Veering triangulations and Birkhoff sections

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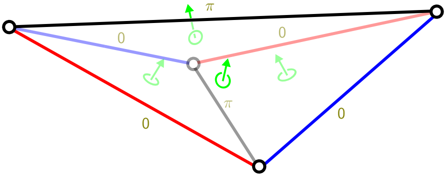
What is a veering triangulation?

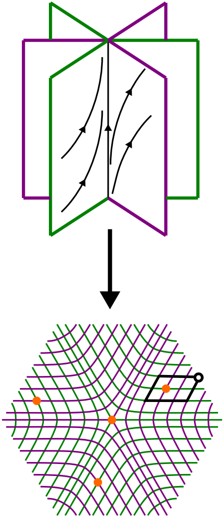
A **veering triangulation** on an orientable 3-manifold M is a finite ideal triangulation along with the following combinatorial data:

- face coorientations
- taut structure
- edge coloring



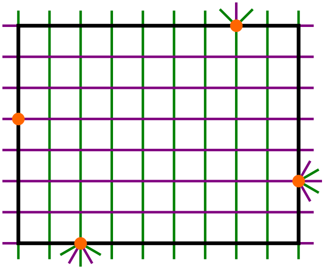
From pseudo-Anosov flows to veering triangulations

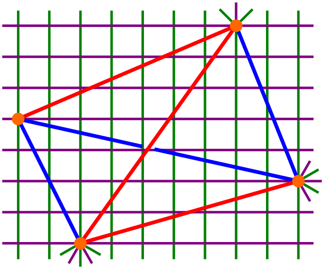
Let ϕ be a (topological) pseudo-Anosov flow on an orientable closed 3-manifold N.
Let $\mathcal{O}$ be the orbit space of $\widetilde{\phi}$.
Let $\mathcal{C}$ be a finite nonempty collection of closed orbits, containing the singular orbits, such that ϕ **has no perfect fits relative to** $\mathcal{C}$, i.e. any perfect fit rectangle must contain points of $\widetilde{\mathcal{C}}$.



From pseudo-Anosov flows to veering triangulations

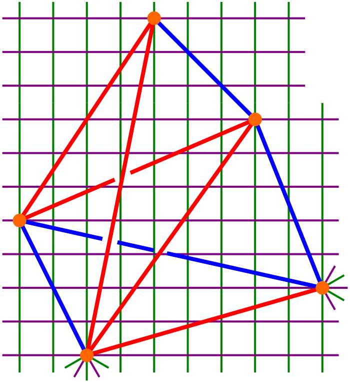
Consider **maximal rectangles** in $\mathcal{O}$ with points of $\widetilde{\mathcal{C}}$ on its sides. Take an ideal tetrahedron lying over each maximal rectangle. Color the edges according to the sign of their slopes.





From pseudo-Anosov flows to veering triangulations

Glue up the faces according to how the maximal rectangles overlap



$\rightsquigarrow$ Veering triangulation on $\widetilde{N} \backslash \widetilde{\mathcal{C}}$.

Quotient by $\pi_1 N$ to get a veering triangulation on $N \backslash \mathcal{C}$.

Correspondence theorem

Theorem (Agol-Gueritaud, Schleimer-Segerman, Landry-Minsky-Taylor, Agol-T.)

This construction defines a bijection

$$\{(\textit{Pseudo-Anosov flow } \phi, \textit{ Orbits } \mathcal{C})\}/\textit{Orbit equivalence}$$

$$\updownarrow$$

$$\{(\textit{Veering triangulation, Filling slopes})\}/\textit{Homeomorphism}$$

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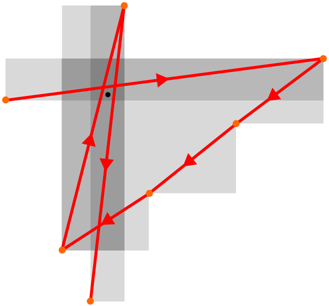
One application:

Theorem (T.)

Any transitive pseudo-Anosov flow admits a Birkhoff section with at most two boundary components.

Winding edge paths

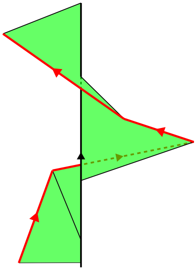
A sequence of oriented red edges of $\tilde{\Delta}$ is a **winding edge path** if their projection winds around a point of $\mathcal{O}$.



Such an edge path can be obtained by lifting a cyclic sequence of red edges of Δ. In this case the edge path is periodic, hence the point it winds around is periodic as well.

Helicoids

A winding edge path bounds a helicoidal partial section $\widetilde{H}$.



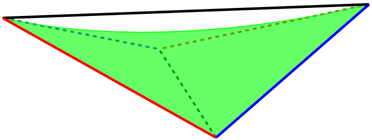
In the periodic case, this quotients down to an (immersed) partial section H with boundary $\partial_v H \cup \partial_h H$.
The vertical boundary $\partial_v H$ lies along a closed orbit while the horizontal boundary $\partial_h H$ is the cyclic sequence of red edges.

Equatorial squares

Strategy:

1. Find a partial section Q that intersects every orbit, such that $\partial_h Q$ consists of red edges.
2. Extract an edge sequence from $\partial_h Q$ to construct helicoid H.
3. Form immersed Birkhoff section $Q \cup H$, then apply Fried's resolution trick to get a honest Birkhoff section.

One approach: Take a positive linear combination of equatorial squares. Each orbit intersects a tetrahedron thus intersects a square.



Birkhoff section with two boundary components

We can add blue edge sequences into the picture:
Construct partial sections $Q_{R/L}$ such that $\partial_h Q_{R/L}$ consist of red/blue edges.
Construct $H_{R/L}$ and apply Fried's resolution to $Q_R \cup H_R \cup Q_L \cup H_L$ to get a Birkhoff section S.
Using freedom of how edge sequences in Δ can be lifted to edge paths in $\widetilde{\Delta}$, we can ensure that S meets orbits in $\mathcal{C}$ in multiples of the meridian. Filling in these meridians shows

Theorem (T.)

Any transitive pseudo-Anosov flow admits a Birkhoff section with two boundary components (namely, $\partial_v H_{R/L}$).