Shifted symplectic structures (Alex Takeda)

PTVV 1111.3209
D. Calaque, Three lectures on der.-sympl. geom + TFT
P. Safronov, 1311.6429 v2 (different shift conventions)

Derived stacks /k char k = 0.

scheme: k-Alg → Set (+ descent)
(art) Stack: k-Alg → Simplicial Sets (+ descent) e.g. BG, X/G
derived scheme: $cdga_{\leq 0}$ → sSet + ... "repr. covers by affines + stuff"
derived stack. $cdga_{\leq 0}$ → sSet + ...

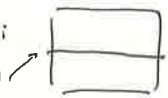
Derived scheme: $(X_0, \mathcal{O}_X)$ ← sheaf of cdgas, s.t. $H^0(\mathcal{O}_X) = \mathcal{O}_{X_0}$, $H^i(\mathcal{O}_X)$ q.coh.

$|\xi| = -1,\ d(\xi) = y$ $\quad \dfrac{k[x,y,\xi]}{y} \simeq \dfrac{k[x,y]}{y} \otimes^L_{k[x,y]} k[x,y] \quad k[x,y]/y$

Example:

$\mathbb{A}^2$

$\mathbb{A}^1$



$$\begin{array}{ccc} \mathbb{A}^1 \overset{R}{\cap} \mathbb{A}^1 & \to & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \to & \mathbb{A}^2 \end{array} \qquad \begin{array}{ccc} \dfrac{k[x,y]}{y} & \xleftarrow{} & k[x,y] \\ \uparrow & & \uparrow \\ \dfrac{k[x,y]}{y} & \leftarrow & k[x,y] \end{array}$$

i.e. $\mathbb{A}^1 \overset{R}{\underset{\mathbb{A}^2}{\cap}} \mathbb{A}^1 \simeq \operatorname{Spec} k[x, \overset{-1}{\xi}]$

Derived critical locus

$f: X \to \mathbb{A}^1$

$$\begin{array}{ccc} R\,Crit(f) & \to & X \\ \downarrow & & \downarrow (id, df) \\ X & \to & T^*X \end{array}$$

example: $X = \mathbb{A}^2 = \operatorname{Spec} k[x_1, x_2]$.
$f(x_1, x_2) = x_1^2$

$\mathcal{O}_{R Crit(f)} = \mathcal{O}_{X} \underset{df}{\overset{L}{\otimes}}{}_{\mathcal{O}_{T^*X}} \mathcal{O}_X$

$= \dfrac{k[x_1,x_2,p_1,p_2]}{(p_1 - 2x_1, p_2)} \otimes_{k[x_1..p_2]} \dfrac{k[x_1,x_2,p_1,p_2]}{(p_1,p_2)}$

$df = 2x_1 \partial x_1 \longrightarrow$

"

$k[x_1, x_2, p_1, p_2, \overset{-1}{\xi_1}, \overset{-1}{\xi_2}]$

$d(\xi_i) = p_i$

$\simeq k[\overset{0}{x_1}, \overset{0}{x_2}, \overset{-1}{\xi_1}, \overset{-1}{\xi_2}]$ $\quad d(\xi_1) = 2x_1$, $\quad d(\xi_2) = 0$

x_2 is direction in classical locus

ξ_2 is "derived" p_2. cotangent dir.

$\simeq k[\overset{0}{x_2}, \overset{-1}{\xi_2}]$

In general, $\mathcal{O}_{\mathbb{R}\mathrm{Crit}(f)} = \mathcal{O}_X \otimes^{\mathbb{L}}_{\mathcal{O}_{T^*X}} \mathcal{O}_{X_{df}} = \left(\mathrm{Sym}_{\mathcal{O}_X}(T_X[1]),\ d : d(\xi_i) = \frac{df}{dx_i}\right)$

$\left(\mathrm{Sym}_{\mathcal{O}_X}(\underset{\xi}{T_X[1]} \oplus \underset{p}{T_X}),\ d = T_X \xrightarrow{id} T_X\right)$

Differential forms

cdga

On affine $X = \mathrm{Spec}(A)$

1-forms: $\Omega^1_X = \mathbb{L}_X = \Omega^1_{\tilde{A}}$

Kähler diff. of $\tilde{A} \to A$ cofibrant res.

$= I/I^2$

$= A\langle ds\rangle/\ldots$

want p-forms: $\Omega^p_X = \Lambda^p \Omega^1_X$. classically.

more natural: $\Omega^p_X = \mathrm{Sym}^p_X(\Omega^1_X[-1])[p]$ in derived setting.

Q: Why $[-1], [p]$, not $[1], [-p]$?
A: doesn't matter

closed forms.

Def: (de Rham alg)

$dR(A) = \bigoplus_{n \geq 0} \Omega^n_A[-n]$.

differential: $d + d_{dR}$.

$\Omega^p \to \Omega^{p+1}$

Filtration:

$F^p dR(A) = \bigoplus_{n \geq p} \Omega^n_A[-n]$

Def: closed p-forms on $\mathrm{Spec}(A)$: $\Omega^{p,cl}_X = F^p dR(\tilde{A})[p] = \bigoplus_{n \geq p} \Omega^n_A[p-n]$

A: classical k-alg, smooth.

$\Omega^{p,\bullet}_A$ in degree 0.

$$\Omega^{p,cl}_A = \overset{0}{\Omega^p_A} \xrightarrow[d_{dR}]{} \overset{1}{\Omega^{p+1}_A} \longrightarrow \cdots$$

$H^0(\Omega^{p,cl}_A)$ = sheaf of closed p-forms in A. so, quasi-iso to $\Omega^{p,cl}_A$.

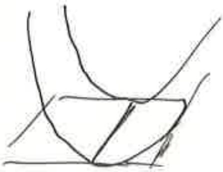
$H^i(\Omega^{p,cl}_A) = 0$ (? ~~...~~ if simply connected... true in analytic case)

Def: Shifted symplectic structure on X

$\omega \in \Omega^{2,cl}_X$ of coh. degree n. giving quasi-iso

~~with~~ ~~...~~ $k \xrightarrow{\omega} \Lambda^2 \mathbb{L}_X[n]$

$\mathbb{T}_X \longrightarrow \mathbb{L}_X[n]$

Ex: 

$\mathcal{O}_{\mathbb{R}Crit(f)} = k[x_i, \xi_i]$

$\mathbb{L}_X =$