X, x_0)$, $[\beta] \in \pi_1(Y, y_0)$

Define $\gamma_s(t) = (\alpha(\max\{\min\{2t-s, 1\}, 0\}), \beta(\max\{\min\{2t-1+s, 1\}, 0\}))$

$\gamma_0(t) = (\alpha(\min\{2t, 1\}), \beta(\max\{2t-1, 0\})) = (\alpha \times \{y_0\}) \cdot (\{x_0\} \times \beta)$

$\gamma_1(t) = (\alpha(\max\{2t-1, 0\}), \beta(\min\{2t, 1\})) = (\{x_0\} \times \beta) \cdot (\alpha \times \{y_0\})$

$\gamma_s(0) = (\alpha(0), \beta(0)) = (x_0, y_0)$

$\gamma_s(1) = (\alpha(1), \beta(1)) = (x_0, y_0)$

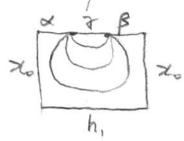
13. ('Homotopic' in this question means path homotopic, i.e. homotopic rel endpoints. Or else all paths are homotopic to constant path at x_0, as long as X is path connected)

($\Leftarrow$) Clear

($\Rightarrow$) Let $\gamma : [0,1] \to X$ with $\gamma(0), \gamma(1) \in A$

Choose $\alpha, \beta : [0,1] \to A$ s.t. $\alpha(0) = x_0$, $\alpha(1) = \gamma(0)$, $\beta(0) = \gamma(1)$, $\beta(1) = x_0$

$[\alpha \cdot \gamma \cdot \beta] \in \pi_1(X, x_0)$



$\therefore \exists h_s : [0,1] \to X$ s.t. $h_0 = \alpha \cdot \gamma \cdot \beta$, $h_1 \subseteq A$

(A reparametrization of) h_s gives a homotopy of γ to $\bar{\alpha} \cdot c_{x_0} \cdot h_1 \cdot c_{x_0} \cdot \bar{\beta} \subseteq A$