What is an ind-coherent sheaf?

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0.1 Introduction

All algebras in this note will be considered over a field k of characteristic zero (an assumption made in [Ga:IC]), so that we may use dg algebras rather than a more complicated model. Further, all rings will be noetherian; there is a discussion in [Ga:IC] Section 2 on what happens in the non-noetherian case for interested readers. All gradings use *cohomological* conventions (i.e. differentials increase degree), differing from [Pr:IndLoop].

1 Homotopy category of injectives

Definition 1. We define some dg-enhanced triangulated categories. Assume X is a noetherian scheme.

- Let $\mathrm{QCoh}(X)$ denote the unbounded derived category of quasicoherent sheaves on X, localized with respect to quasi-isomorphisms.
- Let $\mathrm{Coh}(X)$ denote the subcategory of $\mathrm{QCoh}(X)$ of bounded derived category of coherent sheaves, i.e. the full subcategory of $\mathrm{QCoh}(X)$ whose objects have bounded and coherent cohomology.
- Let $\mathrm{Perf}(X)$ denote the subcategory of $\mathrm{QCoh}(X)$ of perfect complexes, i.e. the full subcategory of $\mathrm{QCoh}(X)$ which are locally quasi-isomorphic to a bounded complex of free modules of finite rank. We also have $\mathrm{Perf}(X) \subset \mathrm{Coh}(X)$.
- Let $\mathrm{IndCoh}(X)$ denote the homotopy category of (possibly unbounded) injective complexes on X. Note that we do not localize with respect to quasi-isomorphisms.

We also have a canonical (left) *completion* functor $\Psi : \mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$ which takes an injective complex and considers it as an object in the derived category. Note that this is well-defined since homotopy equivalences are quasi-isomorphisms.

Example 2 (The difference between QCoh and IndCoh.)**.** Let $R = k[x]/x^2$. Since R is a Frobenius ring, projectives and injectives are the same. There is an injective complex in $\mathrm{IndCoh}(\mathrm{Spec}\, R)$

$$\cdots \longrightarrow R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \xrightarrow{x} R \longrightarrow \cdots$$

whose completion is zero in $\mathrm{QCoh}(X)$.

Theorem 3 (Krause, [Kr])**.** *The triangulated category* $\mathrm{IndCoh}(X)$ *is compactly generated by* $\mathrm{Coh}(X)$*. In particular, the functor* Ψ *induces an equivalence* $\mathrm{IndCoh}(X) \simeq \mathrm{Coh}(X)\text{-mod}$.

Proof. Since coherent sheaves are compact objects in the abelian category of quasicoherent sheaves, they are compact objects in the homotopy category of complexes, and therefore the homotopy category of injectives. To show that $\mathrm{Coh}(X) \subset \mathrm{IndCoh}(X)$ generates, i.e. that if $\mathrm{Hom}_{\mathbf{K}(\mathrm{QCoh}(X))}(\mathcal{F}^\bullet, \mathcal{I}^\bullet)$ for every $\mathcal{F}^\bullet \in \mathrm{Coh}(X)$, then $\mathcal{I}^\bullet \simeq 0$ (homotopy equivalent). From now let us restrict to the affine case for simplicity. In the case when there is n such that $H^n(\mathcal{I}^\bullet) \neq 0$, (say $n = 0$ for simplicity), we can just take $M \to Z^0(\mathcal{I}^\bullet) \to \mathcal{I}^\bullet$, where $M \to Z^0(\mathcal{I}^\bullet)$ is a map which induces a nonzero map $M \to H^0(\mathcal{I}^\bullet)$. In the case when $\mathcal{I}^\bullet$ is acyclic, we choose an n such that $Z^n(\mathcal{I}^\bullet)$ is not injective (say $n = -1$, by shifting) – this is possible because otherwise the complex would split and is evidently homotopy equivalent to zero. By Baer's criterion[1], we can choose an ideal I such that $\mathrm{Ext}^1(A/I, Z^{-1}(\mathcal{I}^\bullet)) \neq 0$. In particular, there is a map $A/I \to \mathcal{J}^\bullet[1]$, where $\mathcal{J}^\bullet$ is an injective resolution of $Z^{-1}(\mathcal{I}^\bullet)$. We can splice the injective resolution $\mathcal{J}$ in the degrees ≥ -1 in $\mathcal{I}^\bullet$ to obtain a nonzero map in H^0.

