Shifted symplectic structures 2 (Alex Takeda) $\mathrm{Char}(k)=0$.

__Derived schemes__ X has:
- $\mathcal{O}_X$ sheaf of cdga $_{\leq 0}$
- ∞-category of ~~[illegible]~~ $\mathrm{Lqcoh}(X)$, f^*, f_*
- $\mathbb{L}$ cotangent complex $\in \mathrm{Lqcoh}(X)_{\leq 0}$

  $E \in \mathrm{Lqcoh}(X)_{\leq 0}$ $\quad X[E] = \mathrm{Spec}_{\mathcal{O}_X}(\mathcal{O}_X \oplus E)$.

  $\mathbb{L}_X$ defined by $\mathrm{Map}_{dSch}(X[E], X) \simeq \mathrm{Map}_{\mathrm{Lqcoh}}(\mathbb{L}_X, E)$.
- $f: Y \to X$ $\quad \mathbb{L}_{X/Y} = \mathrm{fib}(f^*\mathbb{L}_X \to \mathbb{L}_Y)$.

__Derived stack__ X has:
- $\mathcal{O}_X$ sheaf of cdga
- $\mathrm{Lqcoh}(X)$
- $\mathbb{L}_X \in \mathrm{Lqcoh}(X)$ may have pos. part $\leq n$ for n-Artin stack.

__Forms__: X derived stack

__p-forms__: $\Omega^p_X = \mathrm{Sym}^p_{\mathcal{O}_X}(\mathbb{L}_X[1])[-p]$ " $= \Lambda^p_{\mathcal{O}_X}\mathbb{L}_X$ "

Notation: $\Omega^p(X,n) = \mathcal{H}^n(X, \Omega^p_X)$.

$$X \begin{cases} \text{derived scheme: } \mathcal{H}^{>0}(X, \Omega^p_X) = 0 \\ \text{stack (underived): } \mathcal{H}^{<0}(X, \Omega^p_X) = 0. \end{cases}$$

__Closed forms__: $\Omega^{p,cl}_X = \Omega^p_A \to \Omega^{p+1}_A \to \cdots$ $\quad$ __Warning__: not injective in any sense.

$\downarrow$ canonical projection

Ω^p_X

Closed p-form of degree n := sequence of $\{\omega_i\}_{i\geq 0}$ s.t. $\begin{cases} \omega_i \in \Omega^{p+i}(X, n-i) \\ d_{dR}(\omega_i) = d_{int}(\omega_{i+1}). \end{cases}$

$d_{dR}: \Omega^p(X,n) \to \Omega^{p+1}(X,n)$

Def^n shifted symplectic structure

n-symplectic structure is $\omega \in \Omega^{2,cl}(X,n)$

$$\begin{array}{c} \omega \in \Omega^{2,cl}(X,n) \\ \big\downarrow_2 \\ \tilde\omega \in \Omega^{2}(X,n) \end{array} \qquad \begin{array}{l} \tilde\omega : k \to \Lambda^2_{\mathcal{O}_X} \mathbb{L}_X[n] \\ \Theta_{\tilde\omega} : \mathbb{T}_X \to \mathbb{L}_X[n]. \end{array}$$

Def^n Lagrangian structure on $f: Y \to X$,

X has n-sympl. str. ω is a homotopy equivalence

$h: f^*\omega \xrightarrow{\sim} 0$ such that the induced map

$\Theta_{\omega,h} : \mathbb{T}_Y \xrightarrow{\sim} \mathbb{L}_f[n-1]$. is equivalence

$$\left(\begin{array}{l} \mathbb{T}_Y \to f^*\mathbb{T}_X \xrightarrow{f^*\Theta_\omega} f^*\mathbb{L}[n] \to \mathbb{L}_Y[n]. \\ \mathbb{T}_Y \to \text{fiber}\,(f^*\mathbb{L}_X[n] \to \mathbb{L}_Y[n]) \text{ is homotopic to } 0. \\ \qquad\qquad\qquad\quad \mathbb{L}_f[n-1] \end{array} \right)$$

Theorem:
$$\begin{array}{ccc} Y \times_X Z & \longrightarrow & Y \\ \big\downarrow & & \big\downarrow \text{Lag} \\ Z & \xrightarrow[\text{Lag}]{} & X \end{array}$$
n-sh. s.s.

$Y \times_X Z$ has canonical $n-1$ s.s..

Examples · quotient stacks Y = affine derived scheme $= \mathrm{Spec}(A)$.

$\circlearrowleft$ G = aff. alg. gp, reductive

$X = Y/G$ - 1-Artin stack.

$L_{qcoh}(X) \cdot L_{qcoh}(Y)$ w/ compatible G-action. (G-equiv.)

get $\mathcal{O}_Y \otimes \mathfrak{g} \to \mathbb{T}_Y$

Fact: $\mathbb{L}_X = \text{fib}(\mathbb{L}_Y \to \mathcal{O}_X \otimes \mathfrak{g}^*)$

$\Omega^p_X = \left(\bigoplus_{i+j=p} \Lambda^i_Y \mathbb{L}_Y \otimes \mathrm{Sym}^j(\mathfrak{g}^*)[-j] \right)$

Take $Y = pt$. $X = BG$. $\mathbb{L}_X = \mathfrak{g}^*[-1]$. (coadjoint rep)

So, $\Omega^p_X \simeq (\mathrm{Sym}^p \mathfrak{g}^*)^G[-p]$
$\mathbb{H}(X, \Omega^p_X)$

$(-)^G$ no higher coh.

Fact: G reductive, ~~[illegible]~~ ~~[illegible]~~, so

only degree p forms on X.

Fact: G reductive, $\Omega^{p,cl} \longrightarrow \Omega^p$ is quasi-iso.

i.e. closed form = forms

2-shifted sympl st. on BG $\Leftrightarrow$

~~[illegible]~~ $\mathrm{Sym}^2(\mathfrak{g}^*)^G$ i.e. Killing forms.

Ex X usual sch. $T^*[n]X$ has n-shifted s.s.

$f \in H^n(X, \mathcal{O}_X)$

$df : X \to T^*[n]X$ Lagrangian.

$$\begin{tikzcd} \mathbb{R}\mathrm{Crit}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "df"] \\ X \arrow[r, "0"] & T^*[n]X \end{tikzcd}$$

$\mathbb{R}\mathrm{Crit}(f)$ has $n-1$ shifted s.s.

$n=0$ case; $X = \mathrm{Spec}(A)$, $\mathbb{R}\mathrm{Crit}(f) = \mathrm{Spec}(B)$ cdga$_{\le 0}$

$B = \mathrm{Sym}_{\mathcal{O}_X}(T[1])$ with df.

$\omega = \sum d_{dR}(x_i) d_{dR}(\xi^i)$

$\delta(\xi_i) = \partial f / \partial x^i$ etc