Veering triangulations and pseudo-Anosov flows

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Pseudo-Anosov homeomorphisms

Let S be a closed surface with genus $\geqslant 2$.

Theorem (Nielsen-Thurston classification)

Every homeomorphism of S is isotopic to a homeomorphism ϕ satisfying one of the following properties:

- *Periodic, i.e.* $\phi^n = id$ *for some* n.
- *Reducible, i.e.* $\phi(c) = c$ *for some closed curve* c.
- *Pseudo-Anosov, i.e. there exists measured singular foliations* (Λ^s, μ^s) *and* (Λ^u, μ^u) *such that* $\phi(\Lambda^s, \mu^s) = (\Lambda^s, \lambda\mu^s)$ *and* $\phi(\Lambda^u, \mu^u) = (\Lambda^u, \lambda^{-1}\mu^u)$ *for some* $\lambda > 1$.

Pseudo-Anosov homeomorphisms

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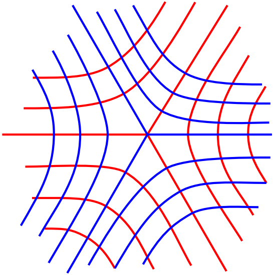

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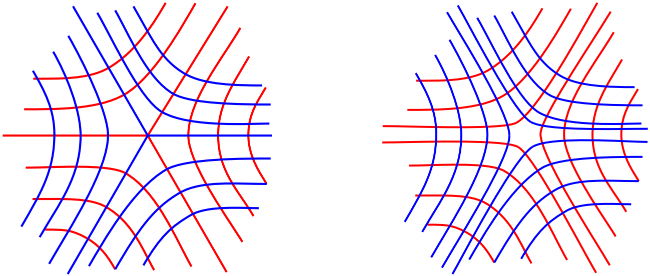
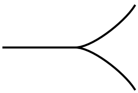
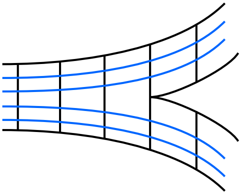


One can blow air into the singular leaves of Λ^s and Λ^u to create laminations.

Train tracks

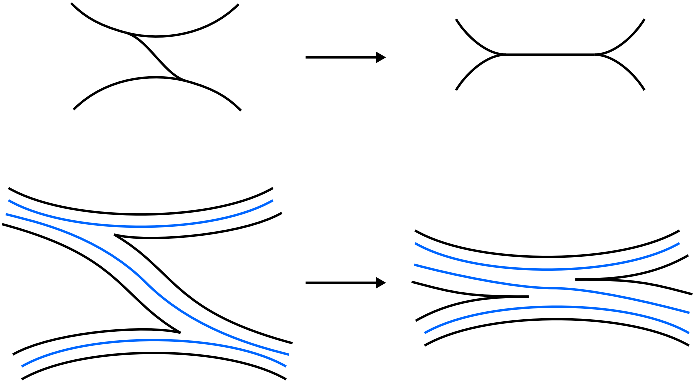
A train track is a trivalent graph embedded in S such that the edges have a common tangent line at each vertex, with one edge on one side and two edges on the other.
A tie neighborhood $N(\tau)$ of a train track τ is a regular neighborhood of τ with a projection map $N(\tau) \to \tau$ with interval fibers, called ties.
A train track τ is said to carry a lamination Λ if there exists a tie neighborhood $N(\tau)$ containing Λ with the ties transverse to the leaves.





Folding sequence of train tracks

There are many train tracks carrying the same lamination.
Example: folding move



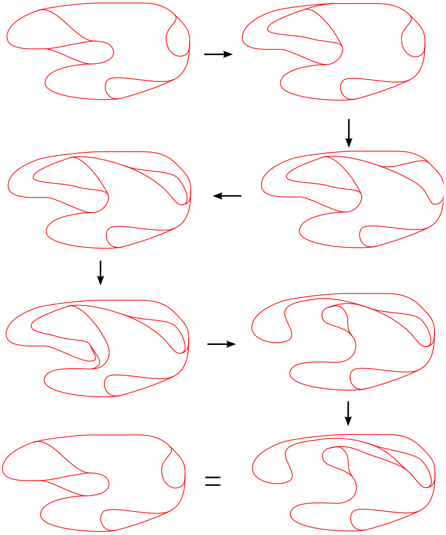
Folding sequence of train tracks

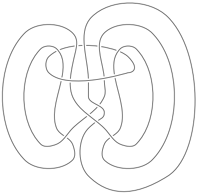
Agol's idea: Given a pseudo-Anosov ϕ, take a periodic folding sequence of train tracks that carry Λ^u (after blowing air into the singular leaves)

I.e. train tracks $\tau_0 \rightsquigarrow \tau_1 \rightsquigarrow \cdots \rightsquigarrow \tau_N$ such that:

- Each τ_i carries Λ^u (after blowing air into singular leaves)
- τ_{i+1} is obtained from τ_i by a folding move
- $\phi(\tau_N) = \tau_0$

