Tannaka duality for derived stacks (Ben Gammage)

Ref: Lurie, Tannak. dual. for geom. stacks. (non-derived)
Lurie, DAG 8 (derived)
Bhatt, HalpernL. Tannaka duality revisited

Classically, reconstruct G from $Rep(G)$?
rep ring $R(G)$: no
cat $Rep(G)$: no
$\otimes$-cat $Rep(G)$: yes.

Thm G affine alg. gp. Then $G = Aut^{\otimes}(Rep(G) \to Vect)$

Thm $Map(pt, BG) \to Fun^{\otimes}(QC(BG), QC(pt))$
$f \mapsto f^*$

is an equivalence of groupoids.

Q: For what sort of stacks (e.g. $X = BG$) do we expect a result of this form?

A: Geometric stacks: an alg. stack if quasicompact, and (representable) & affine diagonal.

Examples: BG G aff alg. gp, X/G quasicpt sch alg. gp.
non-ex: BA A abelian variety

why are these nice? X quasicompact $\Rightarrow \operatorname{Spec}(A) \to X$ cover
$\overset{\shortparallel}{U}$
Δ affine $\Rightarrow$ $U \times_X U$ affine.

So, X represented by gpd $U \times_X U \rightrightarrows U$.
i.e. $\Gamma(-)$: {geom stacks} $\hookrightarrow$ {Hopf algebroids}

(Note: $X/G \simeq Y/H$ where Y affine for all X, G... so X/G geometric)

Thm: "geometric stacks have enough quasi-coherent sheaves"

Thm X geom stack. $D^+(QC(X)) \cong D^+_{qc}(\mathcal{O}_X\text{-Mod})$

Idea: show $QC(X)$ has enough injective objects.

Take smooth cover from affine $U \xrightarrow{p} X$ let $N \in QC(X)$

inj res ~~$\exists$~~

$p^*N \hookrightarrow I$ ⤳ $N \hookrightarrow p_* I$.

② Exts w/ injective vanish. $Ext(M, p_* I)$ acyclic

Main theorem: S affine. X geom stack

$Hom(S, X) \longrightarrow Hom^{\otimes, L}(QC(X), QC(S))$ is fully faithful

with essential image functors preserving a flatness condition

Derived version: same, w/ X = spectral stack w/ affine diag.

essential image = functors preserving connective objects + flat objects.

Bhatt, HL: weaken "affine" to "quasiaffine"

Thm X spectral stack w/ quasiaffine diagonal, $D(X)$ compactly generated. Then, (-----), essential image = functors preserving ~~compact~~ connective objects.

Idea: How to describe quasi-affine maps $\to X$ in terms of $D(X)$.

Thm 1) $QAff^{op}_{/X} \longrightarrow CAlg(D(X))$ fully faithful symm monoidal

$Y \longmapsto f_* \mathcal{O}_Y$.

2) $\mathrm{Op}_{\mathrm{fc}/X}^{op} \longrightarrow \mathrm{CAlg}(D(X))$ has essential image bounded above algebras A which are compact localizations of $\mathcal{O}_X$

3) essential image of $\mathrm{QAff}^{op}_{/X} \to \mathrm{CAlg}(D(X))$ is compact-localizations of connective A'

Def^n $A \to B$ comm. alg. is localization if $B \otimes_A B \xrightarrow{\sim} B$.
"compact" = compact object

Applications

① algebraization of formal points

Thm A I-adically complete ring. X = nice $\overset{\text{classical}}{\text{DM}}$ stack.

Then $X(A) = \lim X(A/I^n)$.

Pf idea: $X(A) = \mathrm{Fun}^{L,\otimes}(\mathrm{QC}(X), \mathrm{QC}(A)) \Rightarrow \mathrm{Perf}(A) \simeq \lim \mathrm{Perf}(A/I^n)$.
$\mathrm{Perf}(X)$, $\mathrm{Perf}(A)$

?

② strange pushouts: can glue along proper birational maps

Ex: $C_1, C_2 \subseteq \mathbb{P}^3$ curves. $Z_i = \mathrm{Bl}_{C_i}(\mathbb{P}^3)$
$X_i = \mathrm{Bl}_{\tilde{C}_i}(Z_i)$
$\tilde{C}_i$ proper transform

$$\begin{array}{ccc} X_0 \simeq X_1 & \longrightarrow & Z_0 \\ \downarrow & & \downarrow \\ Z_1 & \longrightarrow & \mathbb{P}^3 \end{array}$$