□

2 Ind-coherent sheaves

Remark 4 (Ind-completion, compact objects)**.** A functor is *continuous* if it commutes with filtered colimits. By the ∞-categorical Adjoint Functor Theorem, a functor is continuous (between presentable categories) if and only if it admits a right adjoint. An object X is *compact* if the functor $\mathrm{Hom}(X, -)$ is continuous. The compact objects of a category $\mathbf{C}$ form a small subcategory, which we denote $\mathbf{C}^\omega$. A right adjoint is continuous if and only if the left adjoint preserves compact objects.

The Yoneda embedding of an ∞-category $\mathbf{C} \to \mathrm{Psh}(\mathbf{C})$ is fully faithful. We can define the *ind-completion* to be the smallest full subcategory of $\mathrm{Psh}(\mathbf{C})$ containing the essential image of $\mathbf{C}$ and also closed under filtered colimits. It follows that the morphisms are given by $\lim_{d\in D} \mathrm{colim}_{d'\in D'} \mathrm{Hom}_{\mathbf{C}}(F(d), F'(d'))$. A category $\mathbf{C}$ is *compactly generated* if $\mathrm{Ind}(\mathbf{C}^\omega) = \mathbf{C}$.

There is a correspondence between compactly generated categories and the corresponding compact categories. There is an equivalence of categories between small stable ∞-categories with exact functors and (large) compactly generated stable ∞-categories with functors preserving compact objects.

Finally, the ind-completion of a dg-category $\mathbf{C}$ is (derived)[2] Morita equivalent to the category $\mathbf{C}$-mod. For calculations this is often the most useful description. A set of (classical) *generators* in a pretriangulated dg category is a set of objects for which the smallest thick subcategory (i.e. closed under shifts, cones, and direct summands) of the homotopy category is the entire homotopy category. A set of objects X_i are (weak) *generators* if $\mathrm{Hom}(X_i, -) = 0$ implies $- = 0$. A set of objects are weak generators for $\mathbf{C}$ if and only if they are classical generators for $\mathbf{C}^\omega$ and $\mathbf{C} = \mathrm{Ind}(\mathbf{C}^\omega)$, i.e. it is compactly generated (see Stacks Project, 13.34.6).

For this talk we will focus only on the case of an affine dg scheme.

Definition 5. An *affine derived scheme* is a connective (i.e. concentrated in nonpositive degrees) dg algebra. The category of affine derived schemes is the opposite category to the category of connective dg algebras. A dg algebra A is called *eventually coconnective* if $\pi_n(A) := H^{-n}(A) = 0$ for all $n \geq N$ for some N. A dg algebra A is *noetherian* if $H^0(A)$ is noetherian and $H^i(A)$ is a finitely generated $H^0(A)$-module for all i.

Definition 6 (Quasicoherent sheaves, coherent sheaves, perfect complexes)**.** Let $X = \mathrm{Spec}(R)$ be an affine derived scheme. The category $\mathrm{QCoh}(X)$ is defined to be the category of dg A-modules.

A dg module over R is *free* if it is the direct sum of shifts of R. A dg module is *semifree* if it has an increasing filtration of dg submodules which is complete (i.e. the colimit is the entire module) and whose subquotients are free. Note that if R is a classical ring, we can filter by cohomological degree so that the subquotients are just free modules concentrated in a single degree; furthermore, surjections into free modules split, so in fact each of these modules are a sum of free modules. In particular, a perfect complex in the classical sense is just a semifree module whose filtration is finite.

The category $\mathrm{Perf}(X)$ is defined to the compact objects of $\mathrm{QCoh}(X)$. For X an affine derived scheme, the perfect complexes are those quasi-isomorphic to a semi-free modules with finite filtration or a retract of such modules. The coherent subcategory $\mathrm{Coh}(X)$ is defined to be the subcategory consisting of complexes whose cohomology is coherent over $\pi_0(R)$ and which have bounded cohomological amplitude. When A is eventually coconnective and noetherian, $\mathrm{Coh}(X)$ is the subcategory of A-modules which are finitely generated.