Example

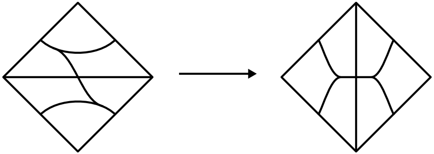




Figures taken from [Ago11]

Ideal triangulation on mapping torus

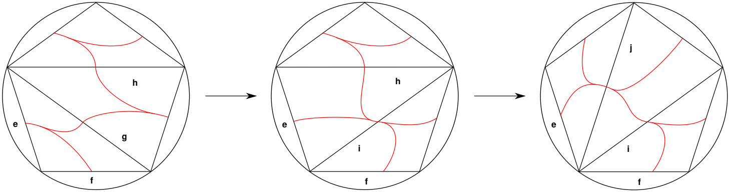
Consider the dual triangulations δ_i to the train tracks τ_i. δ_{i+1} is obtained from δ_i by a diagonal switch.



Construct an ideal triangulation by starting with δ_0 and placing 'flat' tetrahedra from the bottom to effect the diagonal switch. Finally glue the bottom to the top.
This gives an ideal triangulation of the 3-manifold obtained from the mapping torus of ϕ by drilling out the singular orbits of Λ^u, called the layered veering triangulation associated to ϕ.

Why 'veering'

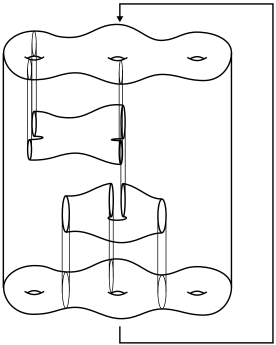
Consider an edge of the triangulation and the faces adjacent to it.



The faces are 'veering' left or right. We can label the edges bLue or Red respectively.

Different fibrations of the same mapping torus

Fact: The mapping tori of two different surface homeomorphisms ϕ, ϕ' can be the same 3-manifold M. Example:
The layered veering triangulations associated to ϕ and ϕ' are the same triangulation on M!



The common feature that unifies ϕ and ϕ' is the suspension flow!

Veering triangulations are associated to flows

The ‘correct’ way of thinking is to regard the layered veering triangulation as associated to the suspension flow on M (as opposed to associated to the surface homeomorphism ϕ).

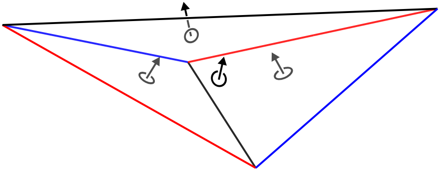
Theorem ((Agol-Guéritaud, Schleimer-Segerman, Landry-Minsky-Taylor, Agol-T.)

Veering triangulations are in one-to-one correspondence with pseudo-Anosov flows without perfect fits.

Definition of veering triangulations

A *veering triangulation* on M is an ideal triangulation along with:

- face coorientations
- taut structure
- edge coloring

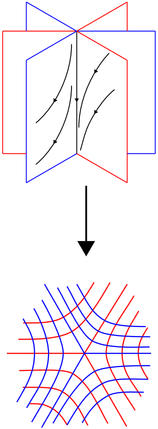


Definition of pseudo-Anosov flows

A pseudo-Anosov flow on a closed 3-manifold N is a flow ϕ_t such that:

- There are two singular foliations Λ^s, Λ^u intersecting transversely along flow lines
- The flow contracts exponentially along Λ^s and expands exponentially along Λ^u
- (Technical conditions about Markov partitions & behavior along singular orbits)

Orbit space of $\widetilde{\phi}_t$ on universal cover $\widetilde{N}$ is homeomorphic to $\mathbb{R}^2$ with two singular foliations.



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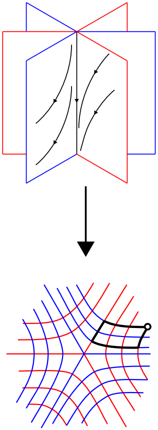


Orbit space of $\widetilde{\phi}_t$ on universal cover $\widetilde{N}$ is homeomorphic to $\mathbb{R}^2$ with two singular foliations. ϕ_t is said to be *without perfect fits* if there are no *perfect fit rectangles* on this orbit space.

The correspondence theorem with more details

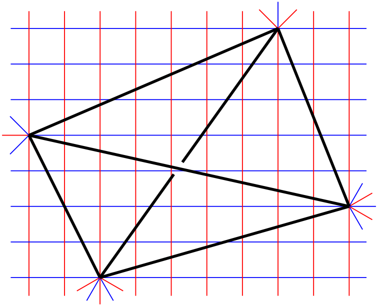
Theorem

1. *Given a pseudo-Anosov flow ϕ on a closed 3-manifold M without perfect fits, let $\{c_i\}$ be the collection of singular orbits of ϕ. Then $M \backslash \cup c_i$ admits a veering triangulation Δ.*
2. *Given a veering triangulation Δ on M, there exists a collection of curves l (called the ladderpole curves) on the ends of M, such that for every collection of slopes s on the ends of M with $|\langle s, l \rangle| \geqslant 3$, the Dehn filling $M(s)$ carries a pseudo-Anosov flow ϕ without perfect fits.*

Moreover, the constructions of (1) and (2) can be chosen to be inverses to each other.

