Topological HH + Witt vectors (Martin Speirs)

$k = \mathbb{Z}, \mathbb{F}_p$

Motivation: A ring $\rightsquigarrow$

$$\begin{tikzcd}
K(A) \arrow[d, "\text{cyclotomic tr}"] & & \\
TC(A) \arrow[r] \arrow[d] \arrow[dl, dotted] \arrow[drr, "tr"] & & HH(A)^{hS^1} \arrow[d] \\
THH(A)^{C_r} \arrow[r, hook] & THH(A) \arrow[r] & HH(A) \circlearrowleft S^1
\end{tikzcd}$$

$C_r \subseteq S^1 \circlearrowleft$

HH recall A (dg) ~~alg~~ algebra$/k$.

$$HH(A) = A \overset{L}{\underset{A \overset{L}{\otimes} A}{\otimes}} A$$

Thm (HKR) A k-algebra smooth

$$HH_*(A) \cong \Omega^{-*}_{A/k}.$$

Def A ring spectrum

$$THH(A) = A \overset{L}{\underset{A \wedge A}{\wedge}} A$$

Spectra

Def A spectrum E is a sequence $E_0, E_1, \dots$ w/ maps $\Sigma E_n \to E_{n+1}$

analogy:

classical alg	spectral alg
Set	Top / sSet
Ab	Spectra
$(\otimes_{\mathbb{Z}}, \mathbb{Z})$	$(\wedge, \mathbb{S})$

~~sp~~ smash prod, sphere spect.

Eilenberg–MacLane functor

$H: Ab \to Sp.$

$A \longmapsto HA = (K(A,0), K(A,1), \dots)$

(on connective sp) fully faithful functor has left adjoint π_0

$\pi_0(E) = \operatorname{colim} \pi_k E_k.$

Def $\pi_n(E) = \operatorname{colim}_k \pi_{n+k}(E_k)$ (even if $n < 0$).

Spectra has - lax monoidal structure. $(HA \wedge HB \to H(A \wedge B))$

- A ring structure means HA also monoidal in Sp

$\pi_* \mathbb{S}$ = stable homotopy gp of spheres
$= \mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, \mathbb{Z}/24, 0, \ldots$

$\mathbb{S} \neq H\mathbb{Z}$!
(but is after $\otimes \mathbb{Q}$.)

$\pi_* HA = A, 0, 0, \ldots$

$H\mathbb{F}_p \wedge H\mathbb{F}_p \neq H(\mathbb{F}_p \otimes \mathbb{F}_p)$ in fact $\pi_*(H\mathbb{F}_p \wedge H\mathbb{F}_p)$
<u>lax</u> monoidal   dual Steenrod alg.

<u>Another def</u>. A ring spectrum ($= HA$)

$$THH(A) = \left| [n] \longmapsto A^{\wedge \Delta^1[-]/\partial\Delta^1[-]} \right|$$

$$= \left| A \leftleftarrows A \wedge A \;\substack{\leftarrow\\ \leftarrow\\ \leftarrow}\; A \wedge A \wedge A \;\cdots \right|$$

<u>Thm</u> (Connes) $THH(A)$ has S^1-action

<u>Thm</u> (Bökstedt periodicity)
$THH(\mathbb{F}_p) = \mathbb{F}_p[\sigma]$ $|\sigma| = 2$
$THH(\mathbb{Z}) =$

$THH(A)$ has special type of S^1-action: <u>cyclotomic</u>.

X top space   Consider $\mathcal{L}X = \mathrm{Maps}(S^1, X) \circlearrowleft S^1$

$C_r \subseteq S^1$ finite cyclic subgp.

$$(\mathcal{L}X)^{C_r} \xrightarrow[\text{equiv.}]{\simeq} \mathcal{L}X$$

$$\circlearrowleft \quad S^1/C_r \xleftarrow{\simeq} S^1$$

Def A cyclotomic spectrum T is a S^1-spectrum w/ maps $\forall\, C_r \subseteq S^1$

$$\Phi^{C_r}T \xrightarrow{\simeq} T$$

"geometric fixed pts" $\Phi^G : Sp^G \to Sp$ monoidal

$$\Phi^G(\Sigma^\infty_G X) = \Sigma^\infty(X^G)$$

+ compatibility.

Other fixed pts: E G spectrum, $E = (E_0, E_1, \dots)$ $E^G = (E_0^G, E_1^G, \dots)$

Fact There is a map

$$(-)^G \longrightarrow \Phi^G$$

Thm (Bökstedt, Hsiang, Madsen) $THH(A)$ admits cyclotomic structure

Consequence:

Restriction maps: T cyclotomic

$$\underbrace{T^{C_{rs}} = (T^{C_r})^{C_s} \longrightarrow (\Phi^{C_r}T)^{C_s} \xrightarrow{\simeq} T^{C_s}}_{r\text{th restriction map}}$$

$THH(A)$, $THH(A)^{C_r}$ form diagram. (e.g. fix p prime. Study one prime at a time)

$$THH(A) \leftarrow \cdots \leftarrow THH(A)^{C_{p^{n+1}}} \underset{F}{\overset{R\ (\text{restrict})}{\rightrightarrows}} THH(A)^{C_{p^n}} \to \cdots \to THH(A)^{C_p}$$

F: Frobenius (inclusion)

$$\begin{tikzcd} \mathrm{THH}(A)^{C_{p^{n+1}}} \arrow[d, shift right=3, "R"'] \arrow[d, "F = \text{Frobenius (inclusion)}"] \\ \mathrm{THH}(A)^{C_{p^n}} \arrow[u, bend left=60, "V"] \end{tikzcd}$$

transfer map

<u>Witt vectors</u> (p-typical)

$W^{(p)} : \mathrm{CRing} \to \mathrm{CRing}$

$W^{(p)}(A) = A^{|\mathbb{N}|}$ w/ exotic ring structure.

e.g. $W^{(p)}(\mathbb{F}_p) = \mathbb{Z}_p$

Finite length Witt vectors

$W_n^{(p)}(A) = A^{n+1}$ w/ exotic ring structure.

$$\begin{tikzcd} W_{n+1}(A) \ni (a_0, \dots, a_n) \arrow[d, shift right=3, "R"'] \arrow[d, "F = \text{R}\circ\text{Frobenius}"] & (0, a_0, \dots, a_{n-1}) \\ W_n(A) \quad (a_0, \dots, a_{n-1}) \arrow[u, bend left=60, "V"] \arrow[ur, "V"'] & \end{tikzcd}$$

$$\begin{tikzcd} \arrow[d, "R"'] \\ \; \end{tikzcd}$$

<u>Thm</u> (Hesselholt Madsen)
A comm ring, $\pi_0 \mathrm{THH}(A)^{C_{p^n}} = W_{n+1}^{(p)}(A)$ $\left(\begin{array}{l} W = \lim_R W_n \\ \mathrm{THH}^{S^1} = \lim \mathrm{THH}^{C_{p^n}} \end{array} \right)$

<u>Thm</u> (H)
A $\mathbb{F}_p$-alg, smooth

$$\pi_* \mathrm{THH}(A)^{C_{p^n}} \simeq W_n \Omega^{\bullet}_{A/\mathbb{F}_p}[\sigma]$$