Remark 7. Let $X = \mathrm{Spec}\, k[u]$ where $|u| = -2$. Then in particular, perfect $k[u]$-modules are not coherent since they have unbounded cohomological support, and we cannot define the functor Ξ. In particular, $\mathrm{Perf}(X)$ consists of finitely generated $k[u]$-modules (the argument is similar to the argument that every graded $k[x]$-module has a finite graded free resolution) and $\mathrm{Coh}(X)$ consists of finitely generated u-torsion modules.

[1]This says that a module M is injective if and only if $\mathrm{Ext}^1_R(R/I, M) = 0$ for all (left) ideals I

[2]In this note, when we write A-mod, we always mean the dg category of A-modules localized with respect to quasi-isomorphisms.

Definition 8. Let X be a quasicompact affine noetherian derived scheme. The *category of ind-coherent sheaves*, denoted $\mathrm{IndCoh}(X)$ is defined to be the ind-completion of the category of *coherent sheaves.* The functor Ψ : $\mathrm{IndCoh}(X) \to \mathrm{QCoh}(X)$ is defined by taking the ind-completion of the exact functor $\mathrm{Coh}(X) \to \mathrm{QCoh}(X)$. For non-noetherian derived schemes, more care is needed to define coherent sheaves; a treatment can be found in [Ga:IC].

Remark 9 (Warning)**.** For a non-quasicompact derived scheme X, the right way to define $\mathrm{IndCoh}(X)$ is via a Kan extension along maps of quasicompact affine derived schemes into X. Done this way, it is not true that $\mathrm{IndCoh}(X) \simeq \mathrm{Ind}(\mathrm{Coh}(X))$. For example, take $X = \mathbb{N}$. By some general formalism, the formation of ind-coherent sheaves should commute with colimits of closed embeddings; using this principle, we expect $\mathrm{IndCoh}(X)$ consists of finite-dimensional vector spaces indexed by $\mathbb{N}$ with finite support on $\mathbb{N}$. However, $\mathrm{Ind}(\mathrm{Coh}(X))$ does not have this finite support condition.

Example 10. Recall that when $R = k[x]/x^2$, the skyscraper module $k_0 = R/x$ has an infinite injective resolution $0 \to R \to R \to \cdots$. In particular, there is a short exact sequence of modules

$$0 \to k_0 \to R \to k_0 \to 0$$

which induces a nonzero map $k_0 \to k_0[1]$, which on injective resolutions looks like

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & R & \longrightarrow & R & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & R & \longrightarrow & R & \longrightarrow & R & \longrightarrow & \cdots \end{array}$$

Thus we have a filtered diagram of modules $k_0 \to k_0[1] \to k_0[2] \to \cdots$ whose colimit is the acyclic injective complex earlier discussed. This complex is *not* isomorphic to zero in $\mathrm{IndCoh}(X)$ because in each finite stage of the colimit the module is nonzero; more precisely, colimit has endomorphisms given by

$$\lim_n \operatorname*{colim}_m \mathrm{Hom}_R(k_0[n], k_0[m]) \simeq \lim_n \operatorname*{colim}_m k \otimes \delta_{m \geq n} = k$$

which is nonzero.

Remark 11. In general, one can compute the category in the following way. Let S be a dg scheme; the category $\mathrm{Coh}(S)$ is generated (in the sense of triangulated categories) by $\mathrm{Coh}(\pi_0(S))$ since every object of $\mathrm{Coh}(S)$ has a finite filtration such that the subquotients are objects of $\mathrm{Coh}(\pi_0(S))$ – in particular because objects of $\mathrm{Coh}(S)$ are required to have finite cohomological support. Since the ind-completion of a dg category can be identified with the category dg modules over that category, $\mathrm{IndCoh}(S)$ can be identified with dg-modules for the dg-category whose objects are objects of $\mathrm{Coh}(\pi_0(S))$, and whose Hom-spaces are given by $R\mathrm{Hom}_S(-,-)$ (i.e. the pushforward to S).