Sketch of proofs - (1)

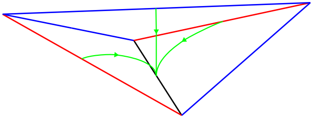
Given a pseudo-Anosov flow ϕ without perfect fits. Consider maximal rectangles in its orbit space $\mathcal{O}$. Place tetrahedra above the rectangles.



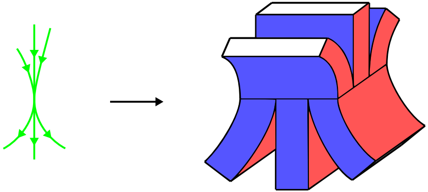
These fit together at the right 'heights' to give an ideal triangulation of the (punctured) universal cover. Quotient by deck transformations to get the veering triangulation.

Sketch of proofs - (2)

Given a veering triangulation. Define its flow graph by:



Thicken up the flow graph, this set has a natural vertical 'flow'. Glue up the stable and unstable faces of this set to obtain a flow on the entire 3-manifold.



Remark on the correspondence theorem: discrete vs continuous objects

Veering triangulations are discrete objects.

See the veering triangulation census [Giannopolous-Schleimer-Segerman]

```
cPcbbbdxm_10 L 1 2 N 1 [2] [2,0,0] Z/5+Z ['m003','otet02_00000']
cPcbbbiht_12 L 1 2 E 1 [4] [2,0,0] Z ['m004','4_1','K2_1','K4a1','otet02_00001']
dLQacccjsnk_200 L 1 1 N 1 [2] [2,0,1] Z ['m016','K3_1','K12n242']
dLQbccchhfo_122 L 1 2 E 1 [4] [2,0,1] Z/2+Z ['m009']
dLQbccchhsj_122 L 1 2 N 1 [2] [2,0,1] Z/6+Z ['m010']
eLAkaccddjsnak_2001 L 1 1 N 1 [2] [2,1,1] Z ['m119']
eLAkbbcdddhwqj_2102 L 1 1 N 1 [2] [2,0,2] Z ['m052']
eLAkbccddhhsqs_1220 L 1 1 N 1 [2] [2,1,1] Z/2+Z ['m146']
eLMkbcddddedde_2100 L 2 1 N 1 [2,2] [4,0,0] Z+Z ['m203','6^2_2','L6a2','otet04_00001']
eLMkbcdddhhhdu_1221 L 1 2 N 1 [2] [2,0,2] Z/7+Z ['m022']
eLMkbcdddhhhml_1221 L 1 2 E 1 [4] [2,0,2] Z/3+Z ['m023']
eLMkbcdddhhqqa_1220 L 1 2 E 1 [4] [2,1,1] Z/2+Z/2+Z ['m136']
eLMkbcdddhhqxh_1220 L 1 2 N 1 [2] [2,1,1] Z/2+Z/4+Z ['m135']
eLMkbcdddhxqdu_1200 L 1 2 E 1 [4] [4,0,0] Z/5+Z ['m206','otet04_00002']
eLMkbcdddhxqlm_1200 L 1 2 N 1 [2] [4,0,0] Z/3+Z/3+Z ['m207','otet04_00003']
eLPkaccddjnkaj_2002 L 1 1 N 1 [2] [2,0,2] Z/3+Z ['m036']
eLPkbcdddhrrcv_1200 L 1 1 E 1 [4] [2,2,0] Z ['m038']
```

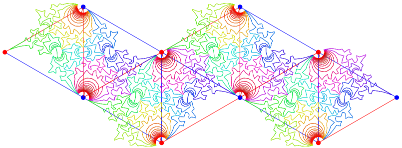
Pseudo-Anosov flows are continuous objects.

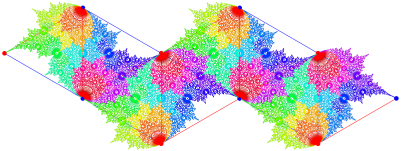
A ~~surprising~~ feature of the correspondence theorem is that it ties together discrete objects with continuous objects.

Remark on the correspondence theorem: discrete vs continuous objects

Theorem [Farb-Leininger-Margalit, Agol, Agol-T.]: There are only finitely many fully punctured mapping tori whose monodromy has normalized dilatation smaller than a given bound.

Manning-Schleimer-Segerman: Approximating Cannon-Thurston sphere filling curves





[Images taken from Segerman's talks.]

Future work: filling out the dictionary

Veering triangulations	$\leftsquigarrow\rightsquigarrow$	Pseudo-Anosov flows
Combinatorial invariants	$\rightsquigarrow$	???
Veering polynomial [Landry-Minsky-Taylor]	$\rightsquigarrow$	???
Veering surgery [Schleimer-Segerman, T.]	$\rightsquigarrow$	???
???	$\leftsquigarrow$	Construction from finite-depth foliations [Gabai, Mosher] (Ongoing project with Michael Landry)
???	$\leftsquigarrow$	Construction from $\mathbb{R}$-covered foliations [Calegari, Fenley]