KONTSEVICH'S FORMULA FOR RATIONAL PLANE CURVES

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This exposition provides an elementary proof of Kontsevich's formula for enumerating rational plane curves in concrete geometric terms, before placing it in the context of genus-0 Gromov Witten invariants. The work assumes some basic algebraic geometry, as well as familarity with the geometry of the moduli space of marked curves $\overline{M}_{g,n}$. An elementary exposition of their geometry in the case $g = 0$ is given by [Cav16]. We generally follow the book of Kock and Vainsecher [KV07], occasionally pulling from the canonical source [FP96] and MIT OpenCourseWare lecture notes [Cos06].

I. The Moduli Space of Stable Maps

I.1. **Motivation and Definition.** Rational curves in projective space are naturally captured by the information of a map $\mu : \mathbb{P}^1 \to \mathbb{P}^r$.

Definition 1. The *degree* of a map $\mu : \mathbb{P}^1 \to \mathbb{P}^r$ is defined as the degree of the pushforward cycle $\mu_\star[\mathbb{P}^1]$. So a constant map has degree zero, a linear embedding has degree 1, etc.

A natural way to parameterize $\mu : \mathbb{P}^1 \to \mathbb{P}^r$ of degree d is to specify $r+1$ binary forms of degree d (up to scalars) which do not simultaneously vanish:

$$\mu(x, y) = (a_{0,0}x^d + a_{0,1}x^{d-1}y + \cdots + a_{0,d}y^d, \ \ \ldots \ \ , \ \ a_{r,0}x^d + a_{r,1}x^{d-1}y + \cdots + a_{r,d}y^d).$$

The collection of such forms, denoted $W(r, d)$, has dimension $(r + 1)(d + 1) - 1 = rd + r + d$.

Remark 2. The space $W(r, d)$ is a fine moduli space for maps $\mu : \mathbb{P}^1 \to \mathbb{P}^r$ of degree d, with universal family $W(r, d) \times \mathbb{P}^1$.

Proposition 3. *The locus of maps birational to their image, $W^\star(r, d) \subset W(r, d)$, is open and ($r \geq 2$) dense.*

The proof of this proposition is omitted, but it should seem plausible: the maps not birational onto their images are exactly the multiple covers, which have many fewer degrees of freedom.

Our interest is in rational curves, so the space $W(r, d)$ is not quite what we are looking for. Distinct parameterizations of the same geometric object are distinct maps in $W(r, d)$, when we would like them to be the same. Additionally, we would like to be able to consider continuous families of rational curves that may not have continuous parameterizations.

To solve these issues, we will want to quotient by conversions between distinct parameterizations, that is, we are structurally interested in

$$W(r, d)/\operatorname{Aut}(\mathbb{P}^1).$$

Lemma 4. *The group of automorphisms fixing a given map $\mu : \mathbb{P}^1 \to \mathbb{P}^r$ is finite, and trivial if μ is birational onto its image.*

Proof. This is trivial at the level of function fields. The pullback forms a finite field extension $K(\mu(\mathbb{P}^1)) \to K(\mathbb{P}^1)$, and automorphisms of $\mathbb{P}^1$ commuting with μ are certainly elements of the corresponding Galois group. Thus there can be only finitely many. If μ is birational onto its image, then $K(\mu(\mathbb{P}^1)) \cong K(\mathbb{P}^1)$ and there can be no nontrivial isomorphisms. ☐

Indeed it turns out that endowing this quotient with geometric structure, we can obtain a coarse moduli space

$$M_{0,0}(\mathbb{P}^r, d) \cong W(r, d)/\operatorname{Aut}(\mathbb{P}^1).$$

We observe the following facts.

(1) We should expect the dimension of this space to be
$$\dim M_{0,0}(\mathbb{P}^r, d) = \dim W(r,d) - \dim \mathrm{Aut}(\mathbb{P}^1) = rd + r + d - 3$$
since the generic fiber is $\mathrm{Aut}(\mathbb{P}^1)$, which has dimension 3.

(2) Open inside this set is $M^\star_{0,0}(\mathbb{P}^r, d)$, the image of $W^\star(r,d)$ under the quotient by automorphisms.

Our goal now is to compactify this space, which will also require us to consider marked points analogously to $\overline{M}_{g,n}$.

I.2. **Pointed maps and Stability.**

Example 5. Consider the pencil of conics satisfying the equation $XY - bZ^2$, where $b \in \mathbb{A}^1$. As $b \to 0$, the family of otherwise smooth conics degenerate to a pair of lines at $b = 0$. We can consider this pencil as a family of parameterizations $\mu : \mathbb{P}^1 \to \mathbb{P}^2$, indexed by b.
$$\mathbb{A}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$$
$$(b, [s:t]) \mapsto [bs^2 : t^2 : st]$$
When $b = 0$, this degenerates. If we want our moduli space to be compact, there must be a unique limit of this family; the natural choice is for the map to have domain consisting of a pair of intersecting lines.