Further, if we can find (e.g. a finite set of) triangulated generators for $\mathrm{Coh}(\pi_0(S))$, this description can be made even simpler by considering only the dg-category whose objects are those generators. For example, if $\pi_0(S) = \mathrm{Spec}(R)$ is smooth and affine, R is a triangulated generator for $\mathrm{Coh}(\pi_0(S)) = \mathrm{Perf}(\pi_0(S))$.

Example 12. Let $A = k[\lambda]$ with $|\lambda| = 1$, a "shifted" version of the dual numbers. Since $\pi_0(A) = k$, and k-mod is generated by the one-dimensional vector space k, we have that $\mathrm{IndCoh}(A) = R\mathrm{Hom}_{k[\lambda]}(k,k)\text{-mod} = k[[u]]\text{-mod} = \mathrm{QCoh}(k[[u]])$ where $|u| = 2$. This is the "large" version of the Koszul duality between finite-dimensional (i.e. coherent) $C_\bullet(S^1;k)$-modules and perfect $C^\bullet(BS^1;k)$-modules.

Remark 13 (Relationship with homotopy category of injectives)**.** A *model* for $\mathrm{IndCoh}(X)$ should be a category $\mathbf{C}$ equipped with a fully faithful functor $\mathbf{C} \to \mathrm{Fun}(\mathrm{Coh}(X)^{op}, \mathbf{Ch})$ whose essential image coincides with the ind-completion of $\mathrm{Coh}(X)$. When X is a classical noetherian scheme, the homotopy category of injectives $\mathbf{K}_{inj}(X)$ is such a model with structure functor $\mathcal{I}^\bullet \mapsto \mathrm{Hom}(-, \mathcal{I}^\bullet)$. This Hom is the literal Hom-complex of complexes (i.e. we do not invert quasi-isomorphisms). Likewise, a model for $\mathrm{IndCoh}(X)^\vee$ should be a category equipped with a fully faithful functor $\mathbf{C} \to \mathrm{Fun}(\mathrm{Coh}(X), \mathbf{Ch})$. Neemen and Murfet [Ne] [Mu] prove that the mock category of projectives does the trick, with structure functor $\mathcal{P}^\bullet \mapsto \mathrm{Hom}(\mathcal{P}^\bullet, -)$.

3 Relating QCoh and IndCoh: t-structures, completion, and convergence

Most of the content of the following section can be found in [Pr:IndLoop] Section 4.

Remark 14 (For my sanity)**.** For my own sanity I need to write down what happens for the various truncations (or I will spend minutes figuring it out each time):

- The soft truncation $\tau^{\geq 0}$ takes the cokernel in degree zero and acts like a quotient functor. The maps go $\mathrm{id} \to \cdots \tau^{\geq -1} \to \tau^{\geq 0}$.
- The soft truncation $\tau^{\leq 0}$ takes the kernel in degree zero and acts like an subobject functor. The maps go $\tau^{\leq 0} \to \tau^{\leq 1} \cdots \to \mathrm{id}$.

Definition 15 (Left-complete t-structure)**.** Let $\mathbf{C}$ be a dg category with a t-structure. The (left) *completion* of a category is the limit $\lim_{n \in \mathbb{N}^{op}} \mathbf{C}^{\geq -n}$, where the maps $\mathbf{C}^{\geq -(n+1)} \to \mathbf{C}^{\geq -n}$ is the (soft) truncation functor. There is a natural functor $\mathbf{C} \to \mathrm{LeftComplete}(\mathbf{C})$. A category is called (left) *complete* if this functor is an equivalence. Spelled out more explicitly, this means that for every $X \in \mathbf{C}$, the natural map $X \to \lim \tau^{\geq -n} X$ is an equivalence.

Remark 16 (Right-completions)**.** We can similarly define the *right completeness* with respect to a t-structure, asking that $\operatorname{colim} \mathbf{C}^{\leq n} \to \mathbf{C}$ is an equivalence. We don't pay much attention to this case because most of the categories we are interested in will be automatically right-complete. For example, the category $\mathrm{QCoh}(X)$ is both left and right complete, and its finite subcategories are also certainly also so. Further, the ind-completion of a right-complete category is automatically right-complete (since colimits commute), but not necessarily left-complete.

