The Weighted Fundamental Lemma for Non-Split Groups

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Abstract. In this work, we prove the Weighted Fundamental Lemma for Lie algebras for general G. The strategy is to adapt the proof of Chaudouard–Laumon from the split case to the general setting. In particular, we define the notion of a ξ-stable Higgs bundle for a quasi-split reductive group scheme G on a curve, and prove that the stack of such form a proper, finite-type Deligne–Mumford stack over the Hitchin base. We then adapt the global cohomological theorem of Chaudouard–Laumon to the quasi-split setting, and perform the required Frobenius-point count to obtain an identity of adelic orbital integrals. A local-to-global argument deduces the WFL. We therefore render the twisted-weighted stable Arthur–Selberg trace formula, the endoscopic classification for classical groups, and an array of downstream consequences in automorphic theory conditional only upon the nonstandard WFL, work in progress of the author.

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CHAPTER 1

Introduction

1.1. What is this document?

This article proves the variant for Lie algebras of the Weighted Fundamental Lemma (WFL) conjectured by Arthur [Art02] and its "nonstandard" variant (the latter work-in-progress). Many results in automorphic theory, including the untwisted and twisted stable trace formulae of Arthur (cf. [Art02; Art01; Art03], [MW16a; MW16b]) and the endoscopic classification for classical groups (see [Art13], [Ato+25]), are currently conditional upon the WFL and nonstandard WFL as proved in [Wal11].

Two broad special cases of the WFL were previously known. In addition to a local unramified reductive G, the WFL depends on a Levi subgroup $M \subset G$ and an elliptic endoscopic group M' for M. The special case $M = G$ is the (Lie algebra version of the) celebrated Langlands–Shelstad Fundamental Lemma introduced in [Lan83], whose standard and non-standard versions were both proven in [Ngô10] for arbitrary (unramified reductive) G. On the other hand, for reductive groups G which are defined and split over $\mathbb{F}_q$, and for endoscopic groups M' which are split, the WFL (though not its non-standard variant) was proved in [CL10; CL12]. This work, then, can be thought of either as (1) the generalization of the work of Ngô from $M = G$ to arbitrary Levis, or as (2) an extension of Chaudouard–Laumon from the case of G and M' constant split to an arbitrary (quasi-split) unramified reductive G and endoscopic group; indeed, it may be accurate to think of this work as the pushout of these two special cases.

Methodologically, our philosophy is to avoid re-inventing the wheel whenever possible. Many results can be directly used, and many arguments can be transferred over *mutatis mutandis*, from the previous work. In the interest of a self-contained and accessible presentation, and because it was necessary to go over through those works with a fine-toothed comb in order to understand where additional arguments were necessary, the current article reproduces many of the arguments of *loc. cit.* in their entirety, as well as referencing the corresponding results. We have made every effort to indicate clearly the difference between results which are fully contained in the prior work (cited in the format "([CL10, Corollaire 3.9.2])"), new results whose proofs proceed closely along the prior work (cited in the format "(cf. [CL10, Corollaire 3.9.2])"), and those places where new inputs or ideas are needed. Our approach should be contrasted with the complementary work of Wang et al. (cf. [Wan23]), which seeks to prove various Fundamental Lemmas directly at the group-level by replacing the Hitchin fibration with multiplicative Hitchin fibrations.

In the remainder of this introduction, we begin with a gentle recollection of the basic theory of local orbital integrals as they arise in applications to the Arthur–Selberg trace formula, and state the WFL as conjectured by Arthur. We then recall in high-level and imprecise terms the nature of the previous work in this area and outline the content of this document over several sections. For the benefit of the reader familiar with the work of Ngô and Chaudouard–Laumon, we highlight some of the new ideas and arguments required for our level of generality.

1.2. The Weighted Fundamental Lemma.

Following the ideas of Arthur, study of the geometric side of the trace formula can often be reduced to very simple kinds of orbital integrals. Let G a connected quasisplit unramified reductive group over a nonarchimedean local field F_v with absolute value $|\cdot|$. Assume that the residue characteristic p is sufficiently large and consider the hyperspecial, Lie algebra, local orbital integral

$$O_\gamma^G := |D^G(\gamma)|^{1/2} \int_{Z_G(\gamma)(F_v)\backslash G(F_v)} 1_{\mathfrak{g}(\mathscr{O})}(\mathrm{Ad}(g^{-1})\gamma)\, dg,$$

where $\gamma \in \mathfrak{g}(F_v)$ is a semisimple regular element and $D^G(\gamma)$ is the Weyl discriminant. For a good choice of Haar measures, O^G_γ depends only on the $G(F_v)$-conjugacy class of γ. The centralizer $J_\gamma = Z_G(\gamma)(F_v)$ is a torus; when this torus is not elliptic, that is, anisotropic mod Z_G, the kinds of global products of such orbital integrals which appear in the trace formula diverge.

Arthur was able to nevertheless extract canonical arithmetic information by introducing regularizing factors, *weight functions* v^G_M on the orbit which vanish as the group element goes "off to infinity in the directions of the split tori." These weight functions depend on a choice of unramified Levi $M \subseteq G$ over F_v; the function is identically 1 when $M = G$, and become harsher truncations as we pass to smaller Levi subgroups. A *weighted orbital integral*, then, is an integral

$$O^G_{M,\gamma_M} = |D^G(\gamma_M)|^{1/2} \int_{J_{\gamma_M}(F_v)\backslash G(F_v)} 1_{\mathfrak{g}(\mathscr{O})}(\mathrm{Ad}(g^{-1})\gamma_M) v^G_M(g)\, dg,$$

where $\gamma_M \in \mathfrak{m}(F_v) \subseteq \mathfrak{g}(F_v)$ is a semisimple G-regular element, so that $Z_G(\gamma_M)$ is a subtorus of M. Calling an element $\gamma \in \mathfrak{g}(F_v)$ *M-elliptic* if the maximal split subtorus $J^{\mathrm{spl}}_\gamma \subset J_\gamma$ is equal to the split connected center of $A^{\mathrm{spl}}_M \subset M$, this gives us a well-behaved notion of orbital integral for all regular semisimple γ.

Orbital integrals and weighted orbital integrals are poorly adapted to the kinds of transfer one expects from Langlands functoriality: say, between pure inner forms of a group, or from an endoscopic group for G to G itself. Since transfer is built atop transfer of geometric data (data defined over an algebraically closed field), this failure is partially reflected in the fact that weighted orbital integrals are constant only on $M(F_v)$-conjugacy classes rather than on *stable conjugacy classes*. The latter, here equal to the set of F_v-points in the $M(\overline{F_v})$-conjugacy class, is the truly algebrao-geometric object, and correspondingly it is natural to seek "stabilized" versions of orbital integrals. For G-elliptic orbital integrals, the stable orbital integral is simply the sum of orbital integrals for the $M(F_v)$-conjugacy classes lying in a given stable class,

$$SO^G_\gamma := \sum_{[\gamma'] \sim_{st} [\gamma]} O^G_{\gamma'}.$$

The set of rational classes in a stable class is finite, and indeed here has the structure of a finite group. In applications to harmonic analysis, one seeks to rewrite expressions in terms of orbital integrals with *stabilized* versions involving only stable orbital integrals. Of course, symmetrizing over a stable conjugacy class naturally creates "error terms," *twisted* sums indexed by characters s of the group of rational classes

$$O^{s,G}_\gamma = \sum_{[\gamma'] \sim_{st} [\gamma]} s(\gamma') O^G_{\gamma'}.$$

The landmark result conjectured by Langlands and proved by Ngô is the *Fundamental Lemma*,

$$O^{s,G}_\gamma = q^{r(\gamma_{H_s})} SO^{H_s}_{\gamma_{H_s}},$$

where H_s is an *elliptic endoscopic group* for G and $r(\gamma_{H_s})$ is an explicit integer. Much has been written about the Fundamental Lemma, see, for example, [Hal05] for a general exposition or [Ngô10, §1.11] for precise constructions in the style of the rest of this thesis.

When $M \neq G$, on the other hand, the combinatorics required to stabilize a weighted orbital integral is more involved. Motivated by the inductive structure that arose when attempting to stabilize the entire trace formula, Arthur conjectured in [Art02] an identity in the same spirit, relating twisted sums of M-elliptic weighted orbital integrals to sums of stable orbital integrals for endoscopic groups. In the remainder of this section, we introduce the requisite notation to state the Lie algebra version of the Weighted Fundamental Lemma as in *loc. cit.*

Let M' be an unramified elliptic endoscopic datum for M, in the sense of [Art96, §2]. For our purposes, this means M' is shorthand for a triple consisting of

- a quasisplit reductive F_v-group M',
- a subgroup of the L-group $\mathsf{M} \subseteq {}^L M$ which is elliptic, i.e., does not lie within any parabolic subgroup, and
- a semisimple element $s_M \in \widehat{M}$.

Here $\widehat{M}$ is the Langlands dual of $\mathbb{M} = M_{\overline{F}_v}$, taken over some algebraically closed field of characteristic 0 (say, $\mathbb{C}$), and the L-group is defined relative to its action by $\Gamma = \mathrm{Gal}(F_v^{\mathrm{s}}/F_v)$ through some finite quotient (indeed through $\mathrm{Out}(\mathbb{M})$, since M is quasisplit). These are required to be compatible in the sense that $\mathscr{M}$ is a split extension of W_{F_v} by $\widehat{M'}$ and $\widehat{M'}$ is the connected centralizer of s_M inside $\widehat{M}$, as well as satisfying a certain vanishing cocycle condition, cf. [MW16a, §1.1.5]. The foundation of transfer from an endoscopic group is then the elementary fact that a fixed maximal torus $T \subset M$ identifies with a maximal torus of M' in such a way that we have a containment of Weyl groups $W^{M'} \subset W^M$. In particular, at the level of *invariants*, i.e., Chevalley spaces, see §2.11 below, we have a morphism

$$\mathfrak{m}'//M' \cong \mathfrak{t}//W^{M'} \to \mathfrak{t}//W^M \cong \mathfrak{m}//M.$$

We write

$$\Delta_{M',M} : \mathfrak{m}'(F_v) \times \mathfrak{m}(F_v) \to \mathbb{C}$$

for the Langlands–Shelstad transfer factor for Lie algebras as in [Wal97], defined on pairs of semisimple G-regular elements (as defined by their images in $\mathfrak{g}//G \cong \mathfrak{t}//W$). The value of $\Delta_{M',M}(\gamma', \gamma)$ is zero unless its inputs have the same image in $\mathfrak{m}//M$, and when $M' = M$ it is exactly the indicator function for the relation $[\gamma'] \sim_{st} [\gamma]$.

Given a triple representing an elliptic endoscopic datum for M, [Art98, §4] defines an *induced* family of representatives for elliptic endoscopic data for G. One begins with $s \in s_M Z(\widehat{M})^\Gamma / Z(\widehat{G})^\Gamma$, and then defines

- $\widehat{G}'_s := Z_{\widehat{G}}(s)^0$,
- $\mathsf{G}_s := \widehat{G'}\mathsf{M}$, and
- G'_s to be a certain quasisplit Langlands dual F_v-group of $\widehat{G}'_s$ determined by G_s.[1]

Then (G'_s, G_s, s) define an endoscopic datum for G, and moreover M' can be identified with a Levi subgroup of G'. Denote $\mathscr{E}_{M'}(G)$ the set of such s for which the resulting endoscopic datum is elliptic, though it depends on the full endoscopic datum for M', not just M' itself.

Now let $\gamma' \in \mathfrak{m}'(F_v)$ be G-regular semisimple. Define the endoscopic sum of orbital integrals

$$O^G_{M',M}(\gamma') := \sum_{\gamma \in M^{G\text{-reg}}(F_v)} \Delta_{M',M}(\gamma', \gamma) O^G_{M,\gamma}$$

with the sum, taken over representatives for the conjugacy classes, finite. One also then has a recursive definition of *stable* weighted orbital integrals for $\gamma \in \mathfrak{m}(F_v)$ semisimple G-regular:

$$SO^G_M(\gamma) := O^G_{M,M}(\gamma) - \sum_{s \neq 1} |Z(\widehat{G}'_s)^\Gamma / Z(\widehat{G})^\Gamma|^{-1} SO^{G'_s}_M(\gamma).$$

the sum taken over $s \in \mathscr{E}_M(G)$, intuitively, over endoscopic groups G'_s for G which are compatible with M.

We prove the following under our characteristic assumptions in Theorem 16.5 below.

Theorem 1.1 (The Weighted Fundamental Lemma for Lie Algebras, cf. [Art02, Conjecture 5.1]). *Assume s_M is not central in $\widehat{M}$, so $M' \neq M$, and pick $\gamma' \in \mathfrak{m}'(F_v)$ semisimple G-regular. Then we have the identity*

$$O^G_{M',M}(\gamma') = \sum_{s \in \mathscr{E}_{M'}(G)} \frac{|Z(\widehat{M}')^\Gamma / Z(\widehat{M})^\Gamma|}{|Z(\widehat{G}'_s)^\Gamma / Z(\widehat{G})^\Gamma|} SO^{G'_s}_{M'}(\gamma')$$

In the case that G and M' are both split groups, this is [CL12, Théoréme 19.1.1].

1.3. The shape of prior work.

We recall now, impressionistically, the framework of the previous works of Ngô and Chaudouard–Laumon on the FL and WFL. For all precise definitions and results we defer to the body of the paper.

Geometric approaches have played a central role in approaches to versions of the Fundamental Lemma since Goresky–Kottwitz–Macpherson [GKM04] observed that the values of local orbital integrals were

[1] To be somewhat more precise, G'_s is the quasisplit F_v-form of the split Langlands dual of $\widehat{G}'_s$ twisted by the Out-torsor defined by the conjugation action in G_s, which is a split extension of $\widehat{G}'_s$ by W_{F_v}. Compare with *loc. cit.* and §14.6 below.

computable in terms of the $\mathbb{F}_q$-points (and so by Grothendieck–Lefschetz, in terms of the cohomology) of affine Springer fibers. A crucial insight of Ngô was that although the WFL is a local identity of orbital integrals, it can be better approached geometrically in the global function field setting. In that setting, Arthur's stabilization of the global trace formula should be incarnate within the relative cohomology of the Hitchin fibration $f_G : \mathscr{M} \to \mathscr{A}$ associated to a quasi-split reductive group scheme G over a curve $X/\mathbb{F}_q$ and a divisor $D > 0$. The space is the moduli stack of pairs $(\mathscr{E}, \theta)$ where $\mathscr{E}$ is a G-torsor on X and $\theta \in H^0(X, \mathrm{Ad}_D(\mathscr{E}))$ is the Higgs field. For classical groups, the Higgs field is a D-twisted endomorphism $\mathscr{E} \to \mathscr{E}(D)$ modeled on the Lie algebra $\mathfrak{g}$. The Hitchin morphism is a global geometric upgrade of the Chevalley map $\mathfrak{g} \to \mathfrak{c}$.

Over the "G-elliptic" locus $\mathscr{A}^{G\text{-ell}} \subset \mathscr{A}$, the Hitchin fibration is a well-behaved Deligne–Mumford stack, and so its relative cohomology admits a Belinson–Bernstein–Deligne–Gabber decomposition after [Bei+18]. The Hitchin fibration and a certain rigidification admit a faithful fiberwise action of a large commutative group stack, the Picard stack $\mathscr{P} \to \mathscr{A}$, which acts on the relative cohomology through its fiberwise groups of components $\pi_0(\mathscr{P})$. Over the G-elliptic locus, $\pi_0(\mathscr{P})$ is fiberwise finite, and so after a rigidification (denoted in this work with $\widetilde{(\ )}$, e.g. $f : \widetilde{\mathscr{M}} \to \widetilde{\mathscr{A}}$) we obtain a well-behaved isotypic decomposition.

Components of this isotypic decomposition on perverse relative cohomology $\pi_0(\widetilde{\mathscr{P}}) \curvearrowright {}^p\mathscr{H}^i(Rf_\star^{G\text{-ell}}\overline{\mathbb{Q}}_l)$ can be identified with endoscopic data s for the group G, i.e., (ignoring all shifts and twists) we have a geometric decomposition over $X_{\overline{\mathbb{F}}_q}$

$$ {}^p\mathscr{H}^i(Rf_{G,\star}^{G\text{-ell}}\overline{\mathbb{Q}}_l) = \bigoplus_{s \in \widehat{T}} {}^p\mathscr{H}^i(Rf_{G,\star}^{G\text{-ell}}\overline{\mathbb{Q}}_l)_s. $$

where the decomposition ranges over elements of a torus $\hat{T}$ in the split dual group $\widehat{G}$ and is supported on finitely many.

The Fundamental Lemma is an identity equating a s-twisted orbital integral for the group G with a stable orbital integral for a corresponding endoscopic group H_s. Given a geometric endoscopic datum (s, ρ), there is a map $i_{(s,\rho)} : \widetilde{\mathscr{A}}_{H_s} \longrightarrow \widetilde{\mathscr{A}}_G$, and Ngô proves a corresponding cohomological version of the Fundamental Lemma: up to semisimplification and an explicit shift and twist,

$$ {}^p\mathscr{H}^i(Rf_{G,\star}^{G\text{-ell}}\overline{\mathbb{Q}}_l)_s \cong \bigoplus_{\rho} i_{(s,\rho),\star}{}^p\mathscr{H}^i(Rf_{H_s,\star}^{H_s\text{-ell}}\overline{\mathbb{Q}}_l)_1, $$

where $f_{H_s} : \widetilde{\mathscr{M}}_{H_s} \to \widetilde{\mathscr{A}}_{H_s}$ is the corresponding Hitchin morphism for the endoscopic group, and the sum ranges over endoscopic data associated with s. Building an appropriate point $a \in \widetilde{\mathscr{A}}^{G\text{-ell}}$ which is interesting at only one place $v \in X$ and applying a product formula, Ngô can deduce the Fundamental Lemma.

The *Weighted* Fundamental Lemma, then, and the non-elliptic parts of Arthur's stabilization, should live geometrically within the cohomology of the part of the Hitchin fibration which is generically regular semisimple (denoted $\mathscr{A}^\heartsuit$ in [Ngô10]), but not G-elliptic. This is the project which, entirely in the case of split constant reductive $G/\mathbb{F}_q$, Chaudouard–Laumon tackle. On the non-G-elliptic complement, which is indexed by proper Levi subgroups $M \subsetneq G$, the Hitchin fibration is worse-behaved in several ways which fundamentally stem from the existence of split central tori in M (the original group, one can reduce to be semisimple). It is not quasi-compact, with the fiber $\mathscr{P}_a$ having a lattice of components of rank equal to that of the split central torus. It is also wildly non-separated. The majority of [CL10; CL12] builds the foundations necessary to bypass these issues and extend Ngô's cohomological results to this locus.

These same infinitude problems induced by split central tori had already been visible in the harmonic analysis work of Arthur, where they appeared as the nonconvergence of orbital integrals resolved by introducing the weight functions. Chaudouard and Laumon had the brilliant realization that in the geometric framework of Ngô, in which adelic orbital integrals geometrize to Hitchin fibers, these weight factors were not a purely analytic phenomenon, but instead should *come from geometry*! In their first paper, they define an open substack $\widetilde{\mathscr{M}}^\xi \subset \widetilde{\mathscr{M}}$ of bundles satisfying a certain stability condition. Point counts on these truncated Hitchin fibers, they show, correspond to weighted orbital integrals closely related to those of Arthur. They prove fundamental geometric properties of the restricted Hitchin map $f^\xi : \widetilde{\mathscr{M}}^\xi \to \widetilde{\mathscr{A}}$ which ensure its cohomological formalism is well-behaved over all of $\mathscr{A}^\heartsuit$.

In the second paper, Chaudouard and Laumon define a corresponding truncation $\widetilde{\mathscr{P}}^1 \subset \widetilde{\mathscr{P}}$ of the (rigidified) Picard stack which preserves $\widetilde{\mathscr{M}}^\xi$. The group of components $\pi_0(\widetilde{\mathscr{P}}^1)$ is always finite, so they obtain isotypic decompositions over the M-elliptic locus for all M. These isotypic decompositions can be identified with endoscopic data s_M for M. The result is a cohomological incarnation of the non-elliptic stabilization of Arthur, generalized to the nonsplit case by Theorem 14.24 below. Computing $\mathbb{F}_q$-points of the s-components, Chaudouard–Laumon show that this induces a global identity of adelic orbital integrals. A systematic procedure of *scindage* breaks this identity down into one consisting of local orbital integrals at only finitely many places, of a type which can be engineered to match that in [MW16b], and a local-to-global construction deduces the WFL.

In the remainder of the introduction, we outline the content of this thesis in an informal way, targeted at the reader familiar with the previous work in this area and emphasizing places of difference.

1.4. Geometry and cocharacters of relative reductive group schemes.

In Section 2, we collect the required facts about parabolics in relative reductive group schemes. Mostly extracted from [Con14], the theory and combinatorics of semistandard parabolics P and their semistandard Levi factors M adapts well, and one can work either in the function field or globally on X. The globalization, we note, is contingent upon the existence of a maximal torus over X, which we assume (and will suffice for our local applications). We assume, as in [Ngô10], that the torus T is split at a rational point $\infty \in X(k)$, and use this splitting to rigidify our Higgs bundles.

As in the theory of Borel and Tits [BT65], one can compare semistandard Levis via the centralizers of their *split* central tori. In many cases, we therefore need to supplement the existing arguments to track the behavior of split tori. See for example the proof of Proposition 3.14.

Stability convexes naturally live in cocharacter spaces $\mathfrak{a}_M = \mathrm{Hom}(X^\star(M), \mathbb{R})$ associated to semistandard Levis M. For such a nonsplit group, there are two notions of this space: the *rational* space $\mathfrak{a}_M$ corresponding only to cocharacters defined over X and the *geometric* space $\underline{\mathfrak{a}}_M$ corresponding to cocharacters over a splitting cover. Given a Γ-Galois étale cover splitting G, the former can be realized as either the Galois invariants or co-invariants of the latter. The Weyl group of the split form $\mathbb{W} = W(\infty)$ acts on the geometric space, while the global Weyl group $W(X) \subset W(\infty)$ acts on the rational space.

In constrast to the discrepancy in splitting behavior between G and the local G_v, the assumption that T be split at ∞ implies that the splitting behavior of T and the cocharacter spaces is independent of extension of the constant field of X; see Proposition 2.20. This is useful when passing between $k_0 = \mathbb{F}_q$ and $k = \overline{\mathbb{F}}_q$ and counting points.

1.5. Stratification of the Hitchin base and relative position.

The reader familiar with the work of Chaudouard and Laumon will recall that in order to remove a Weyl-group ambiguity from the stability convex, it is necessary to work over the étale open $\mathscr{W}_\infty : \widetilde{\mathscr{A}} \to \mathscr{A}$ consisting of choosing an ordering t of the roots of the characteristic polynomial a at the point ∞, or a point of the cameral cover (see §3.5). This ordering allows one to distinguish the "relative position" of different parabolic reductions of a G-torsor by comparing t to the order in which the $\theta(\infty)$-eigenspaces are included in the parabolic reduction. Correspondingly, the stability conditions in Chaudouard–Laumon's convex are cones which point in the directions of different Weyl-translates of the obtuse cone spanned by the coroots, or equivalently defined by positive conditions on the fundamental weights.

The previous work defines a stratification $\bigcup_M \widetilde{\mathscr{A}}_{M\text{-ell}}$ over semistandard Levis based on the "Levi factorization type" of the polynomial a (that is, element of the F-valued points of the Chevalley space). This stratification characterizes how many and which parabolic reductions the Higgs bundles $(\mathscr{E}, \theta)$ in the fiber admit. Chaudouard and Laumon interpret the reductions in different relative positions to t as reductions to the different semistandard parabolics which contain the Levi M. Crucially, their stratification of the base is the coarsest such that the set of parabolic reductions and their relative positions remains constant within a stratum, so is the appropriate stratification for studying weighted stability.

When adapting this, we run into a technical obstruction: the group G, by necessity, is more split at the point ∞ than generically. However, it is the generic behavior of G which determines the set of parabolic reductions that a torsor admits. For example, if $m = (\mathscr{E}, \theta)$ is a Higgs bundle living in the T-elliptic locus (i.e., $\chi(\theta)$ lifts to $\mathfrak{t}$ in the function field), then there will be a $W(X)$-torsor of Borel reductions for m. If we choose t arbitrarily from among the $W(\infty)$-torsor of possible orderings, then the relative position of the parabolic reductions may lie in some nontrivial $W(F)$-coset, and so cannot be naturally bijected with semistandard parabolics containing M.

We evade this obstruction in Section 3 by refining our stratification of the Hitchin base beyond that in [CL10] to one indexed by pairs

$$(w, M) \in W(\infty) \times \{\text{Levis } M \supseteq T\},$$

up to a natural equivalence relation, in which M reflects the elliptic type of the Higgs field θ and w measures the relative between t and θ. As a consequence of our more general analysis, we can then define a *compatible locus*, an open A inside $\widetilde{\mathscr{A}}$, which surjects onto $\mathscr{A}^{\infty\text{-}r.s.}$. On this locus these questions of relative position can be ignored, and the parabolic reductions of an M-elliptic Hitchin triple can be identified bijectively with semistandard parabolics containing M as in [CL10]. In §3.5, we show that our stratification is still subsumed by the stratification by "monodromic invariants" of Ngô, and recall the basic functoriality of the Hitchin base in endoscopic groups.

In Section 4, we recall the definition and basic properties of the Hitchin fibration $f : \mathscr{M} \to \mathscr{A}$ and its rigidification $f : \widetilde{\mathscr{M}} \to \widetilde{\mathscr{A}}$ over the locus of Higgs bundles $(\mathscr{E}, \theta)$ for which θ is generically regular-semisimple. We define the relative position of a parabolic reduction for a rigidified Higgs bundle (which we call "Hitchin triples" after Chaudouard–Laumon). Our primary playground for the rest of the thesis is the part of the Hitchin fibration over the compatible locus, which we denote $f : \mathsf{M} \to \mathsf{A}$.

1.6. Weighted stability.

In Section 5, we define degree vectors associated to the parabolic reductions. These degrees fit together into stability convexes, certain polyhedral convexes in the style of Behrend [Beh91] inside the geometric and relative *cocharacter spaces* $\underline{\mathfrak{a}}_T$ and $\mathfrak{a}_T$. Using them, we define the notion of ξ-stability. The moduli of ξ-stable Higgs bundles $\widetilde{\mathscr{M}}^{\xi} \subset \widetilde{\mathscr{M}}$ defines an open substack.

Our formalism for weighted stability is ultimately a mixture of rational (i.e., relative) techniques which work directly on X with the relative root datum $\Phi(G, S)$ and geometric (i.e., absolute) techniques that work over a splitting cover X' with the absolute root datum $\Phi(G_{X'}, T_{X'})$.

We define first a "geometric" stability convex C_m, which lives within the geometric cocharacter space $\mathfrak{a}_T$ but imposes only the stability conditions corresponding to parabolic reductions which descend to X (in particular, is *not* equal to the stability convex for the pulled-back Hitchin triple, so our definition does not reduce to the split case).

Our analysis of the base implies that the relative positions lie within a $W(\infty)$-translate of the rational space, so we can then relate our convex to a "rational" convex C_m defined within the relative space $\mathfrak{a}_T$. This comparison is cleanest over the compatible locus. Ultimately, we need the flexibility of both perspectives: Arthur's weight functions are defined and analyzed entirely in the relative setting, and so comparison to the rational stability convex allows us to apply the point counting arguments of Chaudouard–Laumon directly, without change. On the other hand, many of the geometric arguments of Chaudouard and Laumon reduce to computations on line bundles corresponding to geometric roots—computations which do not easily adapt to relative weight spaces, and so naturally occur over a splitting cover.

We emphasize that all definitions and constructions related to ξ-stability given in this paper are novel outside the split case. The previous work on stability convexes in Chaudouard–Laumon, as well as the complementary polyhedra of Behrend, are defined *only* for torsors for split groups (equivalently, for group schemes over X which are inner forms).[2]

A crucial warning is that although the notion of degree of an *individual* parabolic reduction is well-behaved under pullback, the stability convexes we define (i.e., the set of *all* parabolic reductions and their degrees) are *not* compatible with descent along a splitting cover. That is, the stability convex for

[2]Behrend's work defines some related concepts (such as the canonical/Harder–Narasimhan parabolic reduction) outside the split case, but the way in which he does so is specific to $\xi = 0$, which has the distinguishing property of being fixed by the Galois action on the geometric cocharacter space.

a G-Hitchin triple is *not* equivalent to one defined by descending the stability convex associated to the pulled-back Hitchin triple along a splitting cover.

This failure to behave well under descent is a reflection of the arithmetic fact that elliptic endoscopic groups can become Levi subgroups upon pullback. Consider, for example, a $U_{X'/X}(n)$-Higgs bundle $(\mathscr{E}, \theta)$ whose characteristic polynomial lies in the endoscopic locus $\widetilde{\mathscr{A}}_{H_s}$ corresponding to elliptic endoscopic group $H_s = U(k) \times U(n-k)$. Such a Higgs bundle is G-elliptic (that is, anisotropic) in Ngô's formalism, and the condition of being ξ-stable should therefore be vacuous. However, the pulled-back Hitchin bundle for $\mathrm{GL}_{n,X'}$ lies inside the locus corresponding to a proper Levi $\mathrm{GL}_k \times \mathrm{GL}_{n-k}$, so we obtain a nontrivial stability condition upstairs which, it turns out, would descend to cut out much of the endoscopic locus entirely.

1.7. Uniformization and adelic characterization.

In order to relate Hitchin fibers with adelic objects, we replace the uniformization results for G-torsors of [DS95] with corresponding relative results, e.g. Steinberg's theorem and [Hei10]. We prove in Section 6 an adelic description of the truncated Hitchin fiber, making essential use of the rational stability convex as the one which corresponds naturally to Beauville–Laszlo data. This comparison is therefore simplest on the compatible locus.

1.8. Geometric properties of the ξ-stable Hitchin morphism.

Outside the G-elliptic locus, the Hitchin fibration acquires some geometric pathologies which obstruct direct adaptation of the cohomological methods of Ngô. The truncation to ξ-stable bundles resolves some of these (the total space is now quasicompact, for example), while introducing new concerns. In particular, the Hitchin fibration $f : \widetilde{\mathscr{M}} \to \widetilde{\mathscr{A}}$ satisfies the existence half of the valuative criterion (though it is very far from separated), and it is not clear that the restriction to the open substack $\widetilde{\mathscr{M}}^{\xi}$ of ξ-stable bundles still contains a lift for all specializations valued in a DVR. Indeed, the ξ-stable locus turns out to truncate the fibration exactly the right amount, as Chaudouard and Laumon realized: the restricted Hitchin map $f^{\xi} : \widetilde{\mathscr{M}}^{\xi} \to \widetilde{\mathscr{A}}$ for a semisimple split group G satisfies *both* halves of the valuative criterion.

In the nonsplit setting, since our stability conditions have been defined in the relative space, the real obstruction is the existence of *split* central tori, and so the natural restriction is to assume G has *anisotropic center*. This is also preserved under passage to elliptic endoscopic groups, while semisimplicity may not be – crucial for the recursive construction of the stable orbital integrals. We carry this hypothesis through the geometric and cohomological parts of the paper, until we can recover our desired generality on the arithmetic side.

In order to prove the valuative criterion, we require a more systematic analysis of not only the ξ-stable (properly, the ξ-semistable) locus, but a stratification of its complement by a version of Harder–Narasimhan type. Much of the geometry of Bun_G, $\mathscr{M}$, and similar moduli stacks lies in not only its semistable locus, but in its full stratification by the type of the Harder–Narasimhan filtration of G-bundles (see for example [GL19], or [Sch15] for a nice exposition in the split case). A systematic formalism for such stratifications has been axiomatized by Halpern–Leistner in [Hal22], and these Θ-*stratifications* allow uniform treatment of many geometric results across GIT and the geometry of bundles.

In Section 7, we define the ξ-canonical reduction, a particular parabolic reduction of a Hitchin triple. We show equivalences between a few characterizations of this reduction in terms of different instability invariants, which seems to be novel in the literature even in the usual Harder–Narasimhan case $\xi = 0$ (though undoubtably well-known to experts). Off the back of these equivalences, we prove that the ξ-canonical reductions define a Θ-stratification. This allows us, in Section 10, to avail ourselves of a general semistable reduction theorem for Θ-stratifications [AHH23]. We thereby streamline and avoid generalizing the corresponding proof from [CL10], a rather technical adaptation of Langton's original semistable reduction algorithm for vector bundles which we found difficult to generalize to the non-split setting.

In Section 8 we show that the substack of ξ-stable Hitchin triples $\widetilde{\mathscr{M}}^{\xi}$ is of finite type. We require some results, more or less orthogonal to the main thrust of the paper, generalizing classical theorems of Harder on the existence of Borel reductions of bounded stability. Our results in this direction are by reduction to the absolutely almost simple case of split rank 1. They may also be able to be extracted from [GL19], with a different proof, but the proof we give here is self-contained. Some parts of the finite-type argument (but not the construction of Borel reductions) follow by descent from the split case. Other parts of the argument work directly with the relative root datum.

In Section 9 we prove the uniqueness half of the valuative criterion of properness, combining with the existence half of 10 to prove that the ξ-stable Hitchin fibration is and in Section 11 we prove that for sufficiently general $\xi \in \mathfrak{a}_T$ (avoiding a countable set) the moduli of ξ-stable Higgs bundles is a Deligne–Mumford stack. The methods of these sections are more or less direct adaptations of the proofs of [CL10]. The geometric properties can be summarized as follows.

Theorem 1.2 (Theorem 8.4, Theorem 9.1, Theorem 10.1, 11.1). *If G/X is a quasisplit reductive group with anisotropic center, and k satisfies the characteristic assumptions of §2.1, then the compatible ξ-stable Hitchin fibration*

$$f : \mathsf{M}^{\xi} \to \mathsf{A}$$

is a proper morphism from a smooth Artin stack of finite type to a smooth scheme of finite type. If ξ is moreover in general position *in the sense of Definition 5.10, then M^{ξ} is Deligne–Mumford.*

1.9. Picard stacks and degree maps.

In Section 12, we recall the Picard stack $\mathscr{P}$ of Ngô, whose action on Hitchin fibers controls their cohomology intimately. We define a degree map out of the restriction $\mathsf{P} \to \mathsf{A}$ of the Picard stack over the compatible locus, which on M-elliptic fibers takes values in $(X^{\star}(M))^{\vee}$, the dual space to the *rational* characters on M (see Proposition 2.11 and §12.4 below). The rank of this invariant, then, is controlled by the rank of the maximal split central torus in M. The fiberwise kernel P^1 of this map preserves the ξ-stable truncation $\mathsf{M}^{\xi} \to \mathsf{A}$.

In the second part of the Section, we then recall Ngô's construction of a map $\mathbb{X}_{\star} = X_{\star}(T_{\infty}) \cong X^{\star}(\hat{T}) \to \pi_0(\mathscr{P})$, the fiberwise-component group of the Picard stack. Using the induced action of $\mathbb{X}_{\star}$ on the relative cohomology of the truncated Hitchin fibration, we extend the isotypic decomposition of Ngô on the relative cohomology beyond the G-elliptic locus. Here several features are visible that were not in the split case, and our description of the Picard stack components $\pi_0(\widetilde{\mathscr{P}})$ is less complete than in the split case. Nevertheless, we obtain in Theorem 12.33 *two* well-behaved isotypic s-decompositions of the perverse cohomology over the compatible locus, one *fine* and one *coarse*. The discrepancy between the two, invisible in the case of split groups, measures the failure of taking Γ-invariants and taking the identity component to commute on $Z_{\widehat{G}}$, and ultimately provides a necessary scalar factor in the WFL.

1.10. Cohomological theorems.

We make use of the machinery we have generalized from the split case to carry over Chaudouard–Laumon's global cohomological theorems from which they deduce the WFL. These are Theorems 14.8, 14.14, and 14.24 below. In terms of methodology, the proofs are virtually unchanged, and these sections are essentially an exposition of certain parts of [Ngô10, §7] and [CL12, §10]. We also adapt in Section 13 the deformation-theoretic proof of a general discriminant criterion for a point of the Hitchin base to lie in Ngô's "good locus," see Theorem 13.24. This computation, which carries over with little change, will be essential in the final step of the proof of the WFL, when we globalize our local parameter and seek to apply our global theorems.

We remark that our ultimate statement of the global cohomological theorem differs slightly from theirs: in order to stay within the category of split groups, Chaudouard–Laumon restrict to an open which avoids the endoscopic loci in the Hitchin base corresponding to nonsplit endoscopic groups. Our theorem more naturally ranges over all endoscopic data for G corresponding to the parameter s for M. On the other hand, however, our cohomological theorem exhibits a crucial asymmetry between coarse and fine s-isotypic components, which ultimately will allow the correct arithmetic consequence.

1.11. Point counts and the global WFL.

In section 15, we perform our version of the Frobenius point counts which convert the global cohomological Theorem 14.24 into an adelic identity which matches the form of Arthur's WFL.

Theorem 1.3 (Theorem 15.16). *Let X_0 be a smooth geometrically integral projective curve over finite field k_0, and let G_0/X_0 a quasisplit reductive group, split at a chosen rational point ∞ and satisfying our running characteristic hypotheses. Let $M_0 \subseteq G_0$ be a semistandard Levi and $\mathfrak{M}' = (s_M, \rho_{s_M})$ be an elliptic endoscopic datum for M_0 with endoscopic group M_0'. Assume moreover that M_0' is split at ∞. For any $a_{M'} \in \mathsf{A}_{M'}^{M'\text{-}ell}(k_0)$ with images $a_M \in \mathsf{A}_M$ and $a \in \mathsf{A}_G^{good}$, we have an identity of adelic orbital integrals*

$$J_M^{G,s_M}(a_M) = |Z_{\widehat{M}'}^{\Gamma}/Z_{\widehat{M}}^{\Gamma}| \sum_{s \in \mathscr{E}_{\mathfrak{M}'}(G)} |Z_{\widehat{G}_s'}^{\Gamma}/Z_{\widehat{G}}^{\Gamma}|^{-1} S_{M'}^{G_s'}(a_{M'}),$$

where $J_M^{G,s_M}(a_M)$ is the global s-orbital integral of Definition 15.5 and $S_{M'}^{G_s'}(a_{M'})$ are the stabilized avatars of Definition 15.12.

This theorem removes the assumption that G_0 have anisotropic center which was required through the geometric part of the theory. Here, the main new ingredients consist of the following.

(1) We carefully confirm that our rational weighted stability conditions match Arthur's formalism for (G, M)-families and weight functions, which he defines always in the relative cocharacter space,
(2) We track the interplay between the various Γ-fixed centers which appear and the various refinements of Arthur's induced parameters $\mathscr{E}_{\mathfrak{M}'}(G)$ which intervene in the cohomological theorems.
(3) We ensure that our formalism for all prior theorems is sufficiently robust to allow them to apply to iterated elliptic endoscopic groups.

1.12. Local-to-global descent and the WFL.

Finally, in §16 we deduce the local WFL from its global adelic version. As in [CL12], this is more difficult than it was for the ordinary Fundamental Lemma because, although the Hitchin fibration satisfies a product formula over places on the curve, the condition of ξ-stability is a global condition. If one knew the existence of a global parameter sufficiently regular at all places except a fixed v, then the local stability condition at v could coincide with the global condition, but such a parameter need not exist. Instead, then, one requires a formalism of scindage to iteratively deduce the WFL on smaller sets of places.

The formalism we provide in §16.2 - §16.4 is, we hope, more minimalist than [CL12, §16-18]. We specialize the general notion of a "stabilizing family" to the condition that the WFL holds on a subset of closed points $V \subseteq |X_0|$, for a fixed G, M, M', etc. Our global theorem then becomes the statement that the WFL holds on $V = |X_0|$, and we prove a general peeling result which allows us to remove sets of places at which we know that the WFL already holds. In accordance with Ngô's principle of "perverse continuation," the proof then reduces to essentially showing that the WFL holds for sufficiently regular parameters. Some of these are simple or reduce to the usual Fundamental Lemma of Ngô, but one is genuinely nontrivial (Lemma 16.19). That result, which is a kind of reduction to the rank 1 case, also requires the greatest number of new ingredients in the nonsplit case.

Finally, we need to globalize an arbitrary local parameter in a way which is sufficiently regular at all other places, and such that the WFL at the place v recovers our desired local statement. With an eye towards brevity, we give a minimal proof that our local s-orbital integrals are locally constant in the characteristic parameter, rather than developing the full geometric theory of truncated affine Springer fibers. We then appeal to Poonen's formalism for Bertini theorems on finite fields [Poo04] in order to show that we can find a desired sufficiently-regular globalization of the local parameter.

1.13. The nonstandard WFL

For arithmetic applications, one generally desires the twisted WFL, which follows from the usual statement and its nonstandard analog. This nonstandard version will follow easily from our geometric formalism, but has been written not even in the split case. This is a work-in-progress and will be added to the current document.

1.14. Acknowledgments

Any cursory survey of this work makes self-evident the depth of its dependence on the ideas of Gérard Laumon. Without his insights this work would not exist, and the author wishes to acknowledge profoundly his memory. I hope to have affirmed the elegance of his ideas.

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Part 1

Background and the definition of weighted stability

CHAPTER 2

Generalities on reductive group schemes

One of the primary goals of this work is to adapt the constructions and arguments of [CL10; CL12] to the case of quasisplit reductive group schemes $G \to X$ over a curve which are neither split nor arise by base change from the scalar field. Thus, it is useful to collect in one place the various results which allow us to work with such schemes combinatorially. In this section, S is an arbitrary scheme except where stated otherwise, while X is always a smooth geometrically integral curve over a field k. A *reductive group scheme* $G \to S$ or $G \to X$ is a smooth group scheme, affine over the base, whose geometric fibers are connected reductive groups. In the latter case, we assume always that k is of good characteristic in the sense defined below relative to $G \to X$.

2.1. Characteristic assumptions.

In this work, "good characteristic" relative to G is a stronger condition than the condition by that name elsewhere in the literature. It means that the characteristic of the underlying field k

(1) does not divide the order of the (split) Weyl group for G, and
(2) the finite monodromy action of $\pi_1(X, \infty)$ on the character lattice $\underline{X}^\star(T)$ of a fixed maximal torus (see below) has order prime to p.

In reality more mild assumptions than the first would perhaps suffice; our primary requirements are

(1) That the Chevalley–Todd–Shepard theorem should hold,
(2) that a Kostant section should exist,
(3) that p does not divide $\pi_1(G)$, and
(4) that centralizers of semisimple elements of the Lie algebra should be (connected) Levi subgroups.

In §11, we will also need the tameness assumption for the torus. For our intended arithmetic applications, it suffices to show the results for sufficiently large characteristic, so we are free to avoid any finite number of characteristics.

Proposition 2.1 ([Ste75, Theorem 3.14], [GKM09, Proposition 14.1.1]). *Let G a reductive group over a field k of characteristic p. Then $Z_G(Y)$ is (geometrically) connected for every semisimple $Y \subseteq \mathfrak{g}(k)$ if and only if p is not a torsion prime for G. Moreover, if p does not divide the order of the Weyl group, then $Z_G(Y)$ is a Levi $L \supset T$ for every semisimple $Y \in \mathfrak{t}(k)$.*

In fact, avoiding torsion primes and $p = 2, 3$ suffices for this result.

Remark 2.2. In [Ngô10], the author takes $p > 2h + 2$, where h is the maximal height of a root, solely to ensure the existence of a Kostant section. By [Ric17, Theorem 3.2.2], our characteristic assumptions are sufficient, see also [BČ22], and indeed it is clear that the results of [Ngô10] hold under the weaker characteristic assumptions.

2.2. Maximal Tori.

For tori, we retain the usual (strong) definition of split: a torus $T \to S$ is *split* if it is isomorphic to $D_S(M)$ for M a $\mathbb{Z}$-lattice. In particular, the rank of a split torus is required to be constant over the base.

Proposition 2.3. *Assume S is connected, Noetherian and normal, and $G \to S$ is a reductive group scheme. Every maximal torus $T \subseteq G$ is split by a* finite, connected *étale cover.*

Proof. The assumptions on S allow us to work at the generic point, where every étale cover is refined by a finite étale one. Let $H \subseteq \pi_1(\eta, \overline{\eta})$ be the corresponding open subgroup of finite index. Its image under the surjection $\pi_1(\eta, \overline{\eta}) \to \pi_1(S, \overline{\eta})$ corresponds to a finite connected étale cover $S' \to S$ which splits T at the generic point. But this implies that T is split over all of S'. ☐

2.3. Quasi-split groups, pinnings, and forms.

For relative reductive group schemes, we generally follow the notation and conventions of [Con14], with the following major exception: following [Ngô10], we use the term *quasi-split* to indicate that a reductive group scheme $G \to S$ arises as the S-form of a reductive $\mathbb{Z}$-group $\mathbb{G}$ associated to an étale $\mathrm{Out}(\mathbb{G})$-torsor ρ_G.

Definition 2.4. Let $G \to S$ a reductive group scheme. A *pinning* of G is a triple (T, B, s), where T is a maximal torus over S, $B \supseteq T$ is a Borel (i.e., all its geometric fibers are Borels), and $s \in H^0(S, \mathrm{Lie}\, B)$, such that over an étale-local cover $S' \to S$ the torus splits and $s = \sum_{\mathfrak{a} \in \Phi_+} s_{\mathfrak{a}}$ for some nowhere-vanishing sections $s_{\mathfrak{a}}$ in the root spaces $\mathfrak{u}_{\mathfrak{a}}$.A pinning is called *split* if the torus T is split.

Any pinning $(\mathbb{T}, \mathbb{B}, \mathbb{s})$ of $\mathbb{G}$ defines a section ε of the natural surjection $\mathrm{Aut}(\mathbb{G}) \to \mathrm{Out}(\mathbb{G})$ consisting of those automorphisms which preserve the pinning; twisting $\mathbb{G}$ along the $\mathrm{Aut}(\mathbb{G})$-torsor $\varepsilon_\star(\rho_G)$ gives rise to a reductive group scheme $G \to S$ which necessarily has a corresponding pinning, in particular G admits a Borel-torus pair.

In other words, we have shown that our notion of quasi-split is equivalent to "pinned quasi-split."

Proposition 2.5. *If $G \to S$ is a form of split group $\mathbb{G}$ defined by an $\mathrm{Out}(\mathbb{G})$-torsor ρ on S, then each pinning $(\mathbb{T}, \mathbb{B}, s)$ induces one for G by splitting. Conversely, if a form $G \to S$ of $\mathbb{G}$ admits a pinning then it is defined by an $\mathrm{Out}(\mathbb{G})$-torsor.*

This definition is strictly stronger than defining quasi-split to mean only the existence of a Borel, as in [Con14]: reductive group schemes with Borel subgroups, even over curves, may only admit maximal tori Zariski-locally, and even if they admit a maximal torus, may require further Zariski-localization in order to obtain a full pinning.

Given a geometric point $\infty \in X$, such a torsor corresponds to a conjugacy class of *pointed twisting data*

$$\rho_G^\bullet : \pi_1(X, \infty) \to \mathrm{Out}(\mathbb{G}),$$

with a particular such datum determined uniquely by picking a point ∞_{ρ_G} in the fiber of ρ_G over ∞. To alleviate notation, we will denote a pointed twisting datum also by ρ_G when no confusion can arise.

Over fields, the forms of a reductive group G are partitioned into inner equivalence classes, and within each inner class is a unique quasi-split form. Over a broader base, uniqueness of the quasi-split inner form fails easily, but existence holds even with our stronger definition of quasi-split.

Proposition 2.6 ([Con14, Prop. 7.2.11]). *Given a reductive group scheme $G \to S$, there is a (not necessarily unique) inner form G^{qs} of G which is quasi-split in our sense, and in particular admits a Borel-maximal torus pair $(B, T \subseteq B)$.*

Note that when k is algebraically closed or finite (our two primary cases of interest), the condition that $G \to X$ admit a Borel is automatic, as in [PR24, Prop. 2.13]. This does not, however, imply that inner classes over X are necessarily trivial. To be precise, they are classified by the following result.

Proposition 2.7 ([Con14, Prop. 7.2.2, Ex. 7.3.8]). *Let $\mathbb{G}$ be a split reductive group over S. The fibers of the map $H^1(S, \mathrm{Aut}_{\mathbb{G}/S}) \to H^1(S, \mathrm{Out}_{\mathbb{G}/S})$ are precisely the S-forms of $\mathbb{G}$ which are inner forms of each other.*

2.4. Character and cocharacter lattices.

Definition 2.8. For any algebraic group scheme $G \to S$, one can define the étale sheaf $\underline{X}^\star(G) := \underline{Hom}_S(G, \mathbb{G}_{m,S})$. We denote $X^\star(G) = \underline{X}^\star(G)(S)$ for its **global sections**, which can be identified with characters of the maximal split sub-S-torus.

For general algebraic groups G, this sheaf is neither representable nor particularly well-behaved (consider $G = \mathbb{G}_a$). The situation is much better for parabolics $P \subseteq G$ inside reductive group schemes. In [Con14, Thm. 5.3.1], the structure theory of reductive G/S is used to construct a derived subgroup $D(G) \subseteq G$ with expected properties:

- $D(G)$ is semisimple (i.e., each geometric fiber is semisimple, or equivalently the center is finite over S), and its construction is compatible with arbitrary base change.
- $D(G) \subseteq G$ is initial among subgroups with the property that $G/D(G)$ is a torus.
- For $A_G \subseteq Z(G)$ the maximal central torus, there is a central isogeny $A_G \times D(G) \to G$.

NOTATION 2.9. Since both will later be useful, we fix the following convention:

- the *maximal central torus* (fiberwise connected center in good characteristic) of a group $H \to S$ will be denoted A_H, and
- the *maximal S-split central torus* of a group $H \to S$ will be denoted A_H^{spl}.

COROLLARY 2.10. *For $P \subseteq G$ parabolic, $\underline{X}^\star(P)$ is representable by a separated étale group S-scheme, also denoted $\underline{X}^\star(P)$, étale-locally isomorphic to the group scheme $\mathbb{Z}^r$ for $r = \operatorname{rank} A_G$ (a locally constant function on S).*

PROOF. By [Con14, Cor. 5.2.5], parabolic subgroups have a satisfactory notion of unipotent radical $R_u(P)$. In particular, $P \to L := P/R_u(P)$ is initial among maps from P to reductive groups. Compatibility with base change then implies

$$\underline{\mathrm{Hom}}_S(P, \mathbb{G}_{m,S}) = \underline{\mathrm{Hom}}_S(L, \mathbb{G}_{m,S}) = \underline{\mathrm{Hom}}_S(L/D(L), \mathbb{G}_{m,S}).$$

The sheaf of characters of a torus is representable: in the split case by $\mathbb{Z}^r$, for locally constant rank $r = \operatorname{rank} L/D(L)$. The central isogeny $A_G \subseteq D(G) \to G$ descends to an isogeny $A_G \to L/D(L)$, so $\operatorname{rank} L/D(L) = \operatorname{rank} A_G$. For the general case étale descent is effective for separated étale schemes: indeed, the descent exists as an algebraic space, and any algebraic space which is separated and locally quasi-finite is representable by a scheme, by [Sta22, Tag 03XX]. $\square$

The fiberwise dual $\underline{\mathrm{Hom}}(\mathbb{G}_{m,S}, L/D(L))$ of $\underline{X}^\star(P)$ is likewise representable, denoted $\underline{X}_\star(P)$ with global sections $X_\star(P)$. Its monodromy representation as a twist of $\mathbb{Z}^r$ is the contragredient of that for $\underline{X}^\star(P)$, and the pairing of the two is perfect locally, but possibly not on global sections.[1]

Instead the dual of the global sections admits a characterization in terms of coinvariants, which we state now. We denote $\mathscr{F} \mapsto H_0(S^{\acute{e}t}, \mathscr{F})$ for the functor of 0-th homology, that is, the *left* adjoint to the constant sheaf functor $\mathsf{Ab} \to \mathsf{Sh}(S, \mathsf{Ab})$ (in contrast to global sections, which is the right adjoint).

PROPOSITION 2.11. *Let T a torus over S. Then the dual of the global sections $X^\star(T) = H^0(S, \underline{X}^\star(T))$ is given by*

$$\mathscr{X}_\star(T) := H_0(S, \underline{X}_\star(T))_{tf}$$

where the subscript tf indicates the quotient by torsion.

If S is connected Noetherian normal and $\overline{s} \to S$ is a geometric point, then H_0 is given by the Galois-coinvariants, and we have

$$\mathrm{Hom}(X^\star(T), \mathbb{Z}) = H_0(S, \underline{X}_\star(T))_{tf} = \underline{X}_\star(T_{\overline{s}})_{\pi_1(S,\overline{s}),tf}.$$

PROOF. By the adjunction and double-duality, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Ab}}(H_0(S, \underline{X}_\star(T)), \mathbb{Z}) &= \mathrm{Hom}_{\mathsf{Sh}(S^{\acute{e}t},\mathsf{Ab})}(\underline{X}_\star(T), \mathbb{Z}_S) \\
&= H^0(S, \mathscr{H}om(\underline{X}_\star(T), \mathbb{Z}_S)) \\
&= H^0(S, \underline{X}^\star(T)).
\end{aligned}$$

Of course, $\mathrm{Hom}_{\mathsf{Ab}}(-, \mathbb{Z})$ kills torsion, at which point both are finite-rank torsion-free and so dualizing gives the Proposition. $\square$

[1] For M a lattice with action of finite group Γ and dual $M^\vee$, there is a natural map $(M^\vee)^\Gamma \to \mathrm{Hom}(M^\Gamma, \mathbb{Z})$ which becomes an isomorphism over $\mathbb{Q}$ but may have finite cokernel.

We will sometimes need to consider cocharacters up to conjugacy.

Lemma 2.12. *Let $G \to S$ a reductive group scheme. The étale sheafification of the presheaf*

$$S' \mapsto \mathrm{Hom}_{S'}(\mathbb{G}_{m,S'}, G)/\{G\text{-}conj.\}$$

is representable by an étale S-scheme, étale-locally isomorphic to the constant scheme with fiber $X_\star(T)/W_G(T)$.

2.5. Parabolics and Levis.

Under our assumption that the quasi-split group admits a Borel pair, the usual dynamic theory of parabolics and Levis goes through unchanged.

Proposition 2.13. *Assume that the reductive group scheme $G \to S$ admits a maximal torus T. Call a parabolic or Levi semi-standard if it contains T.*

(1) Each semi-standard parabolic P has a unique standard $L = L_P$ splitting the natural surjection $P \to P/R_u(P)$.
(2) The pair (P, L) is constructed dynamically (that is, $P = P_G(\lambda), L = Z_G(\lambda)$ arise from the limit construction for a cocharacter $\lambda : \mathbb{G}_{m,S} \to T$.
(3) If $\pi : S' \to S$ is a finite Galois cover with Galois group Γ such that $(G_{S'}, T_{S'} = D_{S'}(M))$ is split, then the semi-standard parabolics of G are in bijection with the Γ-stable parabolics of $G_{S'}$. That is, they correspond to Galois-stable parabolic sets of roots for $G_{S'}$.
(4) If S is a field, the map $L \mapsto A_L^{\mathrm{spl}}$ is a containment-reversing injection from the set of standard Levi subgroups $L \supseteq T$ to the set of split sub-tori of T, with one-sided inverse $S \subseteq T \mapsto Z_G(S)$.

Proof. The claim (1) is [Con14, Prop. 5.4.5] (the uniqueness allows reduction to the split, strictly Henselian case). The claim (2) is [Gil15, Thm. 7.3.1.(1)], which holds without the connectedness assumption therein: simply build the desired cocharacter on each connected component separately (it will not be uniform over the base in the sense expected of (co)root data, but still defines a valid character). The claim (3) follows from the previous parts and étale descent. For the fourth claim, if $L \subseteq M$ are standard Levis, then the maximal tori in the centers of each A_M, A_L are contained in $T \subseteq L \subseteq M$. Therefore they satisfy the opposite containment $A_L \supseteq A_M$, and likewise their maximal split subtori. On the other hand, $Z_G(A_L^{\mathrm{spl}}) = L$ over a field by [BT65, Thm. 4.15]. ☐

Remark 2.14. The semistandard Levi subgroups within a quasi-split reductive group $G \to S$ are also quasi-split.

Proposition 2.15. *Given two semistandard Levis $L \subset M$ containing T, there exist parabolics $P \subset Q$ such that L is a Levi for P and M is a Levi for Q.*

Proof. By (2) above, all parabolics and Levis arise from the dynamic method. Moreover, just as in the case over a field, the assignments $\lambda \mapsto Z_G(\lambda)$ and $\lambda \mapsto P_G(\lambda)$ are constant on the facets of the Weyl fan (see §6.1 for a definition). They define inclusion-reversing bijections from the set of faces[2] (resp. closures of cones) of the fan to semistandard Levis (resp. parabolics). Define

$$W_L = \{v \in X_\star(S)_{\mathbb{R}} : \langle \alpha, v \rangle \ \forall \alpha \in \mathfrak{l} = \mathrm{Lie}(L)\}.$$

Then it is clear that $Z_G(\lambda) \supset L$ if and only if $\lambda \in W_L$, and $Z_G(\lambda) = L$ for any $\lambda \in W_L$ avoiding a finite number of hyperplanes. Take $\mu \in W_L$ a cocharacter with $Z_G(\mu) = M$. Choose a full-dimensional cone in W_L containing μ on its boundary, and let λ lie in the interior of that cone. Then $P = P_G(\lambda)$ is a parabolic with semistandard Levi L, $Q = P_G(\mu)$ is a parabolic with Levi M, and $P \subset Q$. ☐

[2]By faces we mean the linear subspaces defined by zero pairing with regards to subsets of the roots.

2.6. Character and cocharacter spaces and their relations.

Until the end of this section, let G a reductive group over connected S with maximal torus T.

DEFINITION 2.16. For any semi-standard parabolic $P \subseteq G$ with Levi L, define the *character space* as the étale sheaf

$$\underline{\mathfrak{a}}_P^\star = \underline{X}^\star(P) \otimes_{\underline{\mathbb{Z}}} \underline{\mathbb{R}},$$

the tensor product taken as abelian sheaves with the constant sheaf $\underline{\mathbb{R}}$. Define the *cocharacter space* $\underline{\mathfrak{a}}_P$ to be the sheaf of dual spaces, i.e., $\underline{\mathfrak{a}}_P = \underline{\mathrm{Hom}}(X^\star(P), \underline{\mathbb{R}})$. Write $\mathfrak{a}_P^\star = \underline{\mathfrak{a}}_P^\star(S)$ and $\mathfrak{a}_P = \underline{\mathfrak{a}}_P(S)$ for the real vector space of global sections. We will frequently abuse notation and write $\mathfrak{a}_L^\star = \mathfrak{a}_P^\star$ and $\mathfrak{a}_L = \mathfrak{a}_P$, reasonable since every character of P factors through L.

These are étale-locally isomorphic to the constant sheaf $\underline{\mathbb{R}}^r$ for r the rank of the maximal central torus A_L, and monodromy acts by elements of $\mathrm{GL}_r(\mathbb{Z})$ (its action on $\mathfrak{a}_P^\star$ and on $\mathfrak{a}_P$ differ by a transpose). These spaces are determined by the maximal central tori.

PROPOSITION 2.17. *For each semi-standard (P, L) in G, we have natural isomorphisms*

$$\underline{\mathfrak{a}}_P^\star \cong X^\star(A_L) \otimes_{\underline{\mathbb{Z}}} \underline{\mathbb{R}}, \qquad \textit{and} \qquad \underline{\mathfrak{a}}_P \cong \mathrm{Hom}_{\underline{\mathbb{Z}}}(X^\star(A_L), \underline{\mathbb{R}})$$

where A_L is the maximal central torus subgroup in L. In particular,

$$\underline{\mathfrak{a}}_P(S) \cong X^\star(A_L^{\mathrm{spl}}) \otimes_{\mathbb{Z}} \mathbb{R} \qquad \textit{and} \qquad \underline{\mathfrak{a}}_P^\star(S) \cong \mathrm{Hom}_{\mathbb{Z}}(X^\star(A_L^{\mathrm{spl}}), \mathbb{R}).$$

PROOF. As noted above, the central isogeny $A_L \times D(L) \to L$ gives rise to an isogeny of tori $A_L \to L/D(L)$. The construction of A_L and the isogeny is compatible with base change, so by functoriality of the presheaf tensor product we have a map of presheaf tensor products

$$\underline{\mathrm{Hom}}(L/D(L), \mathbb{G}_{m,S}) \otimes_{\underline{\mathbb{Z}}}^{pre} \underline{\mathbb{R}} \longrightarrow \underline{\mathrm{Hom}}(A_L, \mathbb{G}_{m,S}) \otimes_{\underline{\mathbb{Z}}}^{pre} \underline{\mathbb{R}}.$$

For any $S' \to S$, this is a bijection on the S'-sections (indeed, it suffices to work with connected S', where this is obvious). This sheafifies to the desired isomorphism. The isomorphism with $\underline{\mathfrak{a}}_P$ follows by dualizing section-by-section. $\square$

These sheaves are compatible with containments of semi-standard parabolics in the following sense, well-known in the split case over a field.

PROPOSITION 2.18. *Let S and (G, T) as above, and $P \subseteq Q$ a containment of semi-standard parabolics. Then $\underline{\mathfrak{a}}_Q^\star$ has a natural structure as a summand in $\underline{\mathfrak{a}}_P^\star$, and likewise $\underline{\mathfrak{a}}_Q$ inside $\underline{\mathfrak{a}}_P$. That is, there are canonical decompositions of sheaves*

$$\underline{\mathfrak{a}}_P = \underline{\mathfrak{a}}_Q \oplus \underline{\mathfrak{a}}_P^Q, \qquad \textit{and} \qquad \underline{\mathfrak{a}}_P^\star = \underline{\mathfrak{a}}_Q^\star \oplus \left(\underline{\mathfrak{a}}_P^Q\right)^\star \tag{2.6.1}$$

where the decompositions are as real vector spaces for each S'. These containments are proper whenever $P \subsetneq Q$.

PROOF. The inclusion of spaces comes from the natural restriction map $X^\star(Q) \to X^\star(P)$. This is injective on sections over any S': let $\chi : Q \to \mathbb{G}_m$ a character which sends P to the identity. Such a character corresponds bijectively to $\overline{\chi} : L_Q/D(L_Q) \to \mathbb{G}_m$, as before, which is a morphism between group schemes of multiplicative type. In particular, whether it is trivial can be checked on geometric fibers. So we reduce to the case of an algebraically closed field, where projectivity of $Q/P \subseteq G/P$ and connectedness gives the result. The induced morphism $\mathfrak{a}_Q^\star(S') \to \mathfrak{a}_P^\star(S')$ is then an inclusion of subspaces for each $S' \to S$.

In the other direction, note that if $P \subseteq Q$ with Levis $L_P \subseteq L_Q$, then the corresponding maximal central tori inside the Levis naturally satisfy $A_Q \subseteq A_P$: we can check containment inside of G on geometric fibers, hence reduce to the case of an algebraically closed field. This corresponds to a linear surjection of character lattices $X^\star(A_P) \twoheadrightarrow X^\star(A_Q)$. The previous proposition then shows there is an induced surjective map $\underline{\mathfrak{a}}_P^\star \to \underline{\mathfrak{a}}_Q^\star$, which is $\mathbb{R}$-linear on sections over any $S' \to S$.

The composition $\mathfrak{a}^\star_Q \hookrightarrow \mathfrak{a}^\star_P \twoheadrightarrow \mathfrak{a}^\star_Q$ corresponds over any S' to restricting characters along the chain $Q \supseteq P \supseteq A_P \supseteq A_Q$, which induces exactly the isomorphism of the previous proposition. Hence, we have canonical decompositions

$$\underline{\mathfrak{a}}^\star_P(S') = \underline{\mathfrak{a}}^\star_Q(S') \oplus \left(\underline{\mathfrak{a}}^Q_P\right)^\star(S')$$

for $\left(\underline{\mathfrak{a}}^Q_P\right)^\star = \ker(\underline{\mathfrak{a}}^\star_P \to \underline{\mathfrak{a}}^\star_Q)$ the complement. These decompositions are compatible with change in S', hence give a decomposition of sheaves. Since for any S', the space $\underline{\mathfrak{a}}_P(S')$ is dual to $\underline{\mathfrak{a}}^\star_P(S')$, we get a dual decomposition for $\underline{\mathfrak{a}}_P$. $\square$

Let G be a reductive S-group with maximal torus T and let $S' \to S$ be a Γ-cover which splits G. By construction, we have $\underline{\mathfrak{a}}_P(S) = \underline{\mathfrak{a}}_P(S')^\Gamma$, that is, that the rational cocharacter (resp. character) space is Galois-invariants inside the geometric cocharacter (resp. character) space. Dualizing these inclusions, we can also realize the rational spaces as Galois-coinvariants, that is, there exist canonical projections $\underline{\mathfrak{a}}_P(S') \to \underline{\mathfrak{a}}_P(S)$.

Proposition 2.19. *The rational cocharacter space $\mathfrak{a}_P(S) = X_\star(T^{\mathrm{spl}})_{\mathbb{R}}$ is equal to the Galois coinvariants $(X_\star(T_{S'})_{\mathbb{R}})_\Gamma$.*

Proof. Recall that there is a natural Γ-invariant surjective map $X^\star(T_{S'})_{\mathbb{R}} \to X^\star(T^{\mathrm{spl}})_{\mathbb{R}}$ given by restriction. The equivalence of categories between character lattices and tori shows the induced map on coinvariants is an isomorphism. But we also have a unique-up-to-scaling inner product on the root system, so we get a canonical splitting. $\square$

2.7. Insensitivity of splitting to base field.

Recall that the *split rank* of a torus T over S is the maximal n such that there exists a subtorus $\mathbb{G}_m^n \subseteq T$ over the base. When S is Noetherian irreducible and normal (hence connected), the anti-equivalence of tori on S and $\pi_1(S, \overline{s})$-lattices gives

$$\mathrm{rank}_{\mathrm{sp}}\, T = \dim_{\mathbb{Z}} L^{\pi_1(S,\overline{s})}$$

where L is the π_1-lattice corresponding to T. In our primary case of interest, the split rank is insensitive to extensions of scalars.

Proposition 2.20. *Let G be a (pinned) quasi-split group scheme over X, as in §2.3, with maximal X-torus T, and assume that there is a point $\infty \in X(k)$ such that T_∞ is split. Then there exists a geometrically connected étale cover $X' \to X$ splitting T. Moreover, the split rank of T is insensitive to any extension of scalars, in the sense that for any L/k field extension, the split rank of T_{X_L} is equal to the split rank of T on X.*

Proof. The torus T corresponds to a discrete $\Gamma = \pi_1(X, \overline{\infty})$-lattice; denote the action ρ. The choice of the point $\infty \in X(k)$ gives rise to a splitting of the sequence

$$1 \longrightarrow \pi_1(X_{k^s}, \overline{\infty}) \longrightarrow \pi_1(X, \overline{\infty}) \longrightarrow \mathrm{Gal}(k^s/k) \longrightarrow 1\ .$$

Since T_∞ is split, the image of the splitting acts trivially. The kernel of the monodromy $K \subset \pi_1(X, \overline{\infty})$ defines a connected étale cover splitting the group. Since $\rho(\pi_1(X, \overline{\infty})) = \rho(\pi_1(X_{k^s}, \overline{\infty}))$, the cover is geometrically connected.

For the latter claim, the split rank can only increase upon base change, so it suffices to show the reverse inequality when L is algebraically closed. First consider the case $L = \overline{k}$. We will pass through the separable closure $k^s \subseteq L$ of k. In particular, writing $N = \pi_1(X_{k^s}, \overline{\infty})$, we have

$$N/K \cap N \xrightarrow{\sim} NK/K = \pi_1(X, \overline{\infty})/K.$$

In particular, the pullback torus $T_{X_{k^s}}$ corresponds to an isomorphic lattice with an isomorphic action of a finite group, and so the split ranks are equal when $L = k^s$. The passage to $L = \overline{k}$ and then to an arbitrary algebraically closed field is a standard spreading-out argument as in [Con14, Prop. 3.2.2]. Alternatively when X is proper, the étale fundamental group is canonically insensitive to choice of algebraically closed field, see [SGAI, Exp. X, 1.8]. $\square$

Corollary 2.21. *Assume that there is a point $\infty \in X(k)$ such that T_∞ is split. Then for any field extension L/k, base change along $X_L \to X$ defines a bijection between semistandard parabolics of G and those of G_L, and likewise for semistandard Levis. For any semistandard parabolic P or Levi M, the spaces $\underline{\mathfrak{a}}_P = \underline{\mathfrak{a}}_M$ and $\mathfrak{a}_P = \mathfrak{a}_M$, their duals, and the maps between them are insensitive to extensions L/k of the base field.*

Proof. By the same reasoning as in the previous proposition, the monodromy image of $\pi_1(X, \overline{\infty})$ in the geometric root datum of $(G_{\overline{F}}, T_{\overline{F}})$ remains unchanged under extension of scalars. The Galois-stable parabolic subsets of the root datum are therefore in bijection, so the sets of parabolics are canonically identified at the generic point and thus globally on X. Since all semistandard Levi factors are uniquely determined by their parabolic i.e., according to the dynamic description, the same holds for Levis.

For the character spaces, every character of P or M factors through the maximal torus quotient, and every X-character through its maximal split quotient. The abelianization is always compatible with base change, and by the proposition above the maximal split quotient is also, so the split rank of M^{ab} and the maps between the character and cocharacter spaces for different (P, M) are canonically identified. □

2.8. Dual groups.

Fix our quasisplit G and Borel-torus (B, T) over the curve X/k. The geometric root datum of G determines a split dual group and torus $(\widehat{G}, \hat{T})$ over algebraically closed field E (which we will typically take to be $\mathbb{C} \cong \overline{\mathbb{Q}}_l$). Choosing a geometric point $\infty : \operatorname{Spec} K \to X$ equips the dual group with an action of $\pi_1(X, \infty)$, which factors through a finite quotient Γ. Each semistandard X-Levi $M \subset G$ defines a Γ-stable sub-geometric root datum, which dualizes to another such. That is, we have a natural bijection

$$T \subset M \subset G \mapsto \hat{T} \subset \widehat{M} \subset \widehat{G}$$

between semistandard X-Levis M in G and Γ-*stable* semistandard Levi subgroups $\widehat{M}$ in $\widehat{G}$. Equivalently, we could consider our target to be rational Levis of the L-group ${}^L G := \widehat{G} \rtimes \Gamma$.

Likewise, identifying parabolic subgroups with parabolic subsets of roots (and coroots) defines a bijection

$$P \mapsto \hat{P}$$

between semistandard parabolics for G and Γ-stable parabolics for $\widehat{G}$. It induces the bijection above on Levi factors.

Proposition 2.22. *For any semistandard Levi M in G and point $\infty \in X$ such that T_∞ is split, there is a natural isomorphism*

$$\hat{T}/Z^0_{\widehat{M}} = \operatorname{Hom}_{\mathbb{Z}}(X_\star(T \cap D(M))_\infty, \mathbb{C}^\times).$$

Proof. The claim is dual to

$$X^\star(\hat{T}/Z^0_{\widehat{M}}) = X_\star(T \cap D(M))_\infty.$$

The claim can be checked after base change of T, M, G to an algebraically closed field K, so we alleviate notation by omitting subscripts and assuming (G, T) are already over K. We have

$$X^\star(\hat{T}/Z^0_{\widehat{M}}) = \ker(X^\star(\hat{T}) \to X^\star(Z^0_{\widehat{M}}))$$

but $Z^0_{\widehat{M}}$ is the dual torus to $M/D(M) = M^{\mathrm{ab}}$, so

$$X^\star(\hat{T}/Z^0_{\widehat{M}}) = \ker(X_\star(T)_\infty \to X_\star(M/D(M))_\infty) = X_\star(T \cap D(M))_\infty.$$

□

2.9. Endoscopic groups.

The fundamental lemma relates orbital integrals for G (over a local field) to orbital integrals for endoscopic groups of G. As we will recall below, the Hitchin fibration for endoscopic groups is compatible with that for G. Let $(\mathbb{G}, \mathbb{B}, \mathbb{T}, \mathbb{s})$ a pinned split reductive group scheme and $(\widehat{G}, \widehat{B}, \hat{T}, \hat{\mathbb{s}})$ the dual pinned quasi-split reductive group. We have therefore a corresponding isomorphism $\mathrm{Out}(\mathbb{G}) \cong \mathrm{Out}(\widehat{G})$.

Write (G, B, T, s) for the pinned quasi-split reductive group over X defined by a torsor $\rho_G \in H^1(X, \mathrm{Out}(\mathbb{G}))$.

Definition 2.23. An *endoscopic datum* for G consists of a triple $\mathfrak{E} = (s, \rho_s, \rho_s \to \rho_G)$ defined as follows.

- $s \in \hat{T}_{tors}$.[3] Given such an element, consider its centralizer $Z_{\widehat{G} \rtimes \mathrm{Out}(\widehat{G})}(s)$ inside the semidirect product $\widehat{G} \rtimes \mathrm{Out}(\widehat{G})$. We write its connected-etale short exact sequence as
$$1 \longrightarrow \widehat{G}'_s \longrightarrow Z_{\widehat{G} \rtimes \mathrm{Out}(\widehat{G})}(s) \longrightarrow \pi_0(s) \longrightarrow 1 \ .$$
The pinning of $\widehat{G}$ induces a pinning of $\widehat{G}'_s$, cf. [GWZ20, Lemma 4.4]. The natural quotient map $Z_{\widehat{G} \rtimes \mathrm{Out}(\widehat{G})}(s) \to \mathrm{Out}(\widehat{G})$ and conjugation map $Z_{\widehat{G} \rtimes \mathrm{Out}(\widehat{G})}(s) \to \mathrm{Out}(\widehat{G}'_s)$ are both maps to finite constant étale groups, hence descend to $\pi_0(s)$:
$$\begin{tikzcd} & \pi_0(s) \arrow[dl, "o_{\widehat{G}'_s}(s)"'] \arrow[dr, "o_{\widehat{G}}(s)"] & \\ \mathrm{Out}(\widehat{G}'_s) & & \mathrm{Out}(\widehat{G}) \end{tikzcd} .$$
- ρ_s is a $\pi_0(s)$-torsor, and
- $\rho_s \to \rho_G$ is an $o_{\widehat{G}}(s)$-equivariant map. Equivalently, there is a canonical isomorphism $\rho_s \times^{\pi_0(s)} \mathrm{Out}(\widehat{G}) \to \rho_G$.

In other words, $\rho_s \to \rho_G$ is a reduction of scalars of ρ_G to $\pi_0(s)$. When no ambiguity can arise we will abbreviate an endoscopic datum as (s, ρ_s), with the structure of ρ_s as a reduction of ρ_G implicit.

Definition 2.24. Given s, the *split endoscopic group* $\mathbb{G}'_s$ is the pinned Langlands dual to $\widehat{G}'_s$, and in particular we have an isomorphism $\mathrm{Out}(\mathbb{G}'_s) \cong \mathrm{Out}(\widehat{G}'_s)$. The *endoscopic group* for G corresponding to the datum $(s, \rho_s, \rho_s \to \rho_G)$ is the quasisplit X-form G'_{ρ_s} of $\mathbb{G}'_s$ defined by twisting along
$$\rho_{G'_s} := o_{\widehat{G}'_s}(s)_\star(\rho_s),$$
viewed as a torsor for $\mathrm{Out}(\mathbb{G}'_s) \cong \mathrm{Out}(\widehat{G}'_s)$. We will denote it as G'_s when no confusion can arise, but it depends on the torsor ρ_s.

We will frequently work with a pointed version of this construction, which renders the third datum redundant. Assume now that we have chosen $\infty : \mathrm{Spec}\, \overline{k} \to X$ and a pointed twisting datum $\rho_G : \pi_1(X, \infty) \to \mathrm{Out}(\mathbb{G})$ defining G.

Definition 2.25. A *pointed endoscopic datum* for ρ_G is a pair $(s, \rho^\bullet_s)$, where $s \in \hat{T}_{tors}$ as above, and
$$\rho^\bullet_s : \pi_1(X, \infty) \to \pi_0(s)$$
is a homomorphism refining $\rho^\bullet_G$, i.e.,
$$\begin{tikzcd} \pi_1(X, \infty) \arrow[r, "\rho^\bullet_s"] \arrow[dr, "\rho^\bullet_G"'] & \pi_0(s) \arrow[d, "o_{\widehat{G}}"] \\ & \mathrm{Out}(\mathbb{G}) \end{tikzcd}$$

[3]In [GWZ20], they prefer the formalism of homomorphisms $\varprojlim_{\gcd(N,p)=1} \mu_N \to \widehat{T}$, each of which determines a finite-order étale subgroup. This is probably more natural.

Given such a datum, the endoscopic group G'_s is the form of $\mathbb{G}'_s$ defined by the pointed twisting datum $o_{\widehat{G'_s}} \circ \rho_s^\bullet : \pi_1(X, \infty) \to \mathrm{Out}(\mathbb{G}'_s)$.

We have the following standard characterization.

Proposition 2.26. *Let (s, ρ_s) be an endoscopic datum for G/X. Then the split root system for $\widehat{G'_s}$ with respect to $\hat{T}$ is*

$$\Phi(\widehat{G'_s}, \hat{T}) = \{\alpha \in \Phi(\widehat{G}, \hat{T}) : \alpha(s) = 1\}.$$

In particular, any proper endoscopic group G'_s satisfies $\dim G'_s \leq \dim G - 2$.

Proof. By construction, $\widehat{G'_s} = Z_{\widehat{G}}(s)^0$ contains $\hat{T}$. The Lie algebra of the centralizer is a direct sum of weight spaces, and by definition contains $\hat{\mathfrak{t}}$ and exactly those weight spaces on which s acts trivially, i.e. $\alpha(s) = 1$. But $Z_{\widehat{G}}(s)^0 \neq \widehat{G}$ (equivalently, $G'_s \neq G$) if and only if s is noncentral and we are missing at least one α. But if $\alpha(s) \neq 1$ then the same is true for $-\alpha$, so

$$\dim G'_s = \dim \mathrm{Lie}\, \widehat{G'_s} \leq \dim \mathrm{Lie}\, \widehat{G} - 2 = \dim G - 2$$

□

The Weyl groups satisfy a natural functoriality in endoscopic groups, just as they do for Levi subgroups.

Proposition 2.27 ([Ngô10, §1.9, 1.10]). *Let (s, ρ_s) be an endoscopic datum for $G \to X$, with endoscopic group $H \to X$. There is a natural injective homomorphism*

$$\mathbb{W}_H \rtimes \pi_1(X, \infty) \to \mathbb{W}_G \rtimes \pi_1(X, \infty),$$

or after passage to any finite Γ through which both actions factor,

$$\mathbb{W}_H \rtimes \Gamma \to \mathbb{W}_G \rtimes \Gamma.$$

The cocharacter spaces satisfy the same decompositions for general endoscopic groups that they do for Levis. Indeed, for the sub-root system of Proposition 2.26 above associated to endoscopic datum (s, ρ_s) with endoscopic group H, identifying the split maximal torus inside the endoscopic group gives direct sum decompositions

$$\underline{\mathfrak{a}}_H = \underline{\mathfrak{a}}_G \oplus \underline{\mathfrak{a}}_H^G,$$

just as for Levi subgroups: the inclusion $i : \underline{\mathfrak{a}}_G \hookrightarrow \underline{\mathfrak{a}}_H$ comes from the inclusion

$$\{x \in X_\star(T)(\infty) : \alpha(x) = 0\ \forall \alpha \in \underline{\Phi}_G\} \hookrightarrow \{x \in X_\star(T)(\infty) : \alpha(x) = 0\ \forall \alpha \in \underline{\Phi}_H\}$$

and the surjection $q : \underline{\mathfrak{a}}_H \to \underline{\mathfrak{a}}_G$ comes from the map

$$\underline{\mathfrak{a}}_T/\mathbb{R}\langle \underline{\Phi}_H^\vee \rangle \twoheadrightarrow \underline{\mathfrak{a}}_T/\mathbb{R}\langle \underline{\Phi}_G^\vee \rangle.$$

Proposition 2.28. *Both the inclusion $\underline{\mathfrak{a}}_G \hookrightarrow \underline{\mathfrak{a}}_H$ and the surjection $\underline{\mathfrak{a}}_H \twoheadrightarrow \underline{\mathfrak{a}}_G$ are equivariant for the homomorphism of Weyl groups $\mathbb{W}_H \rtimes \Gamma \to \mathbb{W}_G \rtimes \Gamma$. In particular, we have a corresponding canonical decomposition of rational cocharacter spaces*

$$\mathfrak{a}_H = \mathfrak{a}_G \oplus \mathfrak{a}_H^G.$$

Proof. All the equivariance is clear from the contruction of Ngô's homomorphism except to observe that under restriction to the centers, although the H-action of Γ and the G-action on $\mathbb{T}$ differ by a Weyl-translate, the split Weyl group acts trivially on any central geometric cocharacter. □

2.10. Geometric and arithmetic endoscopic data.

The definitions of the previous section are ambivalent about what field k the curve X is defined over. If k is algebraically closed, we call the above data *geometric* endoscopic data. Taking k_0 an arbitrary perfect field with algebraic closure k and X_0 a curve over k_0 with base change X/k, one has the canonical short exact sequence

$$1 \to \pi_1(X,\infty) \to \pi_1(X_0,\infty) \to \mathrm{Gal}(k/k_0) \to 1,$$

split if $\infty \in X(k_0)$. In particular, a pointed endoscopic datum is a pair consisting of $s \in \hat{T}_{tors}$ and a homomorphism

$$\pi_1(X,\infty) \rtimes \mathrm{Gal}(k/k_0) \to \pi_0(s),$$

such a datum is the same as a $\mathrm{Gal}(k/k_0)$-fixed conjugacy class of pointed geometric endoscopic datum $\pi_1(X,\infty) \to \pi_0(s)$. For example, one could have a geometric endoscopic datum which is fixed on the nose by $\mathrm{Gal}(k/k_0)$, though not all arithmetic endoscopic data arise in this way.

2.11. Twisting and Chevalley spaces.

We recall the elementary theory of the space of "characteristic polynomials" $\mathfrak{c}$. In this section let G_0 a split reductive group scheme over a field with pinning (B_0, T_0, s_0) and Weyl group $W_0 = \mathbb{W}$. Then the theorem of Chevalley–Todd–Shepard implies an isomorphism of affine quotients

$$\mathfrak{c}_0 := \mathfrak{t}_0 /\!/ W_0 \cong \mathfrak{g}_0 /\!/ G_0,$$

and that these are isomorphic to affine space. We call $\chi : \mathfrak{g}_0 \to \mathfrak{c}_0$ and various derived maps *characteristic polynomial maps*, and somewhat abusively call an element of $\mathfrak{c}_0$ a *characteristic polynomial*, though the invariant space may have additional generators (i.e., Pfaffians). The scaling $\mathbb{G}_m$-action on $\mathfrak{g}_0$ commutes with the adjoint action, so descends to an action on $\mathfrak{c}_0$ making χ equivariant. Explicitly, the action is given by the exponents of the group, the (canonically defined) degrees of homogeneous polynomial generators for $k[\mathfrak{t}_0]^W$.

After the theorem of Kostant (cf. [BČ22]) whenever the characteristic p is sufficiently large[4] there exists a section $\varepsilon : \mathfrak{c}_0 \to \mathfrak{g}_0^{G\text{-reg}} \subseteq \mathfrak{g}_0$. The restriction $\chi = \chi_G^T : \mathfrak{t}_0 \to \mathfrak{c}_0$ is finite-flat, and is an étale W_0-torsor over a nonempty open subset $\mathfrak{c}_0^{\mathrm{rs}} \subseteq \mathfrak{c}_0$.

Let (G, T, B, s) a pinned reductive group over X defined by twisting (G_0, T_0, B_0, s_0) with $\rho_G \in H^1(X, \mathrm{Out}(G_0))$. As in [Ngô10, §1.3], the constructions above can all be twisted by ρ_G to obtain corresponding constructions for G. In particular, we have X-schemes $\mathfrak{g}$ and $\mathfrak{c}$ (both vector bundles) along with a characteristic polynomial map $\chi : \mathfrak{g} \to \mathfrak{c}$ over X, which is $\mathbb{G}_m$-equivariant for an appropriate fiberwise action on $\mathfrak{c}/X$. In particular, given any $\mathbb{G}_m$-torsor $\mathscr{O}(D)$ on X, one can twist to obtain

$$\mathfrak{t}_D := \mathfrak{t} \times_X^{\mathbb{G}_m} \mathscr{O}(D) \qquad \text{and} \qquad \mathfrak{c}_D := \mathfrak{c} \times_X^{\mathbb{G}_m} \mathscr{O}(D),$$

along with a *characteristic polynomial map* $\chi : \mathfrak{t}_D \to \mathfrak{c}_D$ which is the finite affine quotient by W.

The scaling $\mathbb{G}_m$-action on $\mathfrak{g}$ likewise defines a twist of the entire Lie algebra $\mathfrak{g}_D := \mathfrak{g} \times_X^{\mathbb{G}_m} \mathscr{O}(D)$, and Chevalley–Todd–Shepard twists to an X-isomorphism

$$\mathfrak{c}_D = \mathfrak{t}_D /\!/ W \cong \mathfrak{g}_D /\!/ G.$$

If we have more strongly a *pointed* twisting datum $\rho_G^\bullet : \pi_1(X, x) \to \Gamma \subseteq \mathrm{Out}(\mathbb{G})$ for $x \in X$, it determines a Γ-Galois cover $\pi : X' \to X$ as well as a point $x' \in X'$ over x and an isomorphism $G_{X'} \cong \mathbb{G} \times_{\mathbb{Z}} X'$.

Definition 2.29 (The Discriminant Divisor). For the split group G_0, define the *discriminant function* D_{G_0} on $\mathfrak{t}_0$ to be the polynomial function

$$\prod_{\alpha \in \Phi^{G_0}} d\alpha.$$

This is clearly W_0-invariant, so descends to a function on $\mathfrak{c}_0$, also denoted D_{G_0}. The *discriminant divisor* $\mathfrak{D}_{G_0} \subset \mathfrak{c}_0$ is the vanishing of this function.

[4]In particular, good in our sense suffices.

By [Ngô10, Lemme 1.10.1] the discriminant divisor is reduced, with complement $\mathfrak{c}_0^{\mathrm{rs}}$, and stable under the action of $\mathrm{Out}(G_0)$. Thus, the divisor twists to give a corresponding discriminant divisor $\mathfrak{D}_G$ on $\mathfrak{c}_D$ satisfying the same properties.

The Chevalley spaces satisfy a basic functoriality in endoscopic groups. Let (s, ρ_s) be an endoscopic datum for $G \to X$, with endoscopic group $H \to X$.

Proposition 2.30 ([Ngô10, §1.9, 1.10]). *Under the homomorphism*

$$\mathbb{W}_H \rtimes \pi_1(X, \infty) \to \mathbb{W}_G \rtimes \pi_1(X, \infty)$$

of Proposition 2.27, identifying split maximal tori in H and G, twisting defines a natural X-map

$$\chi_G^H : \mathfrak{c}_{H,D} \to \mathfrak{c}_D,$$

Definition 2.31. The *resultant divisor* $\mathfrak{R}_H^G$ for (s, ρ_s) on $\mathfrak{c}_{H,D}$ is defined by

$$(\chi_G^H)^\star \mathfrak{D}_G = \mathfrak{D}_H + 2\mathfrak{R}_H^G.$$

This satisfies a simple transitivity property:

Proposition 2.32. *Let (s, ρ_s) be an endoscopic datum for G defining a group H, and let (s_H, ρ_{s_H}) be an endoscopic datum for H defining a group K. Then under the maps of Proposition 2.30,*

$$\mathfrak{R}_K^G = \mathfrak{R}_K^H + (\chi_H^K)^\star \mathfrak{R}_H^G.$$

Proof. Immediate from the definition and functoriality of pullback for the discriminant. ☐

After passing to a splitting cover and identifying maximal tori, the resultant divisor is given by

$$\mathfrak{R}_H^G = \mathrm{div}\left(\prod_{\alpha \in \Psi} d\alpha\right),$$

where $\Psi \subset \Phi^G - \Phi^H$ contains one of $\alpha, -\alpha$ for each pair of roots, cf. [Ngô10, Lemme 1.10.2]. In particular, when $H = M$ happens to be a Levi subgroup of G, such a choice is given by a parabolic $P = MN$ for M; then

$$R_{M,P}^G := \prod_{\alpha \in \Phi(N,T)} d\alpha$$

is locally an equation defining $\mathfrak{R}_M^G$. It is independent of P up to sign.

We obtain a simple determinantal formula for the resultant relative to a Levi, whose proof follows immediately by descent from the split case and we leave as an exercise.

Proposition 2.33. *Let $Y \in \mathfrak{m}$ be any section of the Lie algebra of M and $P = MN$ a Levi for M. Then the function* $\det(\mathrm{ad}(Y)|_{\mathfrak{n}})$ *is M-invariant, so descends to a corresponding section $R_{M,P}^G$ of $\mathfrak{c}_{M,D}$. We have*

$$\mathrm{div}(R_{M,P}^G) = \mathfrak{R}_M^G.$$

CHAPTER 3

The Hitchin base and ellipticity

In this section, let k be an arbitrary field. Let X be a smooth, geometrically integral curve over k, and let $G \to X$ be a quasi-split reductive group scheme (cf. §2.3). We assume k is of good characteristic for G. We assume there exists $\infty \in X(k)$ such that G_∞ is split—this will ensure that our combinatorics in terms of Levi subgroups remains unchanged when extending k, cf. Corollary 2.21.

We denote by W the Weyl group, viewed as a finite étale group scheme on X. So $W(X) = W(F)$ is the Weyl group of G_F, while $W(\infty) = \mathbb{W}$ is the split Weyl group.

3.1. The Hitchin base.

Definition 3.1. For each semistandard Levi $M \subseteq G$, define the *characteristic scheme* or *Hitchin base* $\mathscr{A}_M$ as the k-scheme corresponding to the vector space $H^0(X, \mathfrak{c}_{M,D})$. Define the *cameral base* over ∞, abusively denoted $\widetilde{\mathscr{A}}_M$, as the fiber product

$$\begin{tikzcd} \widetilde{\mathscr{A}}_M \arrow[r] \arrow[d, "\mathscr{W}_{M,\infty}"'] & \mathfrak{t}_D^{G\text{-reg}} \arrow[d, "\chi_M"] \\ \mathscr{A}_M \arrow[r, "\mathrm{eval}_\infty"] & \mathfrak{c}_{M,D} \end{tikzcd}$$

That is, an S-point of $\widetilde{\mathscr{A}}_M$ consists of a pair (a, t), where $a \in H^0(X_S, \mathfrak{c}_D(\mathscr{E}))$ and $t \in \mathfrak{t}_D^{G\text{-reg}}(S)$ (these are points in the *total space* of the vector bundle $\mathfrak{t}_D$), such that

$$a(\infty_S) = \chi_M(t).$$

Note this condition implies that t lives over the constant S-point ∞_S. Note also that the regularity condition implies that $\widetilde{\mathscr{A}}_M$ depends upon G, not just M.[1] When no confusion is possible, we write $\mathscr{A} := \mathscr{A}_G$ and $\widetilde{\mathscr{A}} := \widetilde{\mathscr{A}}_G$.

Equivalently, defining the *cameral cover* $X_a \to X \times S$ by the Cartesian square

$$\begin{tikzcd} X_a \arrow[r] \arrow[d] & \mathfrak{t}_D \arrow[d, "\chi_G^T"] \\ X \times S \arrow[r, "a"] & \mathfrak{c}_{G,D} \end{tikzcd},$$

the data t consists of a geometric point $\widetilde{\infty} \in X_a$ lying over ∞.

The map $\mathscr{W}_{M,\infty} : \widetilde{\mathscr{A}}_M \to \mathscr{A}_M$ is a $W_M(\infty)$-Galois finite étale cover of the open locus $\mathscr{A}_M^\infty \subset \mathscr{A}_M$ consisting of points which are G-regular semisimple at ∞ (and hence generically). In particular, it is a smooth k-scheme. We record also $\mathscr{A}_M^\heartsuit \subset \mathscr{A}_M$, the larger open subset of points which are generically G-regular semisimple.

Proposition 3.2 ([CL10, Proposition 3.4.1], [Ngô10, Lemme 4.13.1]). *If* $\deg(D) > 2g - 2$, *the schemes* $\mathscr{A}_M$ *and* $\widetilde{\mathscr{A}}_M$ *are irreducible of dimension*

$$\dim(\mathscr{A}_M) = \dim(\widetilde{\mathscr{A}}_M) = \deg(D)(\#\Phi^M/2 + \operatorname{rk} M) + (1 - g)\operatorname{rk} M$$

where Φ^M *is the set of roots of the split form of* M.

[1]For example, if $M = T$ is the standard torus inside $G = \mathrm{GL}_n$, then the condition of T-regularity would be trivial, while the condition of G-regularity means the components of $\mathfrak{t}_D$ are pairwise distinct.

Proof. We recall the proof. For irreducibility one remarks that the map forgetting a is a surjective open map onto $\mathfrak{t}^{G\text{-reg}}$ with fibers isomorphic to affine spaces. To compute the dimension of $\mathscr{A}_M$, one uses a splitting Galois cover $\pi : X' \to X$. After pullback one has $\pi^\star \mathfrak{c}_D = \bigoplus_{i=1}^r \mathscr{O}(D)^{\otimes e_i}$, where the e_i are the exponents for the split group $\mathbb{G}$. The degree condition implies the vanishing of $H^1(X', \pi^\star \mathfrak{c}_{M,D})$ and thus its summand $H^1(X, \mathfrak{c}_{M,D})$. The simple form of Riemann–Roch on X then gives the desired formula, noting that

$$e_1 + \cdots + e_r - r = \sum_{i=1}^r (e_i - 1) = \#\Phi^M/2$$

Since $\widetilde{\mathscr{A}}_M$ is an étale cover of a nonempty open in affine space, its dimension is the same. □

3.2. Example: the quasi-split unitary group.

Our goal is to adapt the stratification of the Hitchin base discussed in [CL10, §3] to the relative quasisplit setting. Many of the arguments translate directly. In particular, the "split at ∞" assumption ensures that $T_{\operatorname{Spec} \mathscr{O}_\infty}$ is split (see [Sta22, Tag 0BQC] and [Con14, B.3.6]), so when working at F_∞ nothing changes from [CL10].

The biggest new ingredient in this part of the theory comes from the fact that the rigidifying data t is fundamentally overdetermined. In order to explain, we illustrate the failure of the converse of [CL10, Corollaire 3.9.2] in the case of the unitary group. Let $G = U(n)$ be the quasi-split unitary group associated with étale double cover $\pi : X' \to X$ (Galois element σ) of curves over a field k. This can be presented as the subgroup of $\operatorname{Res}_{X'/X} \operatorname{GL}_n$ consisting of matrices A satisfying

$$\overline{A}^T \Phi A = \Phi, \qquad \Phi = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$

and its Lie algebra with the matrices which are skew-Hermitian for the same form.

Denote the extension of function fields corresponding to by F'/F. With respect to the presentation above, T denotes the standard diagonal maximally split maximal torus. Pick $\infty \in X(k)$ a point which splits in X', denoting the fiber $\pi^{-1}(\infty) = \infty' \sqcup \infty''$. So G_∞ is a split unitary group over k, isomorphic to $\operatorname{GL}_{n,k}$ by a choice of basepoint ∞' or ∞'' over ∞, and the maximal torus pulls back to the diagonal maximal torus T'. The isomorphism is given by choosing one of the points in the fiber, without loss of generality ∞', and restricting sections to it.

Example 3.3. Let $n = 3$ and $M = T$, and let (a, t) be a pair such that a admits a reduction to $M = T$. That is, a_M consists of an ordered factorization $(X - r_1)(X - r_2)(X - r_3)$ with $r_i \in H^0(X', \mathscr{O}(D))$ which satisfy $\sigma^\star(r_i) = -r_{4-i}$. This skew-symmetry condition implies that the values taken by the r_i at ∞'' are determined by those at ∞', and the same is true for t. we must have

$$\prod_i (X - r_i|_{\infty'}) = \prod_i (X - t_i|_{\infty'})$$

so there exists a permutation $\pi' \in W_G(\infty) = S_3$ such that $r_i|_{\infty'} = t_{\pi'(i)}|_{\infty'}$. However, this is data for the split group $G|_\infty \cong \operatorname{GL}_3|_{\infty'}$, so there is no requirement that this permutation lie in the *global* Weyl group $W_G(X) = W_G(F) = \langle (1\ 3) \rangle \leq W_G(\infty)$.

For a counterexample, choose an etale cover π and rational functions $x \in H^0(X', \mathscr{O}(D))^{\sigma=1} \subset F, y \in H^0(X', \mathscr{O}(D))^{\sigma=(-1)} \subset F'$ such that

$$x(\infty') = 1 = y(\infty') \quad \text{ and so } \quad x(\infty'') = 1 = -y(\infty'').$$

Consider the elements

$$a(\lambda) = (\lambda + x - y)(\lambda - y)(\lambda - x - y), \qquad t(\infty') = (1, 2, 0), \quad t(\infty'') = (0, -2, -1).$$

We have a compatible pair $(a, t) \in \widetilde{\mathscr{A}}_G$ because

$$\chi_G^T(t)(\infty') = (X - 1)(X - 2)(X - 0) = a(\infty')$$

and likewise over ∞''. The element a is clearly T-elliptic on its own (with the factorization given above or in reverse order), but all T-factorizations are incompatible with t, differing by a permutation $(1\ 2\ 3) \in W_G(\infty) \setminus W_G(X)$. Thus, $(a,t) \in \widetilde{\mathscr{A}}_G \setminus \widetilde{\mathscr{A}}_T$. We claim that the conclusion of the converse of [CL10, Corollaire 3.9.2] is false for $M = G$: no (regular semisimple) element $Y \in \mathfrak{g}(F)$ with characteristic polynomial a is G-elliptic. If it were, then considering the element

$$Z = \begin{pmatrix} y-x & & \\ & y & \\ & & y+x \end{pmatrix},$$

we proceed as in the proof of [CL10, Corollaire 3.9.2]. The two elements $Y, Z \in \mathfrak{g}(F)$ are conjugate by $g \in G(F^{\mathrm{s}})$, so their split centralizing tori are isomorphic: $A_Y^{\mathrm{spl}} \cong A_Z^{\mathrm{spl}}$. But $Z \in \mathfrak{t}(F)$, so its split centralizer is A_T^{spl} and $A_Y^{\mathrm{spl}} \neq A_G^{\mathrm{spl}} = 1$.

We note that this is still compatible with [CL10, Proposition 3.9.1]: the element (a,t) is not in $\widetilde{\mathscr{A}}_T$, and although we have found an element $Z = \mathrm{diag}(y-x, y, y+x)$ such that $\chi_G^T(Z) = a_\eta$, the element t_a agrees with $\mathrm{diag}(y, x+y, y-x)$ at ∞', and these are certainly not conjugate in $T(F_\infty) = T'(E_{\infty'})$.

The essential feature of the above example is that there are $\#W_G(\infty)$ possible orderings of the eigenvalues t, only $\#W_G(F)$ of which are *globally compatible* in the sense that they render (a,t) in the image of $\widetilde{\mathscr{A}}_T$. For the other values of t, the pair (a,t) should still be considered as T-elliptic (and indeed the image of a Kostant section in the function field will be T-elliptic), but we must introduce a twist living in the split Weyl group $W_G(\infty)$. Indeed, any Higgs bundle m with $f(m) = a$ will have a $W_G(F)$-torsor of Borel reductions, regardless of the ordering t. This motivates the formalism of the next section.

3.3. Weyl-twisted stratification and the compatible locus.

In this section, we construct the finest stratification of the Hitchin base over which ξ-stability is well-behaved. The combinatorics of this stratification is governed by semistandard Levis $X' \to X$. We warn the reader that while the stratification is stable under extending the scalar field k, it is *not* stable under a general étale base change $X' \to X$. This can be seen from the intervention of the endoscopic loci defined below: if H_s is an elliptic endoscopic group whose pullback to a splitting cover becomes a Levi subgroup (for example, $H_s = U(1) \times U(1)$ inside $U(2)$), then points in $\widetilde{\mathscr{A}}_{H_s}$ should still be considered G-elliptic, although they will not remain so upon pullback.

For each semistandard Levi M and each element $w \in W_G(\infty)$, there is a twisted closed embedding

$$\begin{aligned} \iota_w : \widetilde{\mathscr{A}}_M &\hookrightarrow \widetilde{\mathscr{A}}_G \\ (a_M, t) &\mapsto (\chi_G^M(a_M), w^{-1}(t)), \end{aligned}$$

defined because $a_M|_\infty = \chi_M(t)$ implies

$$\chi_G^M(a_M)|_\infty = \chi_G(t) = \chi_G(w^{-1}(t)).$$

We define $\widetilde{\mathscr{A}}_{(w,M)}$ the image under this closed embedding, characterized as the set of (a,t) such that $a|_\infty = \chi_M(w(t))$.

Proposition 3.4. *The loci $\widetilde{\mathscr{A}}_{(w,M)}$ satisfy the following redundancies:*

(1) $\widetilde{\mathscr{A}}_{(w,M)} = \widetilde{\mathscr{A}}_{(vw,M)}$ for all $v \in W_M(\infty)$, and
(2) $\widetilde{\mathscr{A}}_{(w,M)} = \widetilde{\mathscr{A}}_{(vw,vM)}$ for all $v \in W_G(F)$.

Proof. For (1), if $(a,t) = \iota_w(a_M, t')$ such that $a_M(\infty) = \chi_T^M(t')$, then the pair $(a_M, v(t'))$ also lies in $\widetilde{\mathscr{A}}_M$, and

$$\iota_{vw}(a_M, v(t')) = (a, w^{-1}v^{-1}v(t')) = (a,t).$$

For (2), let $(a,t) = \iota_w(a_M, t')$. Given $v \in W_G(F)$, we have a well-defined conjugate semi-standard Levi $L = vMv^{-1}$. The map $\mathfrak{t}_D \to \mathfrak{t}_D$ given by the action of v is equivariant for the map $W_M(F) \to W_L(F)$ given by conjugation by v, so descends to a map $\overline{v} : \mathfrak{c}_{M,D} \to \mathfrak{c}_{L,D}$. Writing $a_L = \overline{v}(a_M)$, we can see

$$a_L(\infty) = \overline{v}(a_M(\infty)) = v(\chi_T^M(t')) = \chi_T^L(v(t')),$$

so $(a_L, v(t'))$ is an element of $\widetilde{\mathscr{A}}_L$. We have

$$\iota_{vw}(a_L, v(t')) = (\chi_L^G(a_L), w^{-1}(t')) = (a, t)$$

because the following diagram commutes.

$$\begin{tikzcd}
\mathfrak{t}_D \arrow[r] \arrow[d, "v"] & \mathfrak{t}_D//W_M(F) \arrow[r] \arrow[d, "\overline{v}"] & \mathfrak{t}_D//W_G(F) \arrow[d, equal] \\
\mathfrak{t}_D \arrow[r] & \mathfrak{t}_D//W_L(F) \arrow[r] & \mathfrak{t}_D//W_G(F)
\end{tikzcd}$$

□

Definition 3.5. Define an equivalence relation $\sim$ on the set of pairs $(w, M) \in W_G(\infty) \times \{M \supseteq T\}$ by the two conditions in Proposition 3.4. Denote $[(w, M)]$ the equivalence class.

Define also a partial order by $[(w, M)] \leq [(w, L)]$ for $L \supseteq M$. Note that this is well-defined on equivalence classes for $\sim$.

We have shown the locus $\widetilde{\mathscr{A}}_{(w,M)}$ depends only on the class $[(w, M)]$. We will show a converse below, that on the *elliptic loci* distinct classes have disjoint images.

Proposition 3.6. *If* $[(w, M)] \leq [(v, L)]$, *then* $\widetilde{\mathscr{A}}_{(w,M)} \subseteq \widetilde{\mathscr{A}}_{(v,L)}$.

Proof. Since $\widetilde{\mathscr{A}}_{(v,L)}$ depends only on the class of (v, L), we may assume that $(v, L) = (w, L)$ for $L \supset M$. Then the desired factorization is clear. □

Definition 3.7. Define the (w, M)-elliptic locus to be the open in $\widetilde{\mathscr{A}}_{(w,M)}$ which is not contained in any smaller locus. That is, if we first define the G-elliptic locus

$$\widetilde{\mathscr{A}}_{G\text{-ell}} = \widetilde{\mathscr{A}}_G \setminus \bigcup_{\substack{(w,M) \\ M \subsetneq G}} \widetilde{\mathscr{A}}_{(w,M)},$$

and $\widetilde{\mathscr{A}}_{M\text{-ell}}$ by likewise removing all twisted images of smaller Levis, then

$$\widetilde{\mathscr{A}}_{(w,M)\text{-ell}} := \iota_w(\widetilde{\mathscr{A}}_{M\text{-ell}}).$$

With this definition we claim the elliptic loci stratify the base.

Proposition 3.8. *If* $[(w_1, M_1)] \nsim [(w_2, M_2)]$, *then* $\widetilde{\mathscr{A}}_{(w_1,M_1)\text{-}ell} \cap \widetilde{\mathscr{A}}_{(w_2,M_2)\text{-}ell} = \emptyset$. *In particular, the* (w, M)*-loci give a stratification of* $\widetilde{\mathscr{A}}_G$.

The proof will appear at the end of the next section, after some preliminaries.

After introducing the complexity of Weyl-twists, we observe that ultimately, it will be unnecessary.

Definition 3.9. Let $e \in W_G(\infty)$ be the identity. Define the *compatible locus* to be

$$\mathsf{A} := \widetilde{\mathscr{A}}^{\text{comp}} = \bigcup_M \widetilde{\mathscr{A}}_{(e,M)\text{-ell}}.$$

That is, over the compatible locus we have a well-behaved stratification indexed purely by Levis

$$\mathsf{A} = \bigsqcup_M \mathsf{A}_{M\text{-ell}}.$$

Compatibility can be characterized as follows, though we won't use this characterization. For any semistandard Levi M and $a \in \mathscr{A}(k)$, we have a *partial cameral cover* defined by the Cartesian square

$$\begin{tikzcd}
X_{a,M} \arrow[r] \arrow[d] & \mathfrak{c}_{M,D} \arrow[d, "\chi_G^M"] \\
X \arrow[r, "a"] & \mathfrak{c}_{G,D}
\end{tikzcd}.$$

The data $\chi_M^T(t) \in \mathfrak{c}_M(\infty)$ determines a point $\widetilde{\infty}_M$ in $X_{a,M}$ over ∞ (and under $\widetilde{\infty} \in X_{a,T}$). An element a lies in the untwisted image $\iota_e(\widetilde{\mathscr{A}}_M)$ if and only if $X_{a,M} \to X$ has an F-section, and t is compatible with a if and only if $\widetilde{\infty}_M$ lies in the image of such a section, or equivalently, lies on an irreducible component of $X_{a,M}$ which maps birationally onto X.

The set of classes (e, M) is upwards-closed, so once we have shown that we indeed have a stratification (Corollary 3.15 below) it follows immediately the union of their elliptic loci is open, stratified by loci $\mathsf{A}_{M\text{-ell}}$ for semistandard Levis M. We lose no part of the geometry of $\mathscr{M} \to \mathscr{A}$ in restricting to the compatible locus, as every $a \in \mathscr{A}$ has a t such that (a, t) is compatible, i.e., every Hitchin fiber is visible over the compatible locus.

3.4. Generic characterization of twisted loci.

Using the existence of the Kostant section, we can characterize the elements in the twisted loci at the generic point in the style of [CL10, §3.9]. For the reader familiar with the arguments in *loc. cit.*, the two new ingredients in this Weyl-twisted setting are

(1) carefully tracking *split* central tori and
(2) fundamental results of [Sol20] on rational conjugacy of Levis.

We recall the basic definitions over a field.

Definition 3.10. Let H a reductive group over a field F of good characteristic for H, and $Y \in \mathfrak{h}(F)$ an element of the Lie algebra.

- We say that Y is *H-regular* if its centralizer $Z_H(Y)$ has minimal dimension.
- We say that Y is *semisimple* if Y is equal to its semisimple part in the additive Jordan decomposition over the algebraic closure $\overline{F}$.
- If Y is both H-regular and semisimple, then $A_Y := Z_H(Y)^0$ is a maximal torus in H (defined over F). Let $A_Y^{\mathrm{spl}} \subseteq Z_H(Y)$ be the maximal F-split subtorus. We say that Y is *H-elliptic* if $A_Y^{\mathrm{spl}} = A_H^{\mathrm{spl}}$, i.e., if $Z_H(Y)$ is anisotropic mod center.

Proposition 3.11 (cf. [CL10, Proposition 3.9.1]). *Let $K/\overline{k}$ a field. An element $(a, t) \in \widetilde{\mathscr{A}}_G(K)$ is in the locus $\widetilde{\mathscr{A}}_{(w,M)}(K)$ if and only if there exist*

- *$Y \in \mathfrak{m}(F)$ semisimple G-regular, and*
- *$m \in M(F_\infty)$,*

such that

(a) $\chi_G(Y) = a_\eta$,
(b) $\mathrm{Ad}(m)Y = w(t_a)$.

Here, $t_a \in \mathfrak{t}^{G\text{-}reg}(\mathscr{O}_\infty)$ is the unique lift of t such that $\chi_G(t_a) = a_{\mathscr{O}_\infty}$.

Proof. The proof in *loc. cit.* can be directly adapted, using the fact that $T|_{\mathscr{O}_\infty}$ is split.[2]

Note that a t_a as in the statement exists by formal étaleness of χ over the regular locus. Assume that (a, t) is in the image of $\mathscr{A}_M^w(K)$ for some w, i.e., there exists a_M a section of $(\mathfrak{m}//M)^{G-reg}$ with $\chi_G^M(a_M) = a$ and $a_M(\infty_K) = \chi_M(w(t))$. So we have

$$\begin{tikzcd}
\mathrm{Spec}(\infty) \arrow[r] \arrow[d] & (\mathfrak{m}//M)^{reg} \arrow[d, "\chi_G^M"] \\
\mathrm{Spec}(\mathscr{O}_\infty) \arrow[ur, shift left, "a_{M,\infty}"] \arrow[ur, shift right, "\chi_M(w(t_a))"'] \arrow[r, "a_\infty"] & (\mathfrak{g}//G)^{reg}
\end{tikzcd}.$$

Since the right map is étale, repeatedly applying the lifting criterion for unramifiedness implies that $a_{M,\infty} = \chi_M(w(t_a))$.

[2]This follows from e.g. [Sta22, Tag 0BQC] applied with $X = A$ and [Con14, B.3.6]).

Now look over the generic point, where $a_{M,\eta} \in (\mathfrak{m}//M)^{reg}(F)$. The Kostant section for M implies that we can find $Y \in \mathfrak{m}(F)$, (necessarily regular semisimple because the Kostant section sends $(\mathfrak{m}//M)^{reg}$ into the regular semisimple locus), such that $\chi_M(Y)|_{F_\infty} = a_{M,\eta}|_{F_\infty} = a_{M,\infty} = \chi_M(w(t_a))$. So by regular semisimplicity there exists L/F_∞ finite Galois and $m \in M(L)$ such that

$$\mathrm{Ad}(m)Y = w(t_a).$$

This descends to $m \in M(F_\infty)$ because, since $T|_{F_\infty}$ is split,

$$H^1(L/F_\infty, T_{F_\infty}(L)) = H^1(L/F_\infty, \mathbb{Z}^n_{triv}) = 1.$$

Conversely, let $(a,t) \in \mathscr{A}_G(K)$ and suppose there exist $Y \in \mathfrak{m}(F)$ G-regular semisimple, $m \in M(F_\infty)$, and $w \in W_M(\infty)$ such that $\chi_G(Y) = a_\eta$ in $(\mathfrak{g}//G)^{reg}$ and $\mathrm{Ad}(m)Y = w(t_a)$ in $\mathfrak{m}(F_\infty)$. Then the element Y defines a generic factorization of $a : X_K \to (\mathfrak{g}//G)_D$ through $(\mathfrak{m}//M)_D$: the solid top line of the square

$$\begin{tikzcd} \mathrm{Spec}(F) \arrow[r] \arrow[d] & (\mathfrak{m}//M)_D \arrow[d, "\chi_G^M"] \\ X_K \arrow[r] \arrow[ur, dashed] & (\mathfrak{g}//G)_D \end{tikzcd}.$$

Now we consider the question of extending this morphism to X_K, which we do in two steps. First, spreading out (e.g. by finite-typeness of the morphism χ_G^M) implies that the morphism extends to some affine open of X_K. Then projectivity of χ_G^M and nonsingularity of X_K implies that the morphism extends to $a_M : X_K \to (\mathfrak{m}//M)_D$.

So (a_M, t) is our candidate for a preimage. We need to check that a_M and t are w-compatible, i.e., that $a_M(\infty_K) = \chi_M(w(t))$ in $(\mathfrak{m}//M)^{G-\mathrm{reg}}(K)$. First note that $a_{M,\eta} = \chi_M(Y)$ in $(\mathfrak{m}//M)(F) \subseteq (\mathfrak{m}//M)(F_\infty)$. The second assumption ensures that $\chi_M(Y) = \chi_M(t_a)$, so a_M and $\chi_M(t_a)$ agree on the generic point of $\mathrm{Spec}(\mathscr{O}_\infty)$. By the uniqueness in the valuative criterion of properness, they agree on all of $\mathrm{Spec}(\mathscr{O}_\infty)$, hence on the special fiber. □

Likewise we have a characterization of the elliptic loci.

PROPOSITION 3.12 (cf. [CL10, Corollaire 3.9.2].). *Let K/k a field. An element $(a,t) \in \widetilde{\mathscr{A}}_G(K)$ is in the locus $\widetilde{\mathscr{A}}_{(w,M)\text{-}ell}(K)$ if and only if there exist*

- $Y \in \mathfrak{m}(F)$ *semisimple G-regular M-elliptic, and*
- $m \in M(F_\infty)$,

such that

(a) $\chi_G(Y) = a_\eta$,
(b) $\mathrm{Ad}(m)Y = w(t_a)$.

PROOF. Assume there exists Y, m as in the proposition. Assume moreover that (a,t) is in the image of $\widetilde{\mathscr{A}}_{(v,L)}(K)$ for some $L \subseteq M$ and $v \in W_G(\infty)$. We will show $L = M$. By the forward direction of Proposition 3.11, there exist $Z \in \mathfrak{l}(F)$, $l \in L(F_\infty)$ satisfying $\chi_G(Z) = a_\eta = \chi_G(Y)$ and $\mathrm{Ad}(l)Z = v(t_a)$. Since both $Y, Z \in \mathfrak{g}(F)$ are G-regular semisimple and have the same characteristic polynomial, they are conjugate by some element $g \in G(F^s)$.

The centralizers of Y, Z in G_F are maximal F-tori A_Y, A_Z. Each element $\sigma \in \mathrm{Gal}(F^{\mathrm{s}}/F)$ satisfies

$$\sigma(g)g^{-1}Zg\sigma(g)^{-1} = \sigma(g)Y\sigma(g)^{-1} = \sigma(gYg^{-1}) = \sigma(Z) = Z,$$

so $\sigma(g)g^{-1}$ lies in $A_Z(F^{\mathrm{s}})$. In particular, $\sigma(g)g^{-1}$ commutes with anything in $A_Z(F^{\mathrm{s}})$, and so the isomorphism $\phi = \mathrm{Inn}(g) : A_{Y,F^{\mathrm{s}}} \to A_{Z,F^{\mathrm{s}}}$ is Galois-equivariant:

$$\sigma(\phi(a)) = \sigma(gag^{-1}) = (\sigma(g)g^{-1})(g\sigma(a)g^{-1})(g\sigma(g^{-1})) = g\sigma(a)g^{-1}.$$

(Note $g\sigma(a)g^{-1} \in A_Z(F^{\mathrm{s}})$ since A_Y is defined over F and so its set of F^{s}-points is Galois-stable.) So the isomorphism ϕ descends to one defined over F. In particular, it induces an isomorphism on the maximal split sub-tori, from $A_Y^{\mathrm{spl}} = A_M^{\mathrm{spl}}$ to A_Z^{spl}. Since $Z \in \mathfrak{l}(F)$, we have $A_Z^{\mathrm{spl}} \supseteq A_L^{\mathrm{spl}}$, so the dimension of A_M^{spl} is at least that of A_L^{spl}.

On the other hand, since $L \subseteq M$ we have $A_M^{\mathrm{spl}} \subseteq A_L^{\mathrm{spl}}$ by Proposition 2.13.(4). We conclude equality holds, and so $M = Z_G(A_M^{\mathrm{spl}}) = Z_G(A_L^{\mathrm{spl}}) = L$.

Conversely, assume that (a,t) is in $\widetilde{\mathscr{A}}_{(w,M)\text{-ell}}(K) \subseteq \widetilde{\mathscr{A}}_{(w,M)}(K)$, so in particular there exist Y, m as in Proposition 3.11. We show that Y is necessarily M-elliptic. Assume for the sake of contradiction Y is not M-elliptic, so that the maximal split torus in the centralizer in G_F of Y satisfies $A_Y^{\mathrm{spl}} \supsetneq A_M^{\mathrm{spl}}$. Taking centralizers in G of both sides and applying Proposition 2.13.(4) shows $A_Y^{\mathrm{spl}} \subseteq M$, so up to conjugating Y and m in M we may assume that $A_Y^{\mathrm{spl}} \subseteq T_F$ and the Levi $L_F = Z_G(A_Y^{\mathrm{spl}}) \subsetneq M_F$ is semistandard.

Since $\mathrm{Ad}(m)Y = w(t_a)$, we have

$$mA_Y m^{-1} = T_{F_\infty}$$

in G_{F_∞}. But on the other hand $T_{F_\infty} = Z_{G_{F_\infty}}(w(t_a))$ and A_{Y,F_∞} are both maximal (split over F_∞!) tori in L_{F_∞}, so are conjugate by an element $l \in L(F_\infty)$. Since lm^{-1} lies in M and normalizes the split torus T_{F_∞}, it acts on the torus by an element $v \in \mathbb{W}_M = W_M(\infty)$. In particular, we have

$$\mathrm{Ad}(l)Y = \mathrm{Ad}(lm^{-1})\,\mathrm{Ad}(m)Y = vw(t_a).$$

Note moreover that since $L = Z_G(A_Y^{\mathrm{spl}}) \supseteq Z_G(Z_G(Y))$ and $\mathrm{Lie}\, Z_G(Z_G(Y)) = (\mathrm{Lie}\, G)^{Z_G(Y)}$ (which would hold even in bad characteristic), we have $Y \in \mathrm{Lie}\, L$. So the pair (Y, l) satisfy the hypotheses of Proposition 3.11 with twist vw, and (a,t) is in $\widetilde{\mathscr{A}}_{(vw,L)}(K)$. ☐

Remark 3.13. Note that it was only necessary to change the twist w by an element of $W_M(\infty)$.

Now we can prove the elliptic loci give a stratification.

Proposition 3.14 (cf. [CL10, §3.10]). *If* $[(w,M)] \nsim [(v,L)]$, *then*

$$\widetilde{\mathscr{A}}_{(w,M)\text{-}ell} \cap \widetilde{\mathscr{A}}_{(v,L)\text{-}ell} = \emptyset$$

inside $\widetilde{\mathscr{A}}_G$.

Proof. It suffices to show the intersection is empty on K-points for algebraically closed K. In particular, we may assume $K/\overline{k}$, and so we can apply the generic characterization above. Let F denote the function field of X_K, so G_F is a (possibly non-split) group over $\mathrm{Spec}\, F$, and F^{s}/F a separable closure. Let $(a,t) \in (\widetilde{\mathscr{A}}_{(w,M)\text{-ell}} \cap \widetilde{\mathscr{A}}_{(v,L)\text{-ell}})(K)$. So we have elliptic elements $Y \in \mathfrak{l}(F), Z \in \mathfrak{m}(F)$ as well as $l \in L(F_\infty), m \in M(F_\infty)$, satisfying

$$\chi_G(Y) = a_\eta = \chi_G(Z), \quad \text{and} \quad l.Y = w(t_a), \quad m.Z = v(t_a).$$

Viewing both Y, Z as elements of $\mathfrak{g}(F)$, they have the same characteristic polynomial $a_\eta \in H^0(F_K, \mathfrak{c}_D)$. Since F is of cohomological dimension ≤ 1, cohomological vanishing as in [Ser94, III.2.3, Thm 1' and Remarks] implies there exists an element $g \in G(F^{\mathrm{s}})$ such that $g.Y = Z$ (cf. [CL10, Lemme 3.8.1] and §6.1 below).

Since Y, Z are elliptic elements, their split centralizers are the maximal split tori in their Levis: $A_Y^{\mathrm{spl}} = A_L^{\mathrm{spl}}$, $A_Z^{\mathrm{spl}} = A_M^{\mathrm{spl}}$. As in the previous proposition, the conjugation map $\phi = \mathrm{Inn}(g) : A_Y \to A_Z$ is Galois-equivariant and descends to an isomorphism of tori defined over F. In particular, it identifies the maximal F-split sub-tori, so

$$\phi(A_L^{\mathrm{spl}}) = A_M^{\mathrm{spl}}.$$

In fact, more is true: taking centralizers implies that $\phi(L) = M$, i.e., that L and M become conjugate over a splitting field, which implies by [Sol20] that the Levis are conjugate by $G(F)$. Since maximal split tori are all conjugate in M, we may post-compose with conjugation in M in order to assume that the conjugating element $g' \in G(F)$ normalizes the maximal split torus. That is, we have shown that L and M are conjugate by $w \in W_G(F)$.

Since we know that the images $\widetilde{\mathscr{A}}_{(w,M)\text{-ell}}$ depend only on the class $[(w,M)]$, whose equivalence allows shifting M and w by elements of $W_G(F)$, we have reduced to showing that the intersection is empty when $L = M$. That is, we wish to show that if

$$(a,t) \in (\widetilde{\mathscr{A}}_{(w,M)\text{-ell}} \cap \widetilde{\mathscr{A}}_{(v,M)\text{-ell}})(K),$$

then w and v differ by an element of $W_M(\infty)$. We have

$$\chi_M(v(t)) = a(\infty) = \chi_M(w(t))$$

so by M-regularity of t we know some $w' \in W_M(\infty)$ takes $v(t)$ to $w(t)$ and that $w'v = w$. $\square$

We have therefore proven that the twisted loci give a stratification of the Hitchin base, one which will be well-adapted for weighted stability.

COROLLARY 3.15. *The loci $\widetilde{\mathscr{A}}_{(w,M)\text{-}ell}$, ranging over classes $[(w, M)]$, define a stratification of $\widetilde{\mathscr{A}}_G$ with closure relations given by $\leq$ in Definition 3.5.*

3.5. Monodromic invariants.

For later geometric arguments, we compare our twisted elliptic stratification to Ngô's much finer stratification of the Hitchin base by "monodromic invariants" (cf. [Ngô10, §5.6]). Let $\tilde{a} = (a, t) \in \widetilde{\mathscr{A}}(\kappa)$ be a point of the Hitchin base over a field. Associated to a one has the *cameral cover* $X_a \to X$, defined as the pullback

$$\begin{array}{ccc} X_a & \longrightarrow & \mathfrak{t}_D \\ \big\downarrow{\scriptstyle \pi_a} & & \big\downarrow \\ X_\kappa & \xrightarrow{h_a} & \mathfrak{c}_D, \end{array} \tag{3.5.1}$$

where a determines a morphism $h_a \in \mathrm{Mor}(X_\kappa, \mathfrak{c}_D)$. The data of t gives a canonical point $\widetilde{\infty}$ in the fiber over ∞. Let $U_a = a^{-1}(\mathfrak{c}^{G\text{-reg}}_{G,D,\kappa}) \subset X$ denote the open on which a is regular and let $V_a = (\pi_a)^{-1}(U_a)$.

REMARK 3.16. In the case of a classical group, sections of $X_a \to X$ correspond to orderings of the factorizations of the characteristic polynomial a. Contrast this with the theory of spectral curves, in which the sections correspond to the factors themselves (and which are not necessarily Galois covers of X).

Following [Ngô10, §5.6], we wish to describe monodromic invariants associated with the cameral cover $X_a \to X$. The étale cover $X_a \times_X U_a \to U_a$ is a $W|_{U_a}$-torsor, which defines a class in $H^1(U_a, W|_{U_a})$. Since t determines a point $\widetilde{\infty}$ upstairs, it determines a representing cocycle

$$c_{\tilde{a}} \in Z^1(\pi_1(U_a, \infty), W_\infty).$$

Such a cocycle is equivalent data to its graph, a section

$$\pi_1(U_a, \infty) \to W_\infty \rtimes \pi_1(U_a, \infty)$$

of the natural projection, and this latter semidirect product acts on the fiber X_a.

DEFINITION 3.17. Define the *(intrinsic) monodromic subgroup* $W_{\tilde{a}} \subset W_{\widetilde{\infty}} \rtimes \pi_1(U_a, \widetilde{\infty})$ to be the graph subgroup of the cocycle $c_{\tilde{a}}$. Denote $W^{\mathrm{red}}_{\tilde{a}} \subset W_\infty$ its projection into W_∞, that is, the image of the cocycle.

By Galois theory, we have the following observation.

PROPOSITION 3.18 ([Ngô10, 1.3.6, 5.4.1, 5.4.2]). *The monodromic subgroup $W_{\tilde{a}}$ is the normalizer of the connected component $C_{\widetilde{\infty}}$ of V_a containing $\widetilde{\infty}'$.*

Given G/X which is split by a (geometrically connected) Γ-Galois cover $\pi : X' \to X$, we can recover Ngô's version of the monodromic subgroup via the further pullback

$$\begin{array}{ccc} X'_a & \longrightarrow & X_a \\ \big\downarrow{\scriptstyle \pi'_a} & & \big\downarrow \\ X'_\kappa & \longrightarrow & X_\kappa \end{array} . \tag{3.5.2}$$

A choice of basepoint ∞' over ∞ determines an isomorphism $G_{X'} \cong G_0 \times X'$ (as well as compatible isomorphisms $T_{X'} \cong T_0 \times X'$ and $W_{X'} \cong \mathbb{W} \times X'$) by restriction to ∞'. In this case X'_a is equipped with a point $\widetilde{\infty}' = (\infty', \widetilde{\infty})$ over ∞. Now $X'_a \to X'_\kappa$ has an action of $\mathbb{W} \cong W(\infty)$, the split Weyl group, and $X'_\kappa \to X_\kappa$ and $X'_a \to X_a$ have compatible actions of Γ. So the composite $X'_a \to X_\kappa$ admits an action of the constant group $\mathbb{W} \rtimes \Gamma$, depending on the pointed splitting cover (X', ∞').

Definition 3.19. Given $\tilde{a} = (a,t)$, define its *split monodromic subgroup* $\mathbb{W}_{\tilde{a}} \subseteq \mathbb{W} \rtimes \Gamma$ to be the normalizer of the irreducible component of X'_a containing $\widetilde{\infty}'$.

Example 3.20. Let $a \in \mathscr{A}_T$ inside of $G = U_{X'/X}(3)$. The generic fiber of X_a is $\operatorname{Spec} F \sqcup \operatorname{Spec} F \sqcup \operatorname{Spec} F' \sqcup \operatorname{Spec} F'$, where F'/F is the extension of function fields corresponding to X'/X. If we choose a basepoint $\widetilde{\infty}$ in one of the two rational components, then the pair (a,t) will be compatible, and the other rational basepoint consists of $(1\ 3)\widetilde{\infty}$.

Pulling back along X'/X, the cover splits into 6 irreducible components with function fields F'. Galois fixes the two rational components (interchanging their fibers over ∞) and exchanges the other four in pairs. On the other hand, the cover $X'_a \to X'$ is a trivial cover, so S_3 acts faithfully on the set of components.

Choosing an $\infty' \in X'$ lying over ∞ so that $(\infty', \widetilde{\infty})$ gives a based $S_3 \rtimes \Gamma$-cover $X'_a \to X$, we can see that

- If $\widetilde{\infty}$ is chosen to be a basepoint on an F-component, then anything in the Galois group preserves that irreducible component, but no element of S_3 does, so $\mathbb{W}_{\tilde{a}} = 1 \rtimes \Gamma$.
- If $\widetilde{\infty}$ is chosen to lie on one of the F'-components, then a non-Γ-stable, order-2 subgroup $\mathbb{W}_{\tilde{a}} = \langle (w,\sigma) \rangle \cong \mathbb{Z}/2$, a conjugate of $1 \rtimes \Gamma$, normalizes that component. In fact $w = (1\ 2\ 3)$ or $(1\ 3\ 2)$.

Remark 3.21. The above example demonstrates the necessity of deleting all the *twisted* images of $\widetilde{\mathscr{A}}_M$ when defining $\widetilde{\mathscr{A}}_{G\text{-ell}}$. Indeed, if we choose $\widetilde{\infty}$ above, equivalent data to t, non-compatibility to lie on one of the F'-components of the cameral cover, then $\tilde{a} = (a, \widetilde{\infty})$ is evidently not in the image of the untwisted closed embedding from the only Levi subgroup $\widetilde{\mathscr{A}}_T \hookrightarrow \widetilde{\mathscr{A}}_G$. However, we argue that such point $(a, \widetilde{\infty})$ should not be considered G-elliptic – indeed, it is not anisotropic in the sense of Ngô [Ngô10, 5.4.7]. Computing the action of $\mathbb{W} \rtimes \Gamma$ on $\mathbb{T}$, we see that $\mathbb{T}^{\mathbb{W}_{\tilde{a}}}$ contains a nontrivial rank 1 torus, though not one that descends to F. Alternatively, this $\mathbb{W}_{\tilde{a}}$ is conjugate to $1 \rtimes \Gamma$, so its fixed locus inside $\mathbb{T}$ is isomorphic to $\mathbb{T}^{1 \rtimes \Gamma} = \{\operatorname{diag}(t, 0, -t)\}$. This is not surprising: anisotropicity corresponds to quasicompactness of the Picard stack for the Hitchin fiber (through finiteness of $\pi_0(\mathscr{P})$, see §12), which is *independent of our choice of point* $\widetilde{\infty}$ because the Hitchin fibration is defined over $\widetilde{\mathscr{A}}$ by pullback from $\mathscr{A}$.

This demonstrates a minor error in the statement of [Ngô10, Proposition 6.3.5]: the anisotropic locus of Ngô (G-elliptic, in our terminology) is the complement of the images of all *twisted* embeddings of Levis as in Definition 3.7; it is not enough to consider only the untwisted embeddings $\widetilde{\mathscr{A}}_M \to \widetilde{\mathscr{A}}_G$. Examining the proof of *loc. cit.*, we see that for a non-anisotropic point, the nontrivial torus $\mathbb{S}$ contained in $\mathbb{T}^{\mathbb{W}_{\tilde{a}}}$ and the Levi $\mathbb{M}$ defined there need not be Galois-invariant except up to conjugacy. That is, we can find a preimage a_M for a, but not necessarily compatible with our given data $\widetilde{\infty}$.

Of course, since we have only deleted a finite number of closed subvarieties isomorphic to $\widetilde{\mathscr{A}}_M$, the codimension estimate deduced from this proposition, which is the only way it is used, remains intact.

The monodromic invariant subgroup is sufficient to detect Levi-reductions of polynomials, as the following shows.

Proposition 3.22 (cf. [CL12, Prop. 8.2.1]). *For a standard Levi M, an element $\tilde{a} = (a,t)$ lies in the image of $\widetilde{\mathscr{A}}_{(e,M)}$ if and only if $W_{\tilde{a}} \subseteq W_{\infty,M} \rtimes \pi_1(U_a, \infty)$, where $W_{\infty,M} \subset W_\infty$ corresponds to the Levi $M_\infty \subset G_\infty$, that is, if and only if $W^{\mathrm{red}}_{\tilde{a}} \subseteq W_{\infty,M}$. Equivalently, after choosing a pointed splitting cover $\pi : X' \to X$, the element $\tilde{a}$ lies in the image of $\widetilde{\mathscr{A}}_{(e,M)}$ if and only if $\mathbb{W}_{\tilde{a}} \subset \mathbb{W}_M \rtimes \Gamma$, where $\mathbb{W}_M = \mathbb{W}_{M'}$ is the subgroup of the* split *Weyl group corresponding to $M' = \pi^{-1}(M)$.*

Moreover, if $\tilde{a}_M = (a_M, t)$ is a preimage in $\widetilde{\mathscr{A}}_M$ for $\tilde{a} = (a,t)$, then $W_{\tilde{a}_M} = W_{\tilde{a}}$ and $\mathbb{W}_{\tilde{a}_M} = \mathbb{W}_{\tilde{a}}$.

Proof. The equivalence between the claims for $W_{\tilde{a}}$ and $\mathbb{W}_{\tilde{a}}$ is clear from the discussion above. We will check the claims about $\mathbb{W}_{\tilde{a}}$. Assume first that $\mathbb{W}_{\tilde{a}} \subseteq \mathbb{W}_M \rtimes \Gamma$. We will then construct a preimage a_M for a.

Combining (3.5.1) and (3.5.2) gives a commutative diagram of X_κ-schemes with Cartesian squares

$$\begin{tikzcd}
\mathrm{Spec}(\kappa) \arrow[r, "\widetilde{\infty}'"] \arrow[dr, "\infty'"] \arrow[ddr, "\infty"'] & V'_a \arrow[r, hook, "i'_a"] \arrow[d] & \mathfrak{t}'^{\mathrm{reg}}_{D,\kappa} \arrow[d, "\chi'"] \\
& U'_a \arrow[r] \arrow[d] & \mathfrak{c}'^{\mathrm{reg}}_{G',D',\kappa} \arrow[d] \\
& U_a \arrow[r, hook, "h_a"] & \mathfrak{c}^{\mathrm{reg}}_{G,D,\kappa}
\end{tikzcd}.$$

Let $(V'_a)^0$ be the connected component of V'_a containing $\widetilde{\infty}'$. By construction, $(V'_a)^0 \to U_a$ is Galois étale of group $\mathbb{W}_{\tilde{a}}$, so the composition of i'_a with the two-step quotient

$$(V'_a)^0 \xrightarrow{i'_a} \mathfrak{t}'^{\mathrm{reg}}_{D,\kappa} \longrightarrow \mathfrak{t}'^{\mathrm{reg}}_{D',\kappa}/\!/\mathbb{W}_M = \mathfrak{m}'^{\mathrm{reg}}_{D',\kappa} \xrightarrow{\pi} \mathfrak{m}'^{\mathrm{reg}}_{D',\kappa}/\!/\Gamma = \mathfrak{m}^{\mathrm{reg}}_{D,\kappa}$$

descends to a lift $h_{a_M} : (V'_a)^0/\!/\mathbb{W}_{\tilde{a}} = U_a \to \mathfrak{m}^{\mathrm{reg}}_{D,\kappa}$. Note that the the Chevalley spaces for G/X are built by descent from those for G'/X', so the corresponding Chevalley maps are compatible with pullback.

The valuative criterion of properness now guarantees the existence of h_{a_M} to a morphism on all of X_κ. This lift h_{a_M} is by construction compatible with the datum $\widetilde{\infty}'$, so by considering the two-step quotient in the other order $V'_a \to V_a \to U_a$ we see that h_{a_M} is compatible with $\widetilde{\infty}$ and so $(a_M, t) \in \widetilde{\mathscr{A}}_M$ is a preimage of (a, t).

Consider any element $w \in \mathbb{W}_G \rtimes \Gamma$ such that $wV'_{a_M} \cap V'_{a_M} \neq \emptyset$. Since

$$V'_a/\!/(\mathbb{W} \rtimes \Gamma) = U_a = U_{a_M} = V'_{a_M}/\!/(\mathbb{W}_M \rtimes \Gamma),$$

it follows that the $\mathbb{W}$-orbit of V'_{a_M} is a disjoint union of $\mathbb{W}_M$-orbits, so the assumption on w implies that $w \in \mathbb{W}_M \rtimes \Gamma$. In particular, $\mathbb{W}_{\tilde{a}} \leq \mathbb{W}_M \rtimes \Gamma$.

Finally, we claim that if $\tilde{a}_M = (a_M, t) \in \widetilde{\mathscr{A}}_M$ has image $\tilde{a} = (a, t)$ then $\mathbb{W}_{a_M} = \mathbb{W}_a$, which implies the converse direction. This follows as in [CL12, Prop. 8.2.2]: the condition on $X_{\kappa(a_M)}$ for a_M to be G-regular in $\mathfrak{c}_{M,D}$ is by definition that its image in $\mathfrak{c}_{G,D}$ is G-regular, so $U_{a_M} = U_a$ and likewise $U'_{a_M} = U'_a$. Lifting this, there is a natural map compatible with the lifts of ∞

$$j : V'_{a_M} = U_a \times_{\mathfrak{c}^{\mathrm{reg}}_{M,D}} \mathfrak{t}'^{\mathrm{reg}}_{D'} \hookrightarrow V'_a = U_a \times_{\mathfrak{c}^{\mathrm{reg}}_{G,D}} \mathfrak{t}'^{\mathrm{reg}}_{D'}.$$

The map j is the base change of the relative diagonal map $\mathfrak{c}^{\mathrm{reg}}_{M,D} \to \mathfrak{c}^{\mathrm{reg}}_{M,D} \times_{\mathfrak{c}^{\mathrm{reg}}_{G,D}} \mathfrak{c}^{\mathrm{reg}}_{M,D}$, hence is an open and closed immersion (χ^M_G is unramified on the G-regular locus and separated). Therefore the connected components V'^0_a and $(V'_{a_M})^0$ containing the lifts of ∞ coincide. By functoriality of pullback j is equivariant for the action of $\mathbb{W}_M \rtimes \Gamma$, so $\mathbb{W}_{\tilde{a}_M}$ and $\mathbb{W}_{\tilde{a}}$ coincide on those elements lying within $\mathbb{W}_M \rtimes \Gamma$. But $\mathbb{W}_{\tilde{a}} \leq \mathbb{W}_M \rtimes \Gamma$. $\square$

The twisted loci too can be characterized in terms of monodromic invariants, though we won't use this.

Corollary 3.23. *An element $\tilde{a} = (a, t) \in \widetilde{\mathscr{A}}_G$ lies in $\widetilde{\mathscr{A}}_{(w,M)}$ if and only if $\mathbb{W}_{\tilde{a}} \subseteq w^{-1}(W_{M,\infty} \rtimes \Gamma)w$.*

Proof. Let $X' \to X$ be a Γ-cover which splits G. Since Γ acts transitively on the points $\infty' \in X'$ over ∞. That is, $t \in \mathfrak{t}(\infty)$ is determined by its restriction to any given ∞', and making such a choice, $t(\infty') \in \mathfrak{t}(\infty')$ corresponds to the choice of basepoint $\widetilde{\infty}' = (\infty', \widetilde{\infty})$.

By definition, (a, t) is in $\widetilde{\mathscr{A}}_{(w,M)}$ if and only if $(a, w(t)) \in \widetilde{\mathscr{A}}_M$. The latter is equivalent to $\mathbb{W}_{(a,w(t))} \subseteq \mathbb{W}_M(\infty) \rtimes \Gamma$, which is the normalizer of the irreducible component containing $w(\widetilde{\infty}')$. Since a is regular at ∞, each of these basepoints is contained only in a single irreducible component, so the stabilizer of the component containing $\widetilde{\infty}'$ is

$$\mathbb{W}_{(a,t)} = w^{-1}\mathbb{W}_{(a,w(t))}w.$$

$\square$

Finally, Ngô has shown that the monodromic invariants are semicontinuous on $\widetilde{\mathscr{A}}$.

Proposition 3.24 ([Ngô10, Lemme 5.4.5]). *Let $\tilde{a} \in \mathsf{A}_{M\text{-}ell}$ and $\tilde{a}' \in \mathsf{A}_{L\text{-}ell}$ be a specialization of a, so that $L \subseteq M$. Then $W_{\tilde{a}'} \subset W_{\tilde{a}}$.*

On the compatible locus, we have a simple characterization of the $W_{\tilde{a}}$-invariants, and a crucial finiteness result we will need in §12. We have a short exact sequence of X-tori

$$1 \longrightarrow T \cap D(M) \longrightarrow T \longrightarrow M^{\mathrm{ab}} \longrightarrow 1 \ .$$

Taking $\underline{X}_{\star}(-)(\infty)$ gives a short exact sequence of $(W_{M,\infty} \rtimes \Gamma)$-modules, where Γ is either the Galois group of a splitting cover, or more invariantly, the intrinsic image of the monodromy. We denote it

$$1 \longrightarrow \mathbb{X}_{\star,D(M)} \longrightarrow \mathbb{X}_{\star} \longrightarrow \mathbb{X}_{\star,M^{\mathrm{ab}}} \longrightarrow 1 \ .$$

Tensoring with $\mathbb{R}$ gives a decomposition of the cocharacter space,

$$1 \longrightarrow \underline{\mathfrak{a}}_{D(M)} \longrightarrow \underline{\mathfrak{a}}_{T} \longrightarrow \underline{\mathfrak{a}}_{M^{\mathrm{ab}}} \longrightarrow 1 \ .$$

Lemma 3.25. *For every Levi M and $a \in \mathsf{A}_{M\text{-}ell}$,*

(a)

$$\underline{\mathfrak{a}}_{D(M)}^{W_{\tilde{a}}} = 0,$$

and moreover integrally

$$(\mathbb{X}_{\star,D(M)})^{W_{\tilde{a}}} = 0.$$

(b) The coinvariants $(\mathbb{X}_{\star,D(M)})_{W_{\tilde{a}}}$ is finite.

(c) We have an equality

$$((T_{\infty}^{W_{\tilde{a}}})_{\mathrm{red}})^0 = (A_M^{\mathrm{spl}})_{\infty}.$$

Indeed, if p does not divide $W_{\tilde{a}}$, then $T_{\infty}^{W_{\tilde{a}}}$ is smooth and

$$(T_{\infty}^{W_{\tilde{a}}})^0 = (A_M^{\mathrm{spl}})_{\infty}.$$

Proof. For (a), since the $W_{\tilde{a}}$-invariants is torsion-free, its vanishing can be checked after tensoring with $\mathbb{R}$ ($\mathbb{Q}$ is enough), so the statement about $\underline{\mathfrak{a}}_{D(M)}$ suffices. Now let $v \neq 0 \in \underline{\mathfrak{a}}_{D(M)}^{W_{\tilde{a}}}$. Note that $\underline{\mathfrak{a}}_{D(M)}$ is the usual geometric reflection representation of $W_{M,\infty}$, and is stratified by the geometric Weyl fan. Since $v \neq 0$, it lives in a face corresponding to a proper geometric parabolic subgroup $Q_{\infty} \subsetneq M_{\infty}$ with Levi $L_{\infty} \subsetneq M_{\infty}$. Since W_a surjects onto Γ, the point v and the face it lies in are Γ-stable, so Q_{∞} and L_{∞} correspondingly descend to rational parabolics and Levis. But this implies W_a lies in the stabilizer of the face, which is $W_{L,\infty} \rtimes \Gamma$. By Proposition 3.22, this contradicts M-ellipticity of $\tilde{a}$.

For (b), we know $(\mathbb{X}_{\star,D(M)})_{W_{\tilde{a}}}$ is finite type, so it is finite if and only if it vanishes when tensored to $\mathbb{R}$. Averaging over $W_{\tilde{a}}$ induces an isomorphism

$$(\mathbb{X}_{\star,D(M)})_{W_{\tilde{a}}} \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} (\mathbb{X}_{\star,D(M)})^{W_{\tilde{a}}} \otimes_{\mathbb{Z}} \mathbb{R},$$

so we are done by (a).

For (c), note that taking $W_{\tilde{a}}$-invariants of the (real) cocharacter spaces is exact. On M^{ab}, $W_{\tilde{a}}$ acts through Γ (and surjects onto it), so

$$\underline{\mathfrak{a}}_{M^{\mathrm{ab}}}^{W_{\tilde{a}}} = \underline{\mathfrak{a}}_{M^{\mathrm{ab}}}^{\Gamma}.$$

So in the short exact sequence

$$1 \longrightarrow \underline{\mathfrak{a}}_{D(M)}^{W_{\tilde{a}}} \longrightarrow \underline{\mathfrak{a}}_{T}^{W_{\tilde{a}}} \longrightarrow \underline{\mathfrak{a}}_{M^{\mathrm{ab}}}^{W_{\tilde{a}}} \longrightarrow 1$$

the left side vanishes and the right side has dimension equal to the split rank of M^{ab}, which is equal to $\dim A_M^{\mathrm{spl}}$. So $\dim((T_{\infty}^{W_{\tilde{a}}})_{\mathrm{red}}) = \dim A_M^{\mathrm{spl}}$. By definition, A_M^{spl} is reduced and fixed pointwise by $W_{\tilde{a}}$, so we have

$$(A_M^{\mathrm{spl}})_{\infty} \subseteq (T_{\infty}^{W_{\tilde{a}}})_{\mathrm{red}}$$

and indeed within its identity component. But we have shown they have the same dimension, so equality holds. Under the characteristic hypotheses, the last claim is clear. □

3.6. The Hitchin base and elliptic endoscopic groups.

We here recall various standard results relating ellipticity and the Hitchin base for endoscopic groups. Various notions of transfer in the Langlands program begin with transfers at the level of coarse quotients. In our setting, the fundamental such transfer is the functoriality of the spaces of invariants, and thus the Hitchin base, in endoscopic groups.

Proposition 3.26 ([Ngô10, §1.9, Proposition 6.3.2] or [GWZ20, §5.1-5.3]). *Let (s, ρ_s) an endoscopic datum for G with endoscopic group H. The natural map of Chevalley spaces from Proposition 2.30 induces on global sections a finite, unramified map*

$$\mathscr{A}_H \to \mathscr{A}_G.$$

The pulled back map

$$\widetilde{\mathscr{A}}_H \to \widetilde{\mathscr{A}}_G$$

is a closed embedding which depends on ρ_s.

Proposition 3.27 (cf. [CL12, 10.6.6, 10.6.10]). *Given an endoscopic datum $(s, \rho_s, \rho_s \to \rho_G)$ with endoscopic group $H = G'_s$, the closed embedding $i_{\rho_s} : \widetilde{\mathscr{A}}_{G'_s} \to \widetilde{\mathscr{A}}_G$ has image which intersects the G-elliptic locus if and only if $(Z_{\widehat{G}}^{\pi_1(X,\infty)})^0 = (Z_{\widehat{G}'_s}^{\pi_1(X,\infty)})^0$, where $\pi_1(X, \infty)$ acts on $Z_{\widehat{G}'_s}$ through the canonical composition*

$$\pi_1(X, \infty) \xrightarrow{\rho_s} \pi_0(s) \to \mathrm{Out}(\widehat{G}'_s),$$

noting that inner automorphisms act trivially on the center.

In the case that these conditions hold, $\widetilde{\mathscr{A}}_{G'_s} \cap i_{\rho_s}^{-1}(\widetilde{\mathscr{A}}_{G\text{-}ell}) = \widetilde{\mathscr{A}}_{G'_s\text{-}ell}$, where the elliptic locus in $\widetilde{\mathscr{A}}_{G'_s}$ is defined the same way as for G.

Proof. This is standard; the proof proceeds as in *loc. cit.*, using the monodromic invariant characterizations of the previous section. □

Definition 3.28. We say an endoscopic datum (s, ρ_s) is *elliptic* if it satisfies the equivalent conditions of Proposition 3.27.

Proposition 3.29 (cf. [CL12, Remarque 10.6.7, Proposition 10.6.10, Remarque 10.7.2]). *For reductive quasisplit G, an endoscopic datum $\mathfrak{G}' = (s, \rho_s)$ is elliptic if and only if the endoscopic group $H = G'_s$ satisfies*

$$A_H^{\mathrm{spl}} = A_G^{\mathrm{spl}}.$$

Such s are automatically torsion mod center. In particular, if G has anisotropic center, the same is true for all its elliptic endoscopic groups.

Proof. Proof similar to *loc. cit.* Up to isogeny, the split center is dual to $(Z^{\pi_1(X,\infty)})^0$, and so comparing dimensions

$$\left(Z_{\widehat{G}'_s}^{\pi_1(X,\infty)}\right)^0 = \left(Z_{\widehat{G}}^{\pi_1(X,\infty)}\right)^0$$

is equivalent to equality of split centers. □

Ellipticity of an endoscopic datum is therefore equivalent to the claim that the surjection of cocharacter spaces

$$\mathfrak{a}_H \to \mathfrak{a}_G$$

of Proposition 2.28 is an isomorphism.

Just as for Levi subgroups above, the endoscopic loci too can be characterized in terms of their monodromic subgroups.

Proposition 3.30 ([Ngô10, §6.3]). *Let $(s, \rho_s, \rho_s \to \rho_G)$ be an endoscopic datum for $G \to X$ with H the corresponding endoscopic group, and assume H is also split by Γ. A point $\tilde{a} = (a, t) \in \widetilde{\mathscr{A}}_G$ lies in the image of $i_{\rho_s}(\widetilde{\mathscr{A}}_H)$ if and only if $W_{\tilde{a}} \subset W_H \rtimes \Gamma$, identified as a subgroup of $W_G(\infty) \rtimes \Gamma$ along the homomorphism of Proposition 2.30.*

CHAPTER 4

The Hitchin fibration and parabolic reductions

In this section, we recall the definition of the moduli of Higgs bundles $\mathscr{M}$ and the Hitchin fibration $f : \mathscr{M} \to \mathscr{A}$ as well as the pullback to the étale open $f : \widetilde{\mathscr{M}} \to \widetilde{\mathscr{A}}$. We analyze the set of parabolic reductions of $m \in \widetilde{\mathscr{M}}$ in terms of $f(m)$.

4.1. The Hitchin fibration.

We denote X a geometrically irreducible smooth curve over arbitrary field k. Let $G \to X$ a quasisplit reductive group scheme with split form $\mathbb{G}$, associated to $\mathrm{Out}(\mathbb{G})$-torsor ρ_G. Recall $\mathrm{Bun}_G := \mathrm{Bun}_{G,X}$, a smooth Artin stack whose S-points consists of the groupoid of G-torsors on $X \times S$.

Definition 4.1. The *moduli of Higgs bundles $\mathscr{M}_G$*, denoted $\mathscr{M}$ when no confusion is possible, is defined to be the k-stack whose groupoid of S-points has objects

$$\{(\mathscr{E} \in \mathrm{Bun}_G(S), \theta \in H^0(X \times S, \mathrm{Ad}_D(\mathscr{E}))\}$$

where $\mathrm{Ad}_D(\mathscr{E}) = \mathscr{E} \times^G \mathfrak{g}_D$, the D-twisted Lie algebra bundle of $\mathscr{E}$. We call a point $(\mathscr{E}, \theta)$ a *Higgs bundle*, and θ its *Higgs field*.

The isomorphisms in $\mathscr{M}(S)$ are isomorphisms $\mathscr{E} \cong \mathscr{E}'$ over $X \times S$ such that under the induced isomorphism $\mathrm{Ad}_D(\mathscr{E}) \cong \mathrm{Ad}_D(\mathscr{E}')$, the Higgs fields θ and θ' correspond.

In other words, $\mathscr{M} = \underline{\mathrm{Map}}_{X/k}(X, [\mathfrak{g}_D/G])$, the k-stack parameterizing X-morphisms $X \to [\mathfrak{g}_D/G]$. The map of forgetting θ induces a map $\mathscr{M} \to \mathrm{Bun}_G$, which can be thought of as post-composition

$$\mathscr{M} = \underline{\mathrm{Map}}(X, [\mathfrak{g}_D/G]) \to \underline{\mathrm{Map}}(X, [\star/G]) = \mathrm{Bun}_G .$$

When $\deg(D) > 2g - 2$, a deformation argument due to Biswas and Ramanan deduces from the simple case of Riemann–Roch that the space $\mathscr{M}$ is smooth ([Ngô10, Lemme 4.14.1]).

Remark 4.2. Classically, one chooses $D = K_X$, after which Serre duality identifies $\mathscr{M}$ with $\mathscr{H}^{-1}$ of the cotangent complex of Bun_G. For that reason, over the complex numbers it sits on one side of the "non-abelian Hodge correspondence." For a general D, the map $\mathscr{M} \to \mathrm{Bun}_G$ is often referred to as a "vector bundle." We make this precise, in a derived sense which will play no further role in this work. Associated to D is a perfect complex

$$C_D = Rp_\star(\mathrm{Ad}_D(\mathscr{E}_{univ})) \in D^b(\mathrm{Bun}_G),$$

where $p : X \times \mathrm{Bun}_G \to \mathrm{Bun}_G$. Since X is a curve, C_D is perfect of Tor-amplitude $[0, 1]$. Taking

$$V_D := \underline{\mathrm{Spec}}_{\mathrm{Bun}_G}(\mathrm{Sym}(C_D^\vee))$$

defines a derived linear stack on Bun_G, whose fiber over $\mathscr{E}$ is the derived vector space $R\Gamma(X, \mathrm{Ad}_D(\mathscr{E}))$, which has locally constant Euler characteristic. The moduli of Higgs bundles is then the underlying classical stack $\pi_0 V_D$. Of course, this now has only semicontinuous fiber dimension.

The moduli of Higgs bundles is also fibered over the Hitchin base $\mathscr{A} = \mathscr{A}_G = \underline{\mathrm{Map}}_{X/k}(X, \mathfrak{c}_D)$, as follows.

Definition 4.3 (The Hitchin fibration). The D-twisted Chevalley map $\chi : \mathfrak{g}_D \to \mathfrak{c}_D$ defined in §2.11 induces a map $\chi : \mathrm{Ad}_D(\mathscr{E}) \to \mathfrak{c}_D$ by descent: over each trivializing cover there is a map $\mathfrak{g}_D \cong \mathrm{Ad}_D(\mathscr{E}) \to \mathfrak{c}_D$, and two such differ by G-conjugation, so descend to a well-defined global map $\mathrm{Ad}_D(\mathscr{E}) \to \mathfrak{c}_D$. There is an induced morphism on global sections, which induces the *Hitchin morphism* for G

$$f : \mathscr{M} \to \mathscr{A}$$
$$(\mathscr{E}, \theta) \mapsto \chi(\theta).$$

In other words, the Chevalley map is G-invariant, so descends to a map $[\mathfrak{g}_D/G] \to \mathfrak{c}_D$. Pushforward along this morphism defines the map

$$f : \mathscr{M} = \underline{\mathrm{Map}}(X, [\mathfrak{g}_D/G]) \to \underline{\mathrm{Map}}(X, \mathfrak{c}_D) = \mathscr{A}.$$

We define the *rigidified Hitchin fibration*, also denoted f, to be the pullback $\widetilde{\mathscr{M}} := \mathscr{M} \times_{\mathscr{A}} \widetilde{\mathscr{A}} \to \widetilde{\mathscr{A}}$, where $\widetilde{\mathscr{A}}$ is the rigidified Hitchin base of Definition 3.1. That is, $\widetilde{\mathscr{M}}$ is the moduli space of triples $(\mathscr{E}, \theta, t)$ such that $f(\mathscr{E}, \theta) = a$ agrees with $\chi_G^T(t)$ at infinity, and the rigidified Hitchin map is

$$(\mathscr{E}, \theta, t) \mapsto (\chi(\theta), t).$$

Remark 4.4. The utility of the Hitchin fibration in arithmetic can be understood as follows. Restricting to the generic point and assuming the bundle $\mathscr{E}$ trivial there (which it is up to finite extension of constants), a Higgs field is a conjugacy class of elements of the Lie algebra over the function field F whose matrix entries have zeroes and poles bounded by D over the curve. So this moduli space provides a geometrically well-behaved global setting within which to probe $\mathfrak{g}(F)$ down to a finite depth. As we allow D to range over all sufficiently positive divisors, we can access all of $\mathfrak{g}(F)$. The Hitchin fibration, then, is a globalized version of the map $\mathfrak{g}(F) \to \mathfrak{c}(F)$, and so provides information about the structure of global semisimple F-conjugacy classes in the Lie algebra.

4.2. Parabolic reductions of Higgs bundles.

We recall that reductions of a G-torsor $\mathscr{E} \to X$ to a parabolic subgroup are in bijection with sections of a smooth projective X-variety.

Proposition 4.5. *Given $\mathscr{E} \to X$ a G-torsor and $P \subseteq G$ a parabolic subgroup, the relative quotient sheaf $\mathscr{E}/P \to X$ is representable by a smooth projective X-variety, and represents the functor of P-reductions of $\mathscr{E}$.*

Proof. By twisting, the functor of P-reductions of $\mathscr{E}$ is isomorphic to that of parabolic subgroups of $G^{/}\mathscr{E} := \mathscr{E} \times^{G,ad} G$ of the same type as P ([SGAIII, Exp. XXVI, 3.20]). So the proposition follows from the fact that $G^{\mathscr{E}}/P^{\mathscr{E}} \cong \underline{\mathrm{Par}}_{t(P)}(G^{\mathscr{E}})$ is representable by a smooth projective X-scheme [SGAIII, Exp. XXVI, 3.20] (whose connected components define the type). $\square$

Definition 4.6. Given $m = (\mathscr{E}, \theta) \in \mathscr{M}_G(S)$ and a semistandard parabolic P, we define a *P-reduction* of m to be a pair $(\mathscr{E}_P, \theta_P)$ consisting of a P-torsor $\mathscr{E}_P$ on X_S and $\theta_P \in H^0(X_S, \mathrm{Ad}_D(\mathscr{E}_P))$ such that

- there exists $\phi : \mathscr{E}_P \times^P G \cong \mathscr{E}$, and
- under the corresponding injection $\mathrm{Ad}(\phi) : \mathrm{Ad}_D(\mathscr{E}_P) \to \mathrm{Ad}_D(\mathscr{E})$, we have $\theta_P \mapsto \theta$.

Since θ is generically regular semisimple, there are at most finitely many parabolic reductions up to the natural notion of equivalence. The rigidifying data t allows us to define a relative position for each parabolic reduction, and therefore to organize corresponding stability conditions on each parabolic reduction.

Definition 4.7. Given $m = (\mathscr{E}, \theta, t) \in \widetilde{\mathscr{M}}_G(S)$ and a semistandard parabolic P, we say that a P-reduction $m_P = (\mathscr{E}_P, \theta_P)$ of $(\mathscr{E}, \theta)$ is in *relative position* $w \in W_G(\infty_S)$ if it satisfies

$$\chi_P(\theta_P)(\infty) = \chi_P(w(t)).$$

We will sometimes call such a reduction a *(w, P)-reduction.*

Remark 4.8. Note that the pair (w, P) hold the same ambiguity as the pairs $[(w, M)]$ of the Hitchin base: w is defined only up to $W_M(\infty)$, where M is the semistandard Levi factor of P, and for any $v \in W_G(F)$, a (w, P)-reduction is equivalent to a (vw, vP)-reduction. In the split case, one can always reduce to considering $w = e$, so our definitions reduce to those of [CL10] in that case.

Remark 4.9. For classical groups the relative position w of a P-reduction is given by comparing the order of the θ-eigenspaces in t with the partial order induced by the partial flag $\mathscr{E}_P$ (or its associated graded).

The essential property of the Hitchin base we require is that the (w, M)-stratification we have defined detects parabolic reductions of Higgs bundles.

Proposition 4.10. *Let $m \in \widetilde{\mathscr{M}}_G(K)$ such that $f(m) = (a_M, t) \in \widetilde{\mathscr{A}}_{(w,M)\text{-}ell}(K)$. For each semistandard parabolic $P \subset G$ and $v \in W_G(\infty)$, there exists a unique (v, P)-reduction of m if and only if $[(w, M)] \leq [(v, M_P)]$ (where M_P is the semistandard Levi factor of P).*

Proof. If there exists a (v, P)-reduction $m_P = (\mathscr{E}_P, \theta_P, t)$ of m for any parabolic with Levi M_P, then its characteristic polynomial $(\chi_P(\theta_P), t)$ is an element of $\widetilde{\mathscr{A}}_{(v,M_P)}(K)$ whose image in $\widetilde{\mathscr{A}}_G(K)$ is $f(m)$. Since $f(m)$ is in the image of $\widetilde{\mathscr{A}}_{(v,M_P)}$, the stratification of Corollary 3.15 implies that $[(w, M)] \leq [(v, M_P)]$.

Conversely, let $(w, M) \leq (w, M_P)$, so $M \subseteq P$. We will construct a P-reduction of $m = (\mathscr{E}, \theta, t)$. Choose a preimage $(a_M, w(t)) \in \widetilde{\mathscr{A}}_M$ for (a, t). First, there exists a nonempty open $U \subseteq X_K$ (containing ∞_K) on which a_M is G-regular. By the Kostant section, there exists a regular semisimple lift $Y \in \mathfrak{m}(U)$. Consider the centralizer $Z_Y = Z_G(Y)$, a U-torus subgroup of M_U (which agrees with T over ∞).

Let $\mathscr{E}' \to U$ be the scheme whose sections on $V \to U$ are sections ϕ of $\mathscr{E}_V$ such that the isomorphism $\mathrm{Ad}_D(\mathscr{E}_V) \to \mathfrak{g}_V$ induced by the section sends θ_V to Y_V. This is a Z_Y-torsor, which has sections étale-locally because Z_Y is étale-locally conjugate to T_U, and because θ_U and Y are both regular semisimple with G-characteristic polynomial $\chi_G^M(a_M)$.

The torsor $\mathscr{E}'$ is a reduction of $\mathscr{E}_U$: there is a map

$$\begin{aligned}(\mathscr{E}' \times G_U)(V) &\to \mathscr{E}(V)\\ (\phi, g) &\mapsto g.\phi\end{aligned}$$

which descends to an isomorphism on the quotient by the Z_Y-action. We claim now that θ_U defines a tautological section of $\mathrm{Ad}_D(\mathscr{E}'_P)$ over U. We have

$$\mathrm{Ad}_D(\mathscr{E}'_P) = \mathscr{E}' \times^{Z_Y} P \times^P \mathfrak{p}_D \cong \mathscr{E}' \times^{Z_Y} \mathfrak{p}_D.$$

All the local sections of this associated product (ϕ, Y) are equivalent, so by uniqueness they descend to a U-section $\theta_{P,U}$ whose image under the natural map $\mathrm{Ad}_D(\mathscr{E}'_P) \to \mathrm{Ad}_D(\mathscr{E}_U)$ is θ_U.

Over ∞, the section $\theta_{P,U}$ has the same M-characteristic polynomial as Y, namely, $a_M(\infty) = \chi_M^T(w(t))$. So the triple $(\mathscr{E}'_P, \theta_{P,U}, t)$ is a (w, P)-reduction over U of m. By the valuative criterion of properness for $\mathscr{E}/P$, the torsor extends to all of X_K and $\theta_{P,U}$ likewise.

Finally, we must prove uniqueness. Let $m_P = (\mathscr{E}_P, \theta_P, t)$ and $m'_P = (\mathscr{E}'_P, \theta'_P, t)$ be two (w, P)-reductions. The torsors $\mathscr{E}_P, \mathscr{E}'_P$ correspond to two different sections

$$\sigma_P, \sigma'_P : X_K \to \mathscr{E}/P,$$

and by uniqueness in the valuative criterion it suffices to check they are isomorphic (and that $\theta_P \leftrightarrow \theta'_P$ under an isomorphism) at a geometric generic point $\overline{\eta} = \operatorname{Spec} F^s$ of the curve. But now the group $G|_{F_s}$ is split constant and the proof in [CL10, Proposition 4.10.1] applies, replacing the reference to [CL10, Proposition 3.9.1] with Proposition 3.11. $\square$

4.3. The compatible Hitchin fibration.

As noted in the previous section, the locus $\mathsf{A} \subset \widetilde{\mathscr{A}}$ of pairs (a, t) which are compatible (in the sense of Definition 3.9) is an open subset which surjects onto $\mathscr{A}$.

Definition 4.11. Define the *compatible Hitchin fibration* to be the restriction of f to the compatible locus, also denoted f:

$$\mathsf{M} := f^{-1}(\mathsf{A}) \to \mathsf{A}.$$

Proposition 4.10 implies that for $f(m) \in \mathsf{A}_{M\text{-ell}}$ *M-elliptic* and *compatible* in the sense of Definition 3.9, all the parabolic reductions can be taken to be compatible (i.e., in relative position $w = e$) and to range over exactly the semistandard Levis $P \supset M$.

Since A surjects onto $\mathscr{A}$, every Hitchin fiber is visible in the compatible locus, and we will see in §5.4 that this restriction is compatible with the weighted stability conditions we aim to impose.

CHAPTER 5

Weighted stability conditions

In this section, we define a ξ-weighted analog of Harder–Narasimhan stability for Hitchin triples. Such a description is possible because, since our Higgs fields are generically regular semisimple, a Hitchin triple has at most finitely many parabolic reductions. On the compatible locus, we showed above we can identify these with distinct semistandard parabolics, and thus it is reasonable to impose positivity conditions pointing in different directions, which can be combined into a polyhedral convex C_m. Generalizing the usual notion of semistability, we say that a Hitchin triple m is ξ-stable if ξ lies within this convex.

After briefly defining the polyhedral convex and ξ-stability over the entire ∞-rigidified Hitchin base $\widetilde{\mathscr{A}}$, we will quickly restrict to the compatible locus A, where the convex is determined by its intersection with the rational cocharacter space $\mathfrak{a}_T$ and the combinatorics is more streamlined. The restriction to the compatible locus will then be in effect for much of the remainder of this work. Since A surjects onto the unrigidified (∞-regular semisimple) Hitchin base, this restriction loses no part of the geometry, cf. Proposition 5.15 below.

5.1. Cocharacter spaces, cones, and measures.

Let G/X a reductive group scheme which has a Borus pair over X and choose a Galois cover $X' \to X$ splitting G with Galois group Γ. Recall the cocharacter space sheaf $\underline{\mathfrak{a}}_P$ associated to each semistandard parabolic $P \supset T$. We have then the *geometric* cocharacter space $\underline{\mathfrak{a}}_P(X')$, which we abusively denote $\underline{\mathfrak{a}}_P$, with Galois action, and the *rational* cocharacter subspace $\mathfrak{a}_P = \underline{\mathfrak{a}}_P^\Gamma = \underline{\mathfrak{a}}_P(X)$. Both are real vector spaces. Note that by Proposition 2.20, these spaces remain canonically isomorphic upon base change to X_L for any extension L/k. If $M \supset T$ is the semistandard Levi factor of P, then we will sometimes write $\underline{\mathfrak{a}}_M = \underline{\mathfrak{a}}_P$ and $\mathfrak{a}_M = \mathfrak{a}_P$; indeed, all characters factor through the Levi quotient and its dual space consists of cocharacters landing in A_M, the central torus of the Levi factor.

As in §2, we have for any containment $P \subseteq Q$ of parabolics canonical decompositions $\underline{\mathfrak{a}}_P = \underline{\mathfrak{a}}_Q \oplus \underline{\mathfrak{a}}_P^Q$ and $\mathfrak{a}_P = \mathfrak{a}_Q \oplus \mathfrak{a}_P^Q$. We will have need to consider various cones and roots in these spaces. Geometric data is generally denoted with an underline, and relative/rational data is generally denoted without. For example, the geometric coroots $\underline{\Phi}^\vee$ live inside $\underline{\mathfrak{a}}_T = \underline{\mathfrak{a}}_B$, and the geometric roots $\underline{\Phi}$ live inside $\underline{\mathfrak{a}}_T^\star = \underline{\mathfrak{a}}_B^\star$.

Write $\underline{\Delta}_B$ (resp. $\underline{\Delta}_B^\vee$) for the set of simple positive geometric roots (resp. coroots) associated to B. For any $P \supset B$ one has corresponding subsets $\underline{\Delta}_{P,B} \subset \underline{\Delta}_B$ (resp. $\underline{\Delta}_{P,B}^\vee \subset \underline{\Delta}_B^\vee$) of those simple roots (resp. coroots) corresponding to $R_u(P)$. The image of $\underline{\Delta}_{P,B}$ under the restriction $\underline{\mathfrak{a}}_B^\star \twoheadrightarrow \underline{\mathfrak{a}}_P^\star$ defines a basis $\underline{\Delta}_P \subset \underline{\mathfrak{a}}_P^\star$ of *simple roots* which is independent of B, and likewise $\underline{\mathfrak{a}}_B \twoheadrightarrow \underline{\mathfrak{a}}_P$ defines a basis of *simple coroots* $\underline{\Delta}_P^\vee \subset \underline{\mathfrak{a}}_P$ which is independent of B. Write $\widehat{\underline{\Delta}}_P$ for the basis of *fundamental weights* in $\underline{\mathfrak{a}}_P^\star$, dual to the coroots $\underline{\Delta}_P^\vee$. These form the extremal rays, or corners, of the dominant chambers for the various geometric Borels over X' containing $T_{X'}$.

We will make much use of certain cones in the cocharacter spaces. Denote the *geometric obtuse cone* associated to P

$$ {}^+\underline{\mathfrak{a}}_P := \{x \in \underline{\mathfrak{a}}_P : \omega(x) > 0 \ \forall \omega \in \widehat{\underline{\Delta}}_P\} \subseteq \underline{\mathfrak{a}}_P $$

and the *geometric narrow cone* associated to P

$$ \underline{\mathfrak{a}}_P^+ := \{x \in \underline{\mathfrak{a}}_P : \alpha(x) > 0 \ \forall \alpha \in \underline{\Delta}_P\} \subseteq \underline{\mathfrak{a}}_P. $$

It is most natural to extend the obtuse cones from $\underline{\mathfrak{a}}_P$ to $\underline{\mathfrak{a}}_T$ by inverse image. Indeed as in [CL10, Lemme 5.3.1], ${}^+\underline{\mathfrak{a}}_P$ is the intersection of ${}^+\underline{\mathfrak{a}}_Q + \mathfrak{a}_P^Q$ ranging over all the maximal proper geometric parabolics $Q \supset P$.

On the other hand, the canonical embedding $\underline{\mathfrak{a}}_P \hookrightarrow \underline{\mathfrak{a}}_T$, we can identify the narrow cone $\underline{\mathfrak{a}}_P^+$ with its image, the degenerate (not full-dimensional) cone consisting of $x \in \underline{\mathfrak{a}}_T$ which satisfy $\alpha(x) > 0$ for $\alpha \in \underline{\Delta}_P$ and $\alpha(x) = 0$ for all $\alpha \in (\underline{\mathfrak{a}}_T^P)^\star$ (that is, the roots in the Levi factor of P; if we choose a $B \subseteq P$, then it suffices to check $\alpha \in \underline{\Delta}_B \setminus \underline{\Delta}_P$). The narrow cones are then disjoint, and together encode the *geometric Weyl fan*

$$\underline{\mathfrak{a}}_T = \bigsqcup_{P \supset T_{X'}} \underline{\mathfrak{a}}_P^+$$

where the union ranges over semistandard X'-parabolics. Its maximal facets are the Weyl chambers for the semistandard Borels. The closure of $\underline{\mathfrak{a}}_P^+$ is the union over cones for parabolics containing P. In other words, this is the simplicial structure on the apartment corresponding to $T_{X'}$ in the spherical Tits building for $G_{X'}$.

The constructions of this section are valid for any root system, and so can be performed identically with respect to the relative root system of a maximal split torus $S \subseteq T$. We call such objects *relative* or *rational* to distinguish them from the geometric objects, and denote them without an underline (so Δ_B, Δ_P, $\widehat{\Delta}_P$, $\mathfrak{a}_P^+$, ${}^+\mathfrak{a}_P$, etc.). All the claims made above hold verbatim for the relative versions, with P ranging over the set of X-parabolics.

Definition 5.1. For the sake of defining orbital integrals, we will need to define a measure on each of the $\underline{\mathfrak{a}}_P$ compatibly. For the sake of convenience in our relative arguments, we will normalize our choice in a slightly contorted way so that the restriction to the *relative* spaces has a particularly simple form. We do this once and for all as follows: on the geometric cocharacter space

$$\underline{\mathfrak{a}}_T = \underline{\mathfrak{a}}_B,$$

consider the inner product, positive-definite on $\underline{\mathfrak{a}}_T^G$, defined by

$$(v, w) := \sum_{\underline{\Phi}_i} c_i^{-1} \sum_{\alpha \in \underline{\Phi}_i} \langle v, \alpha \rangle \langle w, \alpha \rangle,$$

where $\underline{\Phi}_i$ ranges over Γ-orbits of irreducible components of the absolute root system

$$\underline{\Phi} = \bigsqcup_i \underline{\Phi}_i, \qquad W = \prod_i W_i,$$

(equivalently, $\Phi = \bigsqcup_i \Phi_i$ relatively) and the scalar c_i is engineered on each component to convert the absolute root-sum into a relative root-sum: for any $W_i(F)$-invariant norm on the relative space $\mathfrak{a}_T$,

$$c_i := \frac{\sum_{\beta \in \Phi_i} m_\beta \|\beta\|^2}{\sum_{\beta \in \Phi_i} \|\beta\|^2}$$

where m_β is the root multiplicity of a relative root $\beta \in \Phi_i$. Fix the measure on $\underline{\mathfrak{a}}_T$ to be the Lebesgue measure with respect to any orthonormal basis for this inner product.

It is immediate that our choice of inner product is invariant under $W_G(\infty) \rtimes \Gamma$, and the scalars are engineered such that

$$(-, -)|_{\mathfrak{a}_T} = \sum_{\alpha \in \Phi} \langle -, \alpha \rangle \langle -, \alpha \rangle,$$

in particular, we get a $W_G(F)$-invariant inner product on $\mathfrak{a}_T$.

By restriction, we then get an induced choice of $W_M(\infty) \rtimes \Gamma$-invariant inner product and measure on each $\underline{\mathfrak{a}}_M \hookrightarrow \underline{\mathfrak{a}}_T$, and a $W_M(F)$-invariant inner product and measure on each $\mathfrak{a}_M$, for each semistandard Levi. The latter will even retain a simple root-sum form, cf. Proposition 7.1 below. Even more, for any endoscopic datum (s, ρ_s) for G with endoscopic group H, each space $\underline{\mathfrak{a}}_H$ and $\mathfrak{a}_H$ are equipped with the restricted inner product and induced measure. Equivariance under Ngô's homomorphism of Weyl groups implies in particular that the geometric (resp. relative) endoscopic cocharacter spaces are all equipped with measures which are $W_H(\infty) \rtimes \Gamma$-invariant (resp. $W_H(F)$-invariant). The decompositions

$$\underline{\mathfrak{a}}_H \cong \underline{\mathfrak{a}}_G \oplus \underline{\mathfrak{a}}_H^G$$

are moreover all orthogonal with respect to the inner product.

5.2. The geometric stability convex.

DEFINITION 5.2. Let $P \subseteq G$ a parabolic subgroup over X and $m = (\mathscr{E}, \theta) \in \mathscr{M}(L)$ for L/k an arbitrary field extension. The *geometric degree* $\deg(m_P)$ of a P-reduction $m_P = (\mathscr{E}_P, \theta_P)$ is defined to be the element of the geometric cocharacter space $\underline{\mathfrak{a}}_P$ which pairs with geometric characters $\chi \in \underline{\mathfrak{a}}_M^\star = \underline{\mathfrak{a}}_P^\star$ (defined over splitting cover X'/X) by the degree of the associated line bundle

$$\deg(m_P)(\chi) = \frac{1}{|\Gamma'|} \deg_{X'}(\mathscr{E}_P \times_{X'}^{\chi} \mathbb{G}_{m,X'}).$$

This is normalized to be independent of choice of splitting cover. It naturally lands in $\mathfrak{a}_M = \underline{\mathfrak{a}}_M^\Gamma \subseteq \underline{\mathfrak{a}}_M$: given two X'-characters of $M' = M \times_X X'$ in the same Galois-orbit $\chi' = \sigma^\star \chi$, we have

$$\sigma^\star(\mathscr{E}_P \times_{X'}^{\chi} \mathbb{G}_{m,X'}) = \mathscr{E}_P \times_{X'}^{\chi'} \mathbb{G}_{m,X'}.$$

Now let $m = (\mathscr{E}, \theta, t) \in \widetilde{\mathscr{M}}(L)$ be a rigidified Higgs bundle, and assume that we are given a P-reduction m_P of $(\mathscr{E}, \theta)$ in relative position w. Then the *twisted cone* of the reduction m_P is defined by

$$\underline{C}_{m_P} = w^{-1}\left(-\deg(m_P) - {}^{+}\underline{\mathfrak{a}}_P + \underline{\mathfrak{a}}_T^P\right) \subseteq \underline{\mathfrak{a}}_T.$$

Note this is well-defined up to the ambiguity in (w, P): for example, a (w, P)-reduction can also be viewed as a (vw, vP)-reduction for $v \in W_G(F)$, and these give the same cone.

The *(geometric) stability convex* of m is defined to be

$$\underline{C}_m = \bigcap_{m_P} \underline{C}_{m_P} \subseteq \underline{\mathfrak{a}}_T,$$

where m_P ranges all parabolic reductions of m. Denote $\overline{\underline{C}}_m$ the closure of $\underline{C}_m$. Given $\xi \in \underline{\mathfrak{a}}_T$, we say that m is *ξ-stable* if $\xi \in \underline{C}_m$, and *ξ-semistable* if $\xi \in \overline{\underline{C}}_m$.

REMARK 5.3. As in [CL10, Lemme 5.5.2], the stability convex is fully determined by the reductions to maximal proper parabolics $P \subset G$, whose cones are half-spaces.

REMARK 5.4. Unlike in [CL10], when $M = T$ the geometric stability convex is *not* a complementary polyhedron in the sense of Behrend: by Proposition 4.10 it will have only as many vertices as there are *rational* Borels containing T, which is in bijection with chambers in the relative root system $\Phi(G, S)$, not $\Phi(G_{\overline{s}}, T_{\overline{s}})$. We will define a more satisfactory rational convex below.

By Proposition 4.10, the stratification $\widetilde{\mathscr{A}}_G = \bigsqcup_{[(w,M)]} \widetilde{\mathscr{A}}_{(w,M)\text{-ell}}$ detects *which* (w, P)-reductions $m \in \widetilde{\mathscr{M}}(L)$ admits.

COROLLARY 5.5. *For m such that $f(m) \in \widetilde{\mathscr{A}}_{(w,M)\text{-ell}}$, the intersection defining the stability convex C_m ranges over (v, P) such that $[(v, M_P)] \geq [(w, M)]$, where M_P is the semistandard Levi factor of P.*

REMARK 5.6. Together with Corollary 2.21, this implies that the stability convex C_m associated to $m \in \widetilde{\mathscr{M}}(L)$ is insensitive to extensions of L/k.

5.3. The rational stability convex.

For the sake of comparison to Arthur's formulation of weighted orbital integrals, which works entirely with relative data A_P^{spl}, Δ_P, etc., it will be essential to understand the interaction of the geometric stability convex over a splitting cover with the X-rational cocharacter spaces $\mathfrak{a}_M$.

Note that if P/X is a parabolic and $\chi \in X^\star(M)$ is a character defined over X, then the associated line bundle in the above definition descends from X' to X, whose degree is

$$\deg(m_P) = \deg_X(m_P) := \deg(\mathscr{E}_P \times_X^{\chi} \mathbb{G}_{m,X}) = \frac{1}{|\Gamma|} \deg_{X'}(\mathscr{E}_P \times_{X'}^{\chi} \mathbb{G}_{m,X'}).$$

If we work over the compatible locus, then all parabolic reductions are in trivial relative position, and we can encode ξ-stability over the compatible locus purely in terms of rational data.

Definition 5.7. For m such that $f(m) \in \widetilde{\mathscr{A}}_{(e,M)\text{-ell}}$ is compatible, we define the *rational stability convex* $C_m \subset \mathfrak{a}_T$ to be the convex given by intersecting

$$C_{m_P} = -\deg_X(m_P) - {}^+\mathfrak{a}_P + \mathfrak{a}_T^P$$

over all reductions to semistandard parabolics m_P. The resulting convex is visibly independent of any splitting cover X', Γ.

Remark 5.8. Since the degree of a P-reduction is defined by its pairing with rational characters, it naturally lands inside the lattice $\mathscr{X}_\star(M) = \mathscr{X}_\star(P) \subset \mathfrak{a}_M = \mathfrak{a}_P$ dual to the global sections $X^\star(M) = X^\star(P)$, cf. Proposition 2.11. For the geometric part of this work, it will suffice to work always in the cocharacter space, but we will later make use of this lattice, cf. Definition 5.10, Definition 12.8, §15.

Definition 5.9. For $\xi \in \mathfrak{a}_T$ we say that a compatible family $m \in \mathsf{M}(S)$ is *ξ-stable* if $\xi \in C_{m_s}$ for all $s \in S$, and *ξ-semistable* if $\xi \in \overline{C}_{m_s}$ for all $s \in S$. This specializes the definition above. For ξ in general position, stability and semi-stability coincide. Denote M^ξ the stack of compatible ξ-stable bundles and $\overline{\mathsf{M}}^\xi$ the stack of compatible ξ-semistable bundles. We will prove below that these are open in M (hence in $\widetilde{\mathscr{M}}$).

We write

$$f^\xi : \mathsf{M}^\xi \to \mathsf{A}$$

for the restriction of the Hitchin map to M^ξ.

To obtain a Deligne–Mumford stack outside the anisotropic locus, [CL10] observed that it is best to work in a setting where ξ is not a lattice point.

Definition 5.10. We say that $\xi \in \mathfrak{a}_T$ is in *general position* if for every proper X-parabolic P, the projection of ξ along every $\mathfrak{a}_T^G \twoheadrightarrow \mathfrak{a}_P^G$ avoids the image of the lattice $\mathscr{X}_\star(P)$.

So, over the compatible locus, we recover exactly the picture of the stability convex as in [CL10] or [Beh95], in the *relative root system*, i.e., a polyhedral convex with facets corresponding to semistandard $P \supseteq M$ cut out by inequality conditions pointing in the direction of different relative Weyl chambers. It is also compatible with the geometric convex defined above in the following senses.

Proposition 5.11. *Given an element $m = (\mathscr{E}, \theta, t) \in \widetilde{\mathscr{M}}(K)$ such that $f(m) \in \widetilde{\mathscr{A}}_{(e,M)\text{-}ell}$ is compatible and $\xi \in \mathfrak{a}_T$, we have for any splitting Γ-cover $\pi : X' \to X$ that C_m is equal to the image of $\underline{C}_m$ along the orthogonal projection $q : \mathfrak{a}_T \twoheadrightarrow \underline{\mathfrak{a}}_{T,\Gamma} = \underline{\mathfrak{a}}_T^\Gamma = \mathfrak{a}_T$, and also equal to the intersection $\underline{C}_m \cap \mathfrak{a}_T$.*

In particular, $\xi \in \mathfrak{a}_T$ lies in C_m if and only if it lies in $\underline{C}_m$. The same claims hold for their closures.

Proof. It suffices to check both compatibilites cone-by-cone,

$$q(\underline{C}_{m_P}) = C_{m_P} = \underline{C}_{m_P} \cap \mathfrak{a}_T^\Gamma,$$

so fix a parabolic reduction m_P in relative position e. The vertex $-\deg_X(m_P)$ naturally lives in $\mathfrak{a}_M$, and is used as the vertex for both cones. The decompositions $\underline{\mathfrak{a}}_T = \underline{\mathfrak{a}}_P \oplus \underline{\mathfrak{a}}_T^P$ are compatible with Γ-invariants (and likewise coinvariants) by Proposition 2.18.

It remains to check compatibility of the obtuse cones ${}^+\underline{\mathfrak{a}}_P$ and ${}^+\mathfrak{a}_P$ under projection and intersection, i.e., that the positive spans of the geometric coroots $\underline{\Delta}_P$ both project onto and intersect to give the positive span of the relative coroots Δ_P. By construction, Δ_P is exactly the image under q of Γ-orbits of geometric coroots $\underline{\Delta}_P$ (cf. [KP23, §2.6]). It follows that ${}^+\mathfrak{a}_P$ is the image of ${}^+\underline{\mathfrak{a}}_P$ under the projection q.

For the intersection characterization, we claim that if $\xi \in {}^+\underline{\mathfrak{a}}_P$, then $q(\xi) \in {}^+\mathfrak{a}_P$. Writing

$$\xi = \sum_{\alpha \in \underline{\Delta}_P} c_\alpha \alpha + \underline{\mathfrak{a}}_T^P$$

with $c_\alpha > 0$ and applying q gives an expression

$$q(\xi) = \sum_{[\alpha] \in \Delta_P} \left(\sum_{\alpha' \in [\alpha]} c_{\alpha'} \right) [\alpha] + \mathfrak{a}_T^P,$$

where we have identified Δ_P with the Galois-quotient of $\underline{\Delta}_P$. From this follows $\supseteq$ for the first equality in the following chain:

$$^+\underline{\mathfrak{a}}_P \cap \mathfrak{a}_T = q(^+\underline{\mathfrak{a}}_P) = {}^+\underline{\mathfrak{a}}_P$$

which implies the result. $\square$

The projection of C_m into $\mathfrak{a}_M$ is a version of Behrend's complementary polyhedron in the relative root system [Beh91], by the following result.

PROPOSITION 5.12 (cf. [CL10, Lemme 5.7.1], [Hei17, Lemma 3.15]). *Let $m \in \mathsf{M}(K)$ such that $f(m) \in \mathsf{A}_{M\text{-}ell}(K)$ is compatible M-elliptic, and let P, Q be two adjacent parabolics (meaning there is a unique coroot $\alpha^\vee$ in $(\Delta_P)^\vee \cap (-\Delta_Q)^\vee)$[1] containing M. Then*

$$\deg(m_Q) - \deg(m_P) \in \mathbb{R}_{\geq 0}\alpha^\vee.$$

PROOF. Note that this is a purely rational condition, since adjacent parabolics may pull back to non-adjacent parabolics over a splitting cover. Let R be the minimal parabolic over X containing both P and Q. $\deg(m_Q)$ and $\deg(m_P)$ agree when evaluated on any X-characters $\mu \in X^\star(R)$, so $\deg(m_Q) - \deg(m_P) \in \mathbb{R}\alpha^\vee$.

We reduce to the case of semisimple split-rank 1 by replacing G with the Levi quotient of R and replacing $\mathscr{E}_P, \mathscr{E}_Q$ with their pushforwards, as in the proof of Theorem 8.4. We may now assume that P and Q are opposite Borels. Writing $\mathfrak{u} = \mathrm{Lie}(R_u(P))$ for the unipotent radical and

$$V_1 = \mathscr{E}_P \times^P \mathrm{Lie}(R_u(P)), \qquad V_2 = \mathscr{E}_Q \times^Q \mathfrak{q},$$

for two subbundles of $V = \mathscr{E}\times^G$, complementary at the generic point, we have an injective morphism of sheaves of the same rank

$$V_1 \to V/V_2$$

and so $\deg(V_1) \leq \deg(V/V_2)$. If α is a multipliable root, then this homomorphism sends the filtration $(V_1)_{2\alpha} \subset V_1$ into the corresponding filtration of V/V_2, so we conclude

$$\deg((V_1)_\alpha) \leq \deg((V/V_2)_\alpha)$$

and likewise for 2α if it is also a root.

The degrees of these vector bundles compute the numerical invariants $\deg(\mathscr{E}_P)(\mu)$ and $\deg(\mathscr{E}_Q)(\mu)$ on the character $\mu = \deg(\mathrm{Ad}|_{\mathfrak{u}}) = \deg(\mathrm{Ad}|_{\mathfrak{g}/\mathfrak{q}}) \in X^\star(P)$, so we compute that

$$0 \leq \deg(m_Q)(\mu) - \deg(m_P)(\mu) = x_\alpha \mu(\alpha^\vee).$$

This determinant character μ is a positive multiple of roots in $\mathfrak{u}$, namely α and possibly 2α, so it pairs positively with $\alpha^\vee$. It follows that $x_\alpha > 0$. $\square$

COROLLARY 5.13 (cf. [CL10, Proposition 5.8.1], [Beh91, Lemma 5.2.5]). *For $m \in \mathsf{M}(K)$ compatible M-elliptic as above, the closed convex $\overline{C}_m$ is equal to the sum*

$$\mathrm{cvx}\left(\{-\deg(m_P)\}_{P \supset M}\right) + \mathfrak{a}_T^M + \mathfrak{a}_G.$$

PROOF. The proof is exactly as in *loc. cit.* $\square$

EXAMPLE 5.14. Consider the case of quasisplit $G = U_{X'/X}(4)$. With respect to the standard presentation of the quasi-split unitary group, as in §3.2, Galois-orbits of geometric coroots for the upper triangular Borel in G are

$$\alpha_1 = (1, -1, 0, 0) \longleftrightarrow \alpha_3 = (0, 0, 1, -1)$$

and

$$\alpha_2 = (0, 1, -1, 0)$$

The projection of $\mathfrak{a}_T = \mathbb{R}^4$ onto the Galois invariants is given by

$$q : (a, b, c, d) \mapsto \frac{1}{2}(a - d, b - c, c - b, d - a),$$

[1] Even if the relative root system is nonreduced, the relative coroot system will only see the underlying reduced root system, so there is no ambiguity about multipliable roots

i.e., the matrix

$$\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

We use the basis for the invariant subspace $\mathfrak{a}_T^\Gamma$ which is the image of the α_i, i.e.,

$$\beta_2 = \alpha_2, \qquad \beta_{13} = \frac{1}{2}(\alpha_1 + \alpha_3)$$

The rational cone ${}^+\mathfrak{a}_B$ corresponding to B is the positive span of these. Its inverse image is the set of (a, b, c, d) such that

$$a - d > 0, \quad b - c > 0.$$

On the other hand, the geometric cone ${}^+\mathfrak{a}_B$ consists of the set of (a, b, c, d) such that

$$a > b > c > d$$

So the rational cone is indeed the image of the geometric cone under the projection, and also its intersection with the invariant subspace. Note that ${}^+\mathfrak{a}_B$ is *not* the inverse image $q^{-1}({}^+\mathfrak{a}_B)$.

5.4. Reduction to the compatible locus.

Recall that the compatible locus A is the set of $(a, t) \in \widetilde{\mathscr{A}}_{(e,M)\text{-ell}}$ for e the identity element, as in Definition 3.9. Our goal is to study truncated Hitchin fibers $f^{\xi-1}(a, t)$. It is generally sufficient to consider only compatible Hitchin fibers by the following observation.

Proposition 5.15. *Let K/k a field extension. If $(a, t) \in \widetilde{\mathscr{A}}(K)$ and $\xi \in \mathfrak{a}_T$ (with respect to some splitting cover), there is isomorphism of groupoids between the truncated Hitchin fibers*

$$(f^\xi)^{-1}(a, t) = (f^{w(\xi)})^{-1}(a, w(t)).$$

Moreover, if (a, t) is compatible, then the truncated Hitchin fiber $(f^\xi)^{-1}(a, t)$ depends only on the image $\xi \mapsto \overline{\xi} \in \mathfrak{a}_T$ under the natural projection $\mathfrak{a}_T \twoheadrightarrow \mathfrak{a}_{T,\Gamma} = \mathfrak{a}_T$.

Proof. Clearly, $m = (\mathscr{E}, \theta, t) \mapsto m_w = (\mathscr{E}, \theta, w(t))$ defines an isomorphism of groupoids between the non-truncated Hitchin fibers. It remains to check the relationship between C_m and C_{m_w}. The set of parabolic reductions is intrinsic to $(\mathscr{E}, \theta)$; if a reduction $(\mathscr{E}_P, \theta_P)$ is a (v, P)-reduction of m, then

$$\chi_P(\theta_P)(\infty) = \chi_P(v(t)),$$

as in Definition 4.7, so the same parabolic reduction will be a (vw^{-1}, P)-reduction of m_w.

Comparing the cones C_{m_P} and $C_{m_{P,w}}$, we see that

$$C_{m_{P,w}} = wv^{-1}(-\deg(m_P) - {}^+\mathfrak{a}_P + \mathfrak{a}_T^P) = wC_{m_P}.$$

So $\xi \in C_{m_P}$ if and only if $w(\xi) \in C_{m_{P,w}}$. Thus, the isomorphism induced by $t \mapsto w(t)$ sends $\widetilde{\mathscr{M}}^\xi(K)$ to $\widetilde{\mathscr{M}}^{w(\xi)}(K)$. $\square$

5.5. Semicontinuity.

Now let S an arbitrary affine k-scheme. For each $s \in S$, we obtain G_s, B_s, T_s on the curve $X_s = X_{\kappa(s)}$. By Proposition 2.20, the spaces $\mathfrak{a}_{P_s}$ and the morphisms between them are constant over S. It is thus reasonable to ask, for an $m \in \mathscr{M}(S)$, how C_{m_s} varies in s. We prove a semi-continuity result on the behavior of the C_{m_s} in families.

Proposition 5.16 (cf. [CL10, Prop. 5.6.1]). *from the split case)] Assume S is Noetherian of finite type. Then the function $s \mapsto C_{m_s}$ is lower-semicontinuous. That is, for any $\Xi \subseteq \mathfrak{a}_T$, the set of points $s \in S$ such that $\Xi \subseteq C_{m_s}$ is open in S.*

The same holds for rational stability convexes C_{m_s} over the compatible locus.

Proof. Fix such a Ξ and $m = (\mathscr{E}, \theta, t) \in \mathscr{M}(S)$. We show first that C_m is constructible and takes only finitely many values. This can be checked on irreducible components, so we reduce to S irreducible. Let η its generic point, X_η the generic fiber. Assume m_η is (w, M)-elliptic. If we show that there exists $U \subset S$ open such that m_u is (w, M)-elliptic for all $u \in U$ and such that $\deg(m_u)$ is constant, then by Noetherian induction the complexes C_{m_s} are constructible. By the characterization of ellipticity above and the definition of the complex C_m, it suffices to check that for any $P \supseteq M$, the (w, P)-reductions of m_η extend to an open $U \subset S$ with constant degrees. But this is clear because P-reductions correspond to sections of $\mathscr{E}/P$, which extend from the generic point to an open, and moreover since the degree of a line bundle is locally constant $\deg(m_{P,u})$ is constant. The relative position w of a particular parabolic reduction is discretely valued and so extends from the generic point to an open. There are only finitely many semi-standard parabolics, so we can intersect to obtain a nonempty open U which satisfies the desired conditions for all $P \supset M$.

Since we have shown constructibility of $\{s \in S : \Xi \subset C_{m_s}\}$, it suffices now to show it is stable under generization. Our hypotheses imply that S is catenary, so we can reduce to a height 1 generization $\{s, \eta\}$. By normalization there always exists a map from a DVR with image these two points, and compatibility with pullback implies that we can check the desired property on DVRs, so we may assume that S is a DVR. We wish to show that $C_{m_s} \subset C_{m_\eta}$. By Proposition 4.10, the set of parabolic reductions of m_η is naturally a subset of those for m_s, so it suffices to show

$$C_{m_{P,s}} \subseteq C_{m_{P,\eta}}.$$

for each $(w, P) \geq (w, M)$. Since the two reductions have the same relative position, the positivity conditions of these cones point in the same direction and so the condition is equivalent to

$$\omega(\deg(m_{P,s}) - \deg(m_{P,\eta})) \geq 0$$

for each geometric fundamental weight $\omega \in \hat{\Delta}_P$.

Since this is now a purely geometric condition on the degree of an individual parabolic reduction, it can be checked after pulling all the data in question back along a splitting cover. So we may now assume that G is split. We can then conclude using the Tannakian argument of [CL10, Prop. 5.6.1]: choose a representation $\rho : G \to V$ of highest weight ω, with P-stable highest-weight line V_ω. We can then compute $\omega(\deg(\mathscr{E}_{P,s}))$ and $\omega(\deg(\mathscr{E}_{P,\eta}))$ as the degrees of the associated sub-line-bundles

$$\mathscr{V}_{\omega,\eta} := \mathscr{E}_{P,\eta} \times^{P,\rho} V_\omega \subset \mathscr{E}_{P,\eta} \times^{P,\rho} V$$

and

$$\mathscr{V}_{\omega,s} := \mathscr{E}_{P,s} \times^{P,\rho} V_\omega \subset \mathscr{E}_{P,s} \times^{P,\rho} V.$$

We deduce the corresponding inequality for degrees of line bundles, from which follows the result.

For the rational stability convexes, the characterization $C_m = C_m \cap \mathfrak{a}_T^\Gamma$ implies the same result: if $\Xi \subset \mathfrak{a}_T^\Gamma$ is a subