Kontsevich's Formula for Rational Plane Curves

An adventure in enumerative algebraic geometry

Connor Halleck-Dubé

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Enumerative Algebraic Geometry

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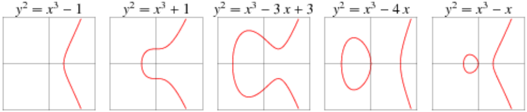

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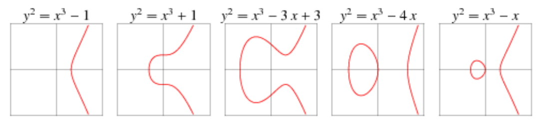

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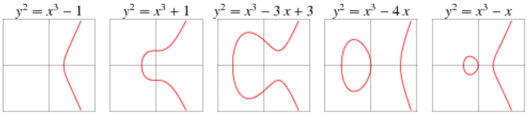

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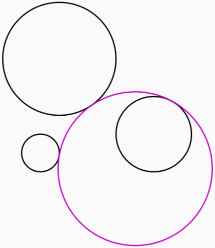

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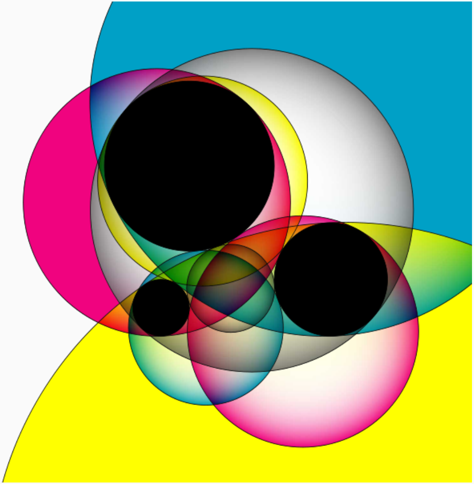

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- E.g. $C' : y^2 - x^2 = 0$ and $(y - x)^2 = 0$ both define degree 2 “curves” (Rather degenerate ones)

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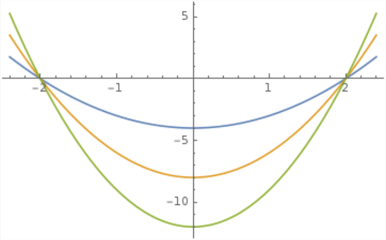

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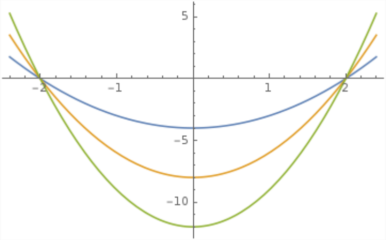

There are $\boxed{\infty}$ degree 2 curves through 2 points.

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Caviat: to get uniform answers, we need to assume the points are in "general position": no three lie on a line.

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- The moduli space of plane curves of degree d is $M_d := \mathbb{P}^N$, $N = \frac{1}{2}(d^2+3d)$.

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- Translated an enumerative problem into a problem about the geometry of a single object, the moduli space

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- (Exercise) The elliptic curve $y^2z = x^3 - xz^2$ is not rational
- Theorem: For degree-d plane curves, rationality is equivalent to the curve having $(d-1)$ singularities (counted appropriately)

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- (Zeuthen 1873) $N_4 = 620$

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$$[x : y] \mapsto [a_0x^d + a_1x^{d-1}y + \cdots + a_dy^d : \cdots : c_0x^d + \cdots + c_dy^d]$$

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- Two problems:
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- (2) some maps are not generically injective
- Solution: consider the quotient $M_d^{rat} := W_d / \mathbf{Aut}(\mathbb{P}^1)$ geometrically (naturally a stack) – the moduli space of rational degree d plane curves

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- Rich combinatorial structure of these objects!

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Let N_d the number of rational curves of degree d passing through $3d-1$ points in general position. Then

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- This is a recursion for all N_d in terms of base case N_1
- Kontsevich was a physicist – these moduli spaces appear in some approaches to string theory/QFT!

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5. Even very abstract algebraic geometry can have shockingly concrete applications

Thank you!