Example 17. The derived category of chain complexes over k, $\mathrm{QCoh}(\mathrm{Spec}(k))$, is both left and right complete; one can check that for chain complexes $V^\bullet, W^\bullet$ that $\mathrm{Hom}(V^\bullet, W^\bullet) \simeq \prod_i \bigoplus_j \mathrm{Hom}(V^i, W^j)[j-i]$, and check compatibility with truncations. For an affine derived scheme, the global sections functor $\Gamma(\mathrm{Spec}(A), -)$ is t-exact, conservative, and preserves limits and filtered colimits, so $\mathrm{IndCoh}(\mathrm{Spec}(A))$ is left and right complete.

Example 18. In $\mathrm{IndCoh}(k[x]/x^2)$, the infinite acyclic complex earlier discussed (here $R = k[x]/x^2$)

$$\cdots \to R \to R \to R \to \cdots$$

is not left-complete; every truncation is quasi-isomorphic to zero, and in particular, homotopy equivalent to zero, since the complexes are injective and bounded below. In particular the limit is zero, not the above complex.

Both $\mathrm{QCoh}(X)$ and $\mathrm{IndCoh}(X)$ come with t-structures. Let us examine the t-structure on $\mathrm{QCoh}(X)$.

Example 19 (t-structures on A-modules)**.** Let A be a dg algebra. Recall two conditions for t-structures: $\mathrm{Hom}(\mathbf{C}^{\leq 0}, \mathbf{C}^{\geq 1}) = 0$ and that every object $X \in \mathbf{C}$ is an extension $X^{\leq 0} \to X \to X^{\geq 1}$.

1. If A is cohomologically concentrated in degree zero, then we have the usual t-structure, i.e. $M^\bullet \in (A\text{-mod})^{\leq 0}$ if and only if $H^i(M) \neq 0$ only when $i \leq 0$.
2. If A is connective (i.e. $A = A_{\leq 0}$ – this is required for affine dg schemes), letting $\theta : A\text{-mod} \to k\text{-mod}$ denote the forgetful functor, we can define a t-structure on A-mod by asking that θ be t-exact. Note that in this case, for any complex of A-modules $M^\bullet$, the soft truncation maps $X \to \tau^{\geq k} X$ and $X^{\leq k} \to X$ are maps of A-modules (i.e. "things below degree k" form a submodule, and "things above degree k" can be obtained via a quotient). Since θ is t-exact, and k-mod is left and right complete, the induced t-structure on A-mod is left and right complete. The heart of this t-structure is the abelian category of $\pi_0(A)$-modules.
3. If A is coconnective (i.e. $A = A_{\geq 0}$ – these examples are not dg schemes, but we include them for fun) this evidently fails. For example, take $A = k[u]$ with $|u| = 2$. Then there is no map of $k[u]$-modules $k[u]/u \simeq k \to k[u]$ nor a map $k[u] \to uk[u]$. In this case, we can define $(A\text{-mod})^{\geq 0} = \theta^{-1}(k\text{-mod}^{\geq 0})$, and we define $(A\text{-mod})^{\leq 0}$ to be the smallest subcategory generated by objects of the form $A \otimes V$ where $V \in (k\text{-mod})_{\leq 0}$ which is closed under colimits and extensions. Note that in this case, the map $\mathbf{C} \to \mathbf{C}^{\geq 1}$ is not the usual truncation

but the quotient by $\mathbf{C}^{\le 0} \to \mathbf{C}$. The truncation $\tau^{\le 0}$ by taking a semi-free resolution and then applying the usual truncation to the generators.

Since θ is right t-exact, we have that A-mod is right complete. It is not in general left complete, as we will shortly see.