As in the case of $\overline{M}_{g,n}$, the natural way to isolate the action of automorphisms is to introduce markings, and demand that the automorphisms preserve marked structure.

Definition 6. An *n-pointed map* is a morphism $\mu : C \to \mathbb{P}^r$, where C denotes a tree of projective lines with n distinct marked points which are smooth points of C. Then an n-pointed map contains in total the data $(C; p_1, \ldots, p_n; \mu)$.

We have a natural notion of isomorphism of such maps: isomorphisms $C \cong C'$ which commute with the maps and preserve marked points. A family of such maps can be compactly denoted
$$\begin{tikzcd} \mathcal{X} \arrow[r, "\mu"] \arrow[d, "\pi"] & \mathbb{P}^r \\ B \arrow[u, bend left, "\sigma_i"] & \end{tikzcd}$$
where π is a flat family of trees of smooth rational curves and σ_i are n disjoint sections avoiding the singularities of the fibers of π. This ensures $\mu_b : \pi^{-1}(b) \to \mathbb{P}^r$ is an n-pointed map.

Definition 7. An n-pointed map $\mu : C \to \mathbb{P}^r$ is *Kontsevich stable* (or just *stable*) if any twig mapped to a point is stable as a pointed curve (i.e. has finite automorphisms, or equivalently, has at least three marked points).

We remark stability is exactly the condition necessary for μ to have finitely many automorphisms in the sense described above, and so we might hope for a coarse moduli space for these objects.

Theorem 8 ([FP96])**.** *There exists a coarse moduli space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ parameterizing isomorphism classes of stable n-pointed maps of degree d. Furthermore, the following properties hold.*

(1) The space $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is a projective normal irreducible variety.

(2) It is locally isomorphic to the quotient of a smooth variety by a finite group.

(3) It contains $\overline{M}^\star_{0,n}(\mathbb{P}^r, d)$ as a smooth open dense subvariety, which parameterizes maps without automorphisms.

Remark 9. So that we can later speak in higher generality, we note that the moduli spaces of maps exist to an arbitrary smooth projective variety X. The datum of degree is naturally generalized by an element $\beta \in H_2(X)$. Given such data there exist coarse moduli spaces $\overline{M}_{0,n}(X, \beta)$, though the spaces need not be well-behaved without further assumptions.

Remark 10. While the fact that we only get a coarse moduli space may appear a defect, this is resolved in the language of stacks, where the stack version of $\overline{M}_{0,n}(\mathbb{P}^r, d)$ is a smooth and proper Deligne-Mumford stack with universal family the stack version of $\overline{M}_{0,n+1}(\mathbb{P}^r, d)$.

Proposition 11. *The dimension of* $\overline{M}_{0,n}(\mathbb{P}^r, d)$ *is* $rd + r + d + n - 3$.

Proof. Since our compactification contains $M_{0,n}(\mathbb{P}^r, d)$ as an open dense subset, this follows from the previous count, along with each marked point incrementing the dimension by 1. $\square$

I.3. **Examples and Basic Properties.** Some examples.

Example 12. The moduli space of stable maps to a point, $\overline{M}_{0,n}(\mathbb{P}^0, 0)$, is exactly the space of stable rational curves $\overline{M}_{0,n}$.

Example 13. The moduli space of stable maps of degree 0 is given by

$$\overline{M}_{0,n}(X, 0) \cong \overline{M}_{0,n} \times X.$$

Example 14. The moduli space of degree one maps to $\mathbb{P}^r$ is the Grassmannian

$$\overline{M}_{0,0}(\mathbb{P}^r, 1) = G(2, r + 1).$$

Example 15. As an exercise to the reader, we observe the following:

Proposition 16. *The space* $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ *is isomorphic to the variety of complete conics.*

Clearly, the moduli spaces are equipped with evaluation maps

$$\begin{aligned} \nu_i : \overline{M}_{0,n}(\mathbb{P}^r, d) &\to \mathbb{P}^r \\ (C; p_1, \ldots, p_n; \mu) &\mapsto \mu(p_i) \end{aligned}$$

Lemma 17. *The evaluation maps are flat morphisms.*

Proof sketch. This follows from generic flatness and the fact that the evaluation maps are invariant under $\mathrm{Aut}(\mathbb{P}^r)$. $\square$

The moduli spaces are also equipped with two kinds of natural forgetful maps,

$$\varepsilon : \overline{M}_{0,n+1}(\mathbb{P}^r, d) \to \overline{M}_{0,n}(\mathbb{P}^r, d) \quad \text{and} \quad \delta : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n}.$$

The first kind, forgetting points, has stabilization properties as in the case of stable rational curves: if a source twig on which μ is constant becomes unstable, it contracts.

Remark 18. The ε-forgetful and evaluation maps commute in the sensible way.

Remark 19. When we restrict to $\overline{M}^\star_{0,n}(\mathbb{P}^r, d)$, the map $\varepsilon : \overline{M}^\star_{0,n+1}(\mathbb{P}^r, d) \to \overline{M}^\star_{0,n}(\mathbb{P}^r, d)$ is a tautological family, proving that indeed we have a fine moduli space for automorphism-free maps.

I.4. **The boundary.** Since $M_{0,n}(\mathbb{P}^r, d) \subset \overline{M}_{0,n}(\mathbb{P}^r, d)$ consists of those stable maps whose source curve has only a single twig, the boundary is exactly the set of stable maps whose domains are reducible curves. For a given partition both of points $A \cup B = \{p_1, \ldots, p_n\}$ and of degrees $d_A + d_B = d$, we obtain an irreducible divisor $D(A, B; d_A, d_B)$. A general point on the divisor has two twigs in its source curve, twig A which contains the points in A and maps to an image curve of degree d_A, and twig B which contains the points in B and maps to an image curve of degree d_B. The boundary of this divisor, in turn, are given by further refinements: maps with more than two twigs which can be obtained by branching the tree further and distributing the points from A and B onto two halves.

We remark that there is a crucial recursive structure on the boundary: a gluing morphism gives a map

$$\overline{M}_{0,A\cup\{x\}}(\mathbb{P}^r, d_A) \times_{\mathbb{P}^1} \overline{M}_{0,B\cup\{x\}}(\mathbb{P}^r, d_B) \quad \to \quad D(A, B; d_A, d_B)$$

which is an isomorphism as long as $A \neq \emptyset$ and $B \neq \emptyset$. The recursive structure allows us to compute intersections with $D(A, B; d_A, d_B)$ in terms of smaller-dimensional moduli spaces.

Let $n \geq 4$, and consider the composition $f : \overline{M}_{0,n}(\mathbb{P}^r, d) \to \overline{M}_{0,n} \to \overline{M}_{0,4}$. Writing $S = \{p_1, \ldots, p_n\}$, we observe that

$$f^\star(D(i, j|k, l)) = \sum_{\substack{A\cup B=S \\ i,j\in A \\ k,l\in B \\ d_A+d_B=d}} D(A, B; d_A, d_B),$$

and a similar formula holds for the other natural boundary divisors $D(i, k|j, l)$ and $D(i, l|j, k)$.

Lemma 20 (The Fundamental Equivalence Relation). *We have*

$$f^{\star}(D(i,j|k,l)) \equiv f^{\star}(D(i,k|j,l)) \equiv f^{\star}(D(i,l|j,k)).$$

Proof. The key observation is that the forgetful map to $\overline{M}_{0,4}$ is flat. Since it is a reduced, irreducible, variety over a nonsingular curve $\overline{M}_{0,4} \cong \mathbb{P}^1$, it suffices for the map to be dominating. This implies that linear equivalence is preserved under pullback, so the result follows. □

II. The Proof of Kontsevich's Formula

II.1. **The Main Proof.** Now we have proven enough properties of $\overline{M}_{0,n}(\mathbb{P}^2, d)$ to derive Kontsevich's formula by considering carefully chosen boundary divisors. Let N_d denote the number of degree d curves passing through $3d-1$ generic points.

Theorem 1. *Let N_d be the number of rational curves of degree d passing through $3d-1$ general points in the plane. Then the following recursive relation holds.*

$$N_d + \sum_{\substack{d_A+d_B=d \\ d_a \geq 1, d_B \geq 1}} \binom{3d-4}{3d_A-1} N_{d_A} N_{d_B} d_A^3 d_B = \sum_{\substack{d_A+d_B=d \\ d_a \geq 1, d_B \geq 1}} \binom{3d-4}{3d_A-2} N_{d_A} N_{d_B} d_A^2 d_B^2.$$

Proof. Let $n = 3d$. Fix l_1, l_2 lines in the plane, and $q_1, \ldots, q_{n-2}$ points in the plane, chosen generically. We will use this geometric data to induce a curve through $\overline{M}_{0,n}(\mathbb{P}^2, d)$ with points labeled $m_1, m_2, p_1, \ldots, p_{n-2}$. Consider the inverse images under the evaluation maps of each of our n points, and define

$$v_{m_1}^{-1}(l_1) \cap v_{m_2}^{-1}(l_2) \cap \bigcap_{i=1}^{n-2} v_{p_i}^{-1}(q_i).$$

That is, $Y = \{\mu \in \overline{M}_{0,n}(\mathbb{P}^2, d) : \mu(m_j) \in l_j, \mu(p_i) = q_i\}$. Since the evaluation maps are flat, taking preimages preserves codimension, so each $v_{m_i}^{-1}(l_i)$ has codimension 1 and each $v_{p_i}^{-1}(q_i)$ has codimension 2. Choosing our points and lines generically ensures that Y has total codimension $2(n-2)+2 = 2n-2 = 6d-2$, that is, it is a curve in $\overline{M}_{0,n}(\mathbb{P}^2, d)$. We have also the following lemma.

Lemma 2 (The Technical Lemma). *The genericity assumption allows us to assume that Y intersects the boundary transversally, and that the intersection occurs within the locus of maps without automorphisms $M^{\star} \subset M$.*

Now we consider the intersection of this curve with the boundary. The linear equivalence previously found implies

$$Y \cap D(m_1, m_2|p_1, p_2) \equiv Y \cap D(m_1, p_1|m_2, p_2).$$

Since Y intersects the boundary transversally and they are of complementary dimension, both intersections consist of a finite number of points, and so the linear equivalence is simply the condition that the number of points on each side agree.

We have

$$D(m_1, m_2|p_1, p_2) \equiv \sum_{\substack{m_1, m_2 \in A \\ p_1, p_2 \in B \\ d_A + d_B = d}} D(A|B).$$

That is, it suffices to count stable maps with source curve consisting of two twigs A and B, such that m_1, m_2 are on A and p_1, p_2 are on B. Let d_A, d_B be the degrees of the two twigs.

(1) If $d_B = 0$, then the image of B must be a point, but $q_1 = \mu(p_1) \neq \mu(p_2) = q_2$, which is impossible, so this case contributes no points to the intersection.
(2) If $d_A = 0$, then this implies $\mu(m_1) = \mu(m_2)$ both land on the intersection point $l_1 \cap l_2$, and so we must count the number of degree d curves through $3d-1$ points. Thus this case introduces N_d points to the intersection.

(3) If $d_A, d_B > 0$, we must partition the $3d-4$ remaining points onto twigs A and B. If we put more than $3d_A - 1$ points from the p_i in total on twig A, then the image would be a degree d_A curve through more than $3d_A - 1$ generic points, which does not exist. If we put fewer than $3d_A - 1$ points from the p_i on twig A, then there would be more than $3d_B - 1$ points on twig B, and the same argument would apply. Thus we must choose $3d_A - 1$ points to place on twig A.

For each partition of points onto the two twigs, there is a unique stable map given by sending twig A to a fixed image curve of degree d_A through the image points and twig B to a fixed image curve of degree d_B. The number of such pairs of curves is $N_{d_A} N_{d_B}$. We must also specify which of the d_A intersections between $\mu(A)$ and l_1 is $\mu(m_1)$, and which of the d_A intersections between $\mu(A)$ and l_1 is $\mu(m_2)$, so we obtain another factor of d_A^2. Finally, we must also specify which of the $d_A d_B$ intersections between $\mu(A)$ and $\mu(B)$ corresponds to the nodal point. Thus, given d_A, d_B, we have a total number of maps given by

$$\binom{3d-4}{3d_A-1} N_{d_A} N_{d_B} d_A^3 d_B.$$

Iterating over all possible values of d_A, we obtain that the intersection $Y \cap D(m_1, m_2 | p_1, p_2)$ consists of

$$N_d + \sum_{\substack{d_A + d_B = d \\ d_A, d_B > 0}} \binom{3d-4}{3d_A-1} N_{d_A} N_{d_B} d_A^3 d_B$$

points.

We perform a similar computation on the other side. Neither $d_A = 0, d_B = 0$ is possible, as then we would have q_1 on l_1 or q_2 on l_2. For fixed $d_A, d_B > 0$, a stable map is identified by a choice of $3d_A - 2$ additional points for twig A, (the complementary $3d_B - 2$ on B), a choice among N_{d_A} image curves for $\mu(A)$ and N_{d_B} image curves for $\mu(B)$, a choice among d_A intersections for $\mu(m_1)$ and d_B intersections for $\mu(m_2)$, and a choice among $d_A d_B$ intersections for the nodal point. Thus the total number of points in the intersection is

$$\sum_{\substack{d_A + d_B = d \\ d_A, d_B > 0}} \binom{3d-4}{3d_A-2} N_{d_A} N_{d_B} d_A^2 d_B^2.$$

The equality on numbers of points holds, which implies the formula. □

Since this expresses N_d as a function of purely smaller values in the sequence, it gives an elementary recursion for N_d, the number of rational curves of degree d through $3d-1$ rational points.

II.2. **The Technical Lemma.** We have deferred the proof of our key technical lemma, which we address now.

Lemma 3 (The Technical Lemma)**.** *For generic choices of irreducible subvarieties $\Gamma_1, \ldots, \Gamma_n \subset \mathbb{P}^r$ with codimension adding to $\dim \overline{M}$, the scheme-theoretic intersection*

$$\bigcap_{i=1}^{n} \nu_i^{-1}(\Gamma_i).$$

consists of a finite number of reduced points, supported in any preassigned nonempty open set and in particular, in $M^\star \subset \overline{M}$ the locus of maps with smooth source and without automorphisms.

Proof. Let $X = \mathbb{P}^r$. It will be convenient to consider the total evaluation map

$$\underline{\nu} = \prod_i \nu_i : \overline{M} \to X^n.$$

Then as schemes, we have

$$\bigcap_{i=1}^{n} \nu_i^{-1}(\Gamma_i) = \underline{\nu}^{-1}(\underline{\Gamma})$$

where $\underline{\Gamma} := \prod_i \Gamma_i$.

Our sledgehammer will be the following theorem of Kleiman.

Theorem 4 (Kleiman, Transversality of the general translate)**.** *Let G be a connected algebraic group and X an irreducible variety with transitive G-action. Let $f : Y \to X$ and $g : Z \to X$ be morphisms between irreducible varieties. For each $\sigma \in G$, denote by Y^σ the variety $Y \xrightarrow{\sigma \circ f} X$.*

Then there exists a dense open subset $U \subset G$ such that for every $\sigma \in U$, either the fiber product $Y^\sigma \times_X Z$ is empty, or

$$\dim(Y^\sigma \times_X Z) = \dim Y + \dim Z - \dim X.$$

We exhibit $\mathbb{P}^r$ as the quotient of SL_r by upper triangular matrices, so $X = \mathbb{P}^r$ has a transitive action of $G = \mathrm{SL}_r$, and $\underline{G} = G^n$ acts transitively on X^n.

We apply the theorem with $Z = (M^\star)^C$, which is a closed subvariety of strictly smaller dimension inside $\overline{M}$, and $Y = \underline{\Gamma}$. Then from the following diagram

$$\begin{tikzcd} & (M^\star)^C \arrow[d, "\underline{\nu}"] \\ \underline{\Gamma} \arrow[r, "\subset"] & X^n \end{tikzcd}$$

there is a dense open set $V_1 \subset \underline{G}$, such that for any $\sigma \in V_1$ the fiber product $\underline{\Gamma}^\sigma \times_{X^n} (M^\star)^C$ is either empty or has dimension

$$\dim(\underline{\Gamma}^\sigma \times_{X^n} (M^\star)^C) = \dim(\underline{\Gamma}) + \dim((M^\star)^C) - \dim(X^n) \leq -1,$$

since $\dim(\underline{\Gamma}) = \sum_i \dim(\Gamma_i) = \dim(X^n) - \dim(\overline{M})$. The second case is a contradiction, so we conclude the fiber product is always empty. However, since $\underline{\Gamma}$ is a subvariety of X^n, it follows that the fiber product is naturally identified with the inverse image of $\underline{\Gamma}^\sigma$ under the total evaluation map from $(M^\star)^C$ – that is, for any $\sigma \in V_1$, the preimage $\underline{\nu}^{-1}(\underline{\Gamma}^\sigma)$ is disjoint from the maps with automorphisms – so is contained in the smooth locus $M^\star$. Since the action of G is transitive, it follows that a generic collection of Γ_i have

$$\bigcap_{i=1}^{n} \nu_i^{-1}(\Gamma_i) = \underline{\nu}^{-1}(\underline{\Gamma}) \subset M^\star.$$

To prove that indeed the intersection is a finite number of reduced points, we apply Kleiman again, to

$$\begin{tikzcd} & M^\star \arrow[d, "\underline{\nu}"] \\ \operatorname{Sing} \underline{\Gamma} \arrow[r, "\subset"] & X^n \end{tikzcd}$$

Then again we compute that the fiber product would have negative dimension, so must be empty. We obtain a dense open set $V_2 \subset \underline{G}$ such that $\underline{\nu}^{-1}((\operatorname{Sing} \underline{\Gamma})^\sigma) = \emptyset$.

A final application of Kleiman with $Y = \underline{\Gamma} \setminus \operatorname{Sing} \underline{\Gamma}$ and $Z = M^\star$ obtains a dense open set $V_3 \subset \underline{G}$ such that the inverse image of each translate Y^σ is smooth and either of correct dimension or empty. Since the expected dimension is zero, this is exactly the condition that the intersection be a finite set of reduced points.

On the dense open set $V_1 \cap V_2 \cap V_3$, we conclude that a translate $\underline{\Gamma}^\sigma$ has inverse image which is contained in $M^\star$ and is a finite set of reduced points. □

II.3. **Did we count the right thing?** In the above proof, we were asked on multiple occasions to count the number of smooth stable maps whose image curve passes through a collection of $3d - 1$ points. We assumed without comment that this value is exactly N_d. This is indeed the case, but not obviously.

Given general subvarieties, say $\Gamma_1, \ldots, \Gamma_n$, the number of stable maps with image curve intersecting all Γ_i could fail to be the number of rational curves through the Γ_i for two reasons. First, a solution curve might have passed multiple times through the same point Γ_i. Then there would be multiple stable maps, as the mark could be placed on either preimage of the intersection point. Second, the curve could have met Γ_i at multiple distinct points, which would similarly ruin the bijection (Bezout's theorem demonstrates the nature of this second failure for hypersurfaces).

In the case of points, the second concern is not possible, but the first could be. Intuitively, a generic point should also be generic on the solution curve: this can be formalized as an additional lemma proved via Kleiman's theorem. More generally, all Γ_i of codimension at least 2 suffices to ensure both degeneracies are avoided.

Lemma 5. *Let $\Gamma_1, \ldots, \Gamma_n \subset \mathbb{P}^r$ be general subvarieties of codimension at least 2 with total codimension $\dim \overline{M}_{0,n}(\mathbb{P}^r, d)$. Then for any $\mu \in \underline{\nu}^{-1}(\underline{\Gamma})$, we have*

(1) the image curve $\mu(C)$ intersects each Γ_i at a single point, and
(2) for each p_i marked point, $\mu^{-1}\mu(p_i) = \{p_i\}$ (the solution curve passes through the intersection point only a single time).

These ideas will be explored in more generality in terms of Gromov-Witten invariants in the sequel.

III. Generalization: The Language of Gromov-Witten Invariants

Throughout this chapter, $r \geq 2$. To introduce the formalism, we work briefly in more generality.

Let $A_i(\mathbb{P}^r) := H_{2i}(\mathbb{P}^r, \mathbb{Q})$, $A^i(X) := H^{2i}(\mathbb{P}^r, \mathbb{Q})$, and $A^\star(\mathbb{P}^r) = \oplus_i A^i(\mathbb{P}^r)$ with the usual cup product as multiplication. For $\overline{M}_{0,n}(\mathbb{P}^r, d)$, our job is not so simple – since the moduli space is singular as a variety, there is no intersection product on cycles.

Fact.

(1) Given a map $f : Y \to X$ from an arbitrary scheme to a smooth variety, there exists a pullback product

$$A^k(X) \otimes A_i(Y) \to A_{i-k}(Y).$$

Thus, the evaluation maps to $\mathbb{P}^r$ allow us to pullback $A^k(X)$ to endomorphisms of $A_\star(Y)$.

(2) There exists a subring of $\operatorname{End}(Y)$, including the Chern classes and all endomorphisms of the above kind, which has the properties we would expect of a cohomology ring.

We denote this subring by $A^\star(\overline{M}_{0,n}(\mathbb{P}^r, d))$ in the case of moduli of stable maps, and use intersection product notation for its action on $A_\star(\overline{M})$. A grading on $A^\star(\overline{M})$ is induced by its action: we say a class $\alpha \in A^k(\overline{M})$ if its action takes i-cycles to $(i-k)$-cycles. All this is to say that cohomology works the way that it should, without necessarily knowing what the cohomology of $\overline{M}$ actually is.

Definition 1 (Gromov-Witten Invariant)**.** Let X be a convex variety. We have a coarse moduli space $\overline{M}_{0,n}(X, \beta)$, which is equipped with n evaluation maps

$$\nu_i : \overline{M}_{0,n}(X, \beta) \to X$$

defined just as in the case of $\mathbb{P}^r$. Given classes $\gamma_1, \ldots, \gamma_n$ in $A^\star(X)$, we have a well-defined product in $A^\star(\overline{M}_{0,n}(X, \beta))$ given by

$$\nu_1^\star(\gamma_1) \cup \cdots \cup \nu_n^\star(\gamma_n).$$

If the total codimension of the γ_i is the dimension of $\overline{M}_{0,n}(X, \beta)$, then this product is an element of the top cohomology, and thus can be evaluated on the fundamental class $[\overline{M}_{0,n}(X, \beta)]$. The *Gromov-Witten invariant*

$$I_\beta(\gamma_1 \cdots \gamma_n) := \int_{\overline{M}_{0,n}(X,\beta)} \nu_1^\star(\gamma_1) \cup \cdots \cup \nu_n^\star(\gamma_n)$$

is defined to be this evaluation. The invariant is written with multiplicative notation to capture that the invariant is invariant under permutation of the γ_i.

If X is homogeneous, then the Gromov-Witten invariants capture generic enumerative information about the number of rational curves in X "through" the γ_i in a suitable sense. The spirit of this theorem is captured by the case of $\mathbb{P}^r$.

Proposition 2 (Enumerative significance of Gromov-Witten invariants)**.** *Let $\gamma_1, \ldots, \gamma_n \in A^\star(\mathbb{P}^r)$ homogeneous classes of codimension at least 2 and total codimension equal to $\dim(\overline{M}_{0,n}(\mathbb{P}^r, d))$. Then for general subvarieties $\Gamma_1, \ldots, \Gamma_n \subset \mathbb{P}^r$ with $[\Gamma_i] = \gamma_i \cap [\mathbb{P}^r]$, the Gromov-Witten invariant $I_d(\gamma_1 \cdots \gamma_n)$ is the number of rational curves of degree d incident to all the subvarieties $\Gamma_1, \ldots, \Gamma_n$.*

Corollary 3. For $\mathbb{P}^2$, we have $I_d(h^2 \cdots h^2) = N_d$, where the product is over $3d-1$ factors of h^2.[1]

[1] We write h for the hyperplane class, so that h^2 in $\mathbb{P}^2$ is naturally the class of a point. More generally, the standard basis for $H^\star(\mathbb{P}^r)$ is $\{h^0, h^1, \ldots, h^r\}$.

We will need three computational facts about these invariants. The first says that we can pull a hyperplane out from the computation.

Proposition 4 (Divisor equation)**.** *If $d > 0$, then*

$$I_d(\gamma_1 \cdots \gamma_n \cdot h) = I_d(\gamma_1 \cdots \gamma_n) \cdot d,$$

where the left-hand invariant is computed in $\overline{M}_{n+1}$ and the right in $\overline{M}_n$.

The second collects base cases.

Proposition 5 (Degree 0)**.** *(1) The only nonzero Gromov-Witten invariants with $d = 0$ are those with three marks and $\sum \operatorname{codim} \gamma_i = r$. Then*

$$I_0(\gamma_1 \cdot \gamma_2 \cdot \gamma_3) = \int (\gamma_1 \cup \gamma_2 \cup \gamma_3) \cap [\mathbb{P}^r].$$

(2) The only nonzero Gromov-Witten invariants with less than three marks are

$$I_1(h^r \cdot h^r) = 1$$

that there exists a unique line through 2 points.

(3) The only nonzero Gromov-Witten invariants containing $1 = h^0 \in A^0(\mathbb{P}^r)$ occur in degree zero, with three marks as above.

That is, we can evaluate the degree zero Gromov-Witten invariants on the level of $\mathbb{P}^r$ rather than the moduli space.

The recursion on the boundary, in particular the gluing isomorphism

$$D(A, B; d_A, d_B) \cong \overline{M}_{0,A\cup\{x\}}(\mathbb{P}^r, d_A) \times_{\mathbb{P}^r} \overline{M}_{0,B\cup\{x\}}(\mathbb{P}^r, d_B),$$

induces a fundamental recursion on Gromov-Witten invariants.

Theorem 6 (Splitting Lemma)**.** *For $\gamma_1, \ldots, \gamma_n$ as above and $D = D(A, B; d_A, d_B)$, we have*

$$\int_D \nu_1^\star(\gamma_1) \cup \cdots \cup \nu_n^\star(\gamma_n) = \sum_{e+f=r} I_{d_A}\left(\prod_{a\in A} \gamma_a \cdot h^e\right) I_{d_B}\left(\prod_{b\in B} \gamma_b \cdot h^f\right),$$

where the recursive Gromov-Witten invariants are evaluated in the moduli spaces $\overline{M}_{0,A\cup\{x\}}(\mathbb{P}^r, d_A)$ and $\overline{M}_{0,B\cup\{x\}}(\mathbb{P}^r, d_B)$

This gives us a recursive procedure for computing Gromov-Witten invariants.

Example 7. Lets say we want to compute the number of rational curves passing through 5 general points, i.e. $N(h^2, h^2, h^2, h^2, h^2)$. We begin with six classes $h = \lambda_1 = \lambda_2$ and $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = h^2$. The pullback class

$$\underline{\nu}^\star(\underline{\gamma}) := \nu_{m_1}^\star(\lambda_1) \cup \nu_{m_2}^\star(\lambda_2) \cup \nu_{p_1}^\star(\gamma_1) \cup \nu_{p_2}^\star(\gamma_2) \cup \nu_{p_3}^\star(\gamma_3) \cup \nu_{p_4}^\star(\gamma_4)$$

is the "dual" of a curve in the moduli space. Rather than intersecting the curve with the boundary divisors, we will evaluate this cohomology class on the equivalent boundary divisors:

$$\int \underline{\nu}^\star(\underline{\gamma}) \cap D(m_1, m_2|p_1, p_2) = \int \underline{\nu}^\star(\underline{\gamma}) \cap D(m_1, p_1|m_2, p_2)$$

We split across the sums $D(m_1, m_2|p_1, p_2) = \sum D(A, B|d_A, d_B)$. The splitting lemma gives

$$\sum_{\substack{A,B\\ d_A,d_B}} \sum_{e+f=2} I_{d_A}(\lambda_1\lambda_2 \prod \gamma_a \cdot h^e) I_{d_B}(\gamma_1\gamma_2 \prod \gamma_b \cdot h^f).$$

If $d_A = 0$, then the only nonzero contribution is with exactly three marks, namely m_1, m_2, x. To satisfy the codimension condition of this result, we must have $e = 0$. The corresponding right term is $I_2(\gamma_1 \cdot \gamma_2 \cdot \gamma_3 \cdot \gamma_4 \cdot h^2) = N_2$. If $d_B = 0$ then there is no contribution, as we already have codimension 4 on the right side.

When $d_A = d_B = 1$, we iterate over the possible distributions of the marks onto each side. If there are no other marks on A, then the moduli space is dimension 5 and the classes have maximal codimension 4. If there is one spare mark on each piece, then no value of e and f makes the codimension counts correct. If both spare marks go on A, then $e = f = 1$ gives

$$I_1(\lambda_1 \cdot \lambda_2 \cdot \gamma_3 \cdot \gamma_4 \cdot h^1) I_1(\gamma_1 \cdot \gamma_2 \cdot h^1).$$

We can throw out the hyperplane classes λ_1, λ_2, h to obtain $I_1(\gamma_3 \cdot \gamma_4) I_1(\gamma_1 \cdot \gamma_2) = 1$. So the left-hand side is $N_2 + 1$. On the right side, a similar computation gives $1 + 1$, so we obtain $N_2 = 1$ as expected.

These recursions grow complicated with the geometry of X, but for $\mathbb{P}^r$, all the Gromov-Witten invariants can be computed from a single value.

Theorem 8 (Kontsevich-Manin '94, Ruan-Tian '95)**.** *All (genus-0) Gromov-Witten invariants for $\mathbb{P}^r$ can be computed recursively from the initial value $I_1(h^r \cdot h^r) = 1$, the number of lines through 2 points.*

Algorithmic outline. Any time a class of codimension 0 or 1 appears, it can be eliminated while decreasing the number of marks. Thus we can assume all classes are of codimension 2. Break the lowest-codimensional class into two of strictly smaller codimension $\gamma_n = \lambda_1 \cup \lambda_2$ (because h generates the cohomology of $\mathbb{P}^r$, so $\gamma_n = h^{\operatorname{codim} \gamma_n}$). Then we have a class of a curve in $\overline{M}_{0,n+1}(\mathbb{P}^r, d)$:

$$\nu^\star_{m_1}(\lambda_1) \cup \nu^\star_{m_2}(\lambda_2) \cup \nu^\star_{p_1}(\gamma_1) \cup \cdots \cup \nu^\star_{p_{n-1}}(\gamma_{n-1}),$$

just as we did in the proof of Kontsevich's formula.

Integrating the class over the two equivalent boundary divisors and applying the splitting lemma gives a sum over $I_{d_A}(\cdot) I_{d_B}(\cdot)$ terms. These are known by the induction hypothesis unless $d_A = 0$ or $d_B = 0$, and the only nonzero terms with zero degree have three marks. Thus we reduce to four possible nonzero terms, given $c_i = \operatorname{codim} \lambda_i$ and $b_i = \operatorname{codim} \gamma_i$,

$$\begin{aligned} I_0(\lambda_1 \cdot \lambda_2 h^{r-c_1-c_2}) I_d(\gamma_1 \cdots \gamma_{n-1} \cdot h^{c_1+c_2}) \\ I_0(\lambda_1 \cdot \gamma_1 \cdot h^{r-c_1-b_1}) I_d(h^{b_1+c_1} \cdot \gamma_2 \cdots \gamma_{n-1} \cdot \lambda_2) \\ I_d(\gamma_1 \cdots h^{c_2+b_2} \cdot \gamma_3 \cdots \gamma_{n-1} \cdot \lambda_1) I_0(\lambda_2 \cdot \gamma_2 \cdot h^{r-c_2-b_2}) \\ I_d(h^{b_1+b_2} \cdot \lambda_2 \cdot \gamma_3 \cdots \gamma_{n-1} \cdot \lambda_1) I_0(\gamma_1 \cdot \gamma_2 \cdot h^{r-b_1-b_2}). \end{aligned}$$

The I_0 factors are all equal to 1, so the first term is the desired value, and the other terms have a smaller-codimensional class at the end. Thus we are done by recursion, with initial value $I_1(h^r \cdot h^r) = 1$. $\square$

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