Let us identify the heart of this t-structure. We claim that the heart can be identified with $\pi_0(A)$-modules, with the functor $M \mapsto M \otimes_{\pi_0(A)} A$ (note that since A is coconnective, $\pi_0(A)$ is a subalgebra of A). Clearly the objects $M \otimes_{\pi_0(A)} A$ are in $A\text{-mod}^{\ge 0}$; to see that they are in the ≤ 0, note that $M \otimes_{\pi_0(A)} A$ is generated in degree 0, so any semi-free resolution ends up with generators in degrees ≤ 0. This functor has a quasi-inverse which takes H^0.

4. Let us do an explicit example, with $A = k[u]$, $|u| = 2$. Then, for example, $A \in (A\text{-mod})^\heartsuit$ but A is not cohomologically concentrated in degree zero. The augmentation module k has semi-free resolution $k[u][-1] \to k[u]$, i.e. with generators in degree 0 and 1, so $k \in (k[u]\text{-mod})^{\le 1}$ but not $(k[u]\text{-mod})^{\le 0}$. Further, $k[u]$-mod is not left complete, since $k(u) \in (k[u]\text{-mod})^{\le k}$ for every k (since we can always choose generators in arbitrarily small degree), so in particular the maps in the limit diagram $\cdots \to u^{-2}k[u] \to u^{-1}k[u] \to k[u]$ are all zero, so the resulting limit is zero but $k(u)$ is nonzero.

The t-structure on $\mathrm{IndCoh}(X)$ is contingent on a good description of $\mathrm{IndCoh}(X)$, which we might not be able to manage in general. However, we have the following.

Proposition 20 ([Ga:IC] 1.2.4)**.** *The functor* $\Psi^{\ge n} : \mathrm{IndCoh}(S)^{\ge n} \to \mathrm{QCoh}(S)^{\ge n}$ *is an equivalence for every* n*.*

Proof. It suffices to prove this for $n = 0$. First, we need to define a t-structure on $\mathrm{IndCoh}(X)$. We claim (without proof) that for any small stable ∞-category $\mathbf{C}$ with a t-structure, there is an induced t-structure on $\mathrm{Ind}(\mathbf{C})$ such that (a) the truncation functors are continuous (i.e. commute with filtered colimits) and (b) the natural map $\mathbf{C} \to \mathrm{Ind}(\mathbf{C})$ is t-exact. Explicitly, it is given by $\mathrm{Ind}(\mathbf{C})^{\ge 0} := \mathrm{Ind}(\mathbf{C}^{\ge 0})$.

The functor is essentially surjective, since $\mathrm{QCoh}(S)^{\ge 0}$ is generated by $\mathrm{Perf}(S)^{\ge 0}$. Since the cohomologies are bounded below, $\mathrm{Perf}(S)^{\ge 0} \subset \mathrm{Coh}(S)^{\ge 0}$, so the image of $\mathrm{Coh}(S)^{\ge 0}$ in $\mathrm{QCoh}(S)^{\ge 0}$ generates. Let us show that the functor is fully faithful, i.e. that

$$\mathrm{Hom}_{\mathrm{IndCoh}(S)}(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{QCoh}(S)}(\Psi(\mathcal{F}), \Psi(\mathcal{G}))$$

for $\mathcal{F}, \mathcal{G} \in \mathrm{IndCoh}(S)^{\ge 0}$. First, since we are dealing with n-coconnective categories, the functor $\Psi^{\ge 0}$ has a left adjoint given by the inclusion of $\mathrm{Perf}(S)^{\ge n} \to \mathrm{Coh}(S)^{\ge 0}$, and therefore commute with limits. Thus we can take $\mathcal{F} \in \mathrm{Coh}(S)$.

Write a filtered colimit $\mathcal{G} = \mathrm{colim}_i\, \mathcal{G}_i$ with $\mathcal{G}_i \in \mathrm{Coh}(S)^{\ge 0}$. The left hand side is $\mathrm{colim}_i\, \mathrm{Hom}_{\mathrm{IndCoh}(S)}(\mathcal{F}, \mathcal{G}_i) = \mathrm{colim}_i\, \mathrm{Hom}_{\mathrm{Coh}(S)}(\mathcal{F}, \mathcal{G}_i)$ by construction of IndCoh. If $\mathcal{F}$ were compact in $\mathrm{QCoh}(S)$, the right hand side would be $\mathrm{colim}_j\, \mathrm{Hom}(\mathcal{F}, \mathcal{G}_j)$. Since these maps are term-by-term isomorphisms, they are isomorphisms in the colimit, completing the proof.

Unfortunately, $\mathcal{F} \in \mathrm{Coh}(S)^{\ge 0}$ may not be compact in $\mathrm{QCoh}(S)^{\ge 0}$. However, affine locally, there is a perfect complex $\mathcal{F}_0 \in \mathrm{Perf}(S)$ (without any coconnectivity assumptions) and a map $\mathcal{F}_0 \to \mathcal{F}$ ("taking a free resolution") such that the cone is in $\mathrm{QCoh}(S)^{\le -1}$. In particular, applying the functor $\mathrm{Hom}_{\mathrm{QCoh}(S)}(-, \mathcal{G})$, we have an exact triangle

$$\mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathcal{F}_0, \mathcal{G}) \to \mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathcal{F}, \mathcal{G}) \to \mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathrm{cone}, \mathcal{G})$$

Since the cone is in $\mathrm{QCoh}(S)^{\le -1}$, there are no degree 0 or -1 Homs to $\mathcal{G} \in \mathrm{QCoh}(S)^{\ge 0}$, so we have

$$\mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathcal{F}_0, \mathcal{G}) \simeq \mathrm{Hom}_{\mathrm{QCoh}(S)}(\mathcal{F}, \mathcal{G})$$

Thus, replacing $\mathcal{F}$ with $\mathcal{F}_0$ completes the proof. □

Corollary 21 ([Ga:IC] 1.3.4)**.** *The category* $\mathrm{QCoh}(X)$ *is complete, and* Ψ *realizes* $\mathrm{QCoh}(X)$ *as the completion of* $\mathrm{IndCoh}(X)$*.*

Remark 22 (Convergence)**.** Let A be a connective dg algebra. It is known that

$$A = \lim \tau^{\ge -n} A = \lim(\cdots \to \tau^{\ge -2} A \to \tau^{\ge -1} A \to \tau^{\ge 0} A)$$

i.e. that the dg scheme $\mathrm{Spec}(A)$ is *convergent* as a prestack. Note that if $S = \mathrm{Spec}(A)$, everything becomes its opposite:

$$S = \operatorname{colim} \tau^{\leq n} S = \operatorname{colim}(\tau^{\leq 0} S \to \tau^{\leq 1} S \to \tau^{\leq 2} S \to \cdots)$$

and the maps are closed immersions (and therefore proper). Another way of saying this is that S is an dg indscheme of eventually coconnective dg schemes.

Proposition 23 (Convergence property, [Ga:IC] 4.3.4)**.** *We have that the natural functor*

$$\operatorname{colim} \mathrm{IndCoh}(A^{\geq n}) = \operatorname{colim}(\ \mathrm{IndCoh}(\tau^{\geq 0} A) \xrightarrow{i_*} \mathrm{IndCoh}(\tau^{\geq -1} A) \xrightarrow{i_*} \cdots\ ) \longrightarrow \mathrm{IndCoh}(A)$$

is an equivalence.

Proof. Since the difference between $A^{\geq -n}$ and A is a nilpotent ideal, the image of the pushforward functors generate. Since the map is a closed immersion and in particular proper, it takes compact objects to compact objects.[3]. Thus, we can check that the map is fully faithful on compact objects. Recall that to show a map of dg categories a dg equivalence, we need to construct a functor on the underlying dg categories (i.e. inducing a map of chain complexes on the Hom spaces) which is an equivalence on H^0.

Since $M^\bullet, N^\bullet$ are bounded, we can take $M^\bullet, N^\bullet \in \mathrm{Coh}(\tau^{\geq -n} A)$. Then, $\mathrm{Hom}_{\tau^{\geq -n}(A)}(M^\bullet, N^\bullet) \simeq \mathrm{Hom}_A(M^\bullet, N^\bullet)$ when $n > a + b$ where $M^\bullet \in \tau^{\leq a}$ and $N^\bullet \in \tau^{\geq -b}$. This is because $A^{< -n}$ takes the "top" of $M^\bullet$ past the "bottom" of $N^\bullet$, and must act by zero anyway. □

Example 24 (Quasicoherent sheaves do not have the convergence property)**.** Take $A = k[u]$ with $|u| = -2$. Then, the colimit of the categories (where the functor is the pushforward functor, i.e. restriction of scalars,

$$k[u]/u\text{-mod} \to k[u]/u^2\text{-mod} \to \cdots$$

is the category of $k[u]$-modules where u acts by torsion. By the previous proposition, this category is $\mathrm{IndCoh}(\mathrm{Spec}(A))$.

4 Applications

We will conclude by mentioning some applications of ind-coherent sheaves. There is a discussion of this in [Ga:IC].

Remark 25 (Functoriality)**.** Let us state some functors which can be constructed given a map $f : X \to Y$ of dg schemes.

- If f is proper, then f_* is defined on $\mathrm{Coh}(X) \to \mathrm{Coh}(Y)$. In particular, the induced functor $f_* : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$ preserves compact objects and is continuous, so we can define a right adjoint $f^!$ which is also continuous. Note that the Grothendieck duality functor as defined on quasicoherent sheaves is *not* continuous, since f_* does not preserve perfect complexes. The functors f_* and $f^!$ satisfy the usual base change formula (see [Ga:IC] 3.4.2).
- If f is an open embedding, then define $f^! = f^*$. If f is smooth, define $f^! = f^*[\dim f]$.

Example 26 (Local complete intersections)**.** Every map $f : X \to Y$ can be factored as a smooth map followed by a closed embedding. For smooth maps, $f^! \simeq f^*[\dim f]$. Closed embeddings are proper, so we can compute it in terms of the adjunction. In particular, say $f : X = \mathrm{Spec}(A/I) \to Y = \mathrm{Spec}(A)$ is a closed embedding of affine schemes. Then, the adjunction in coherent sheaves says

$$R\,\mathrm{Hom}_A(A/I, A) \simeq R\,\mathrm{Hom}_{A/I}(A/I, f^! A) = f^! A$$

If A/I is a complete intersection, we can take a Koszul resolution of A/I over A and compute that $f^! A = A/I[-\mathrm{codim}(X/Y)]$.

[3]We will discuss this later; a map between dg schemes is proper if the map on classical schemes is. Since the cohomology sheaves of coherent sheaf over a dg scheme is coherent over π_0, the claim follows from the statement in classical algebraic geometry.

Remark 27 (Ind-schemes)**.** Let $Z \subset X$ be a (dg) closed subscheme. There is a description of the formal scheme $\widehat{Z}$ as an ind-scheme, i.e. a filtered colimit of closed subschemes of X. The category of ind-coherent sheaves is well-behaved with respect to ind-schemes, and the category $\mathrm{Coh}(\widehat{Z})$ can be described as coherent sheaves on X supported on Z.

Say $X = \mathrm{Spec}(A)$ is affine; then $Z = \mathrm{Spec}(A/I)$. We can write $\hat{Z}$ as a colimit of $Z_n = \mathrm{Spec}(A/I^n)$. The category $\mathrm{QCoh}(\hat{Z})$ consists of the data: a sheaf $\mathcal{F}_n$ on each Z_n and quasi-isomorphisms $i_{m,n}^* \mathcal{F}_n \simeq \mathcal{F}_m$, along with higher coherences. In particular, $\mathrm{QCoh}(\hat{Z})$ is a limit of the $\mathrm{QCoh}(Z_n)$. The description for IndCoh is the same, except with $i_{m,n}^!$; however, in this case, $\mathrm{IndCoh}(\widehat{Z})$ also admits a description as a colimit of $\mathrm{IndCoh}(\hat{Z})$ under the functors $i_{m,n,*}$. In particular, since proper pushforwards preserve coherent sheaves, this category is compactly generated, and further, admits a description as coherent sheaves supported on Z.

Remark 28 (Proper descent)**.** Theorems 8.2.2 and 8.2.3 in [Ga:IC] show that IndCoh satisfies descent with respect to !-pullback for proper maps surjective on geometric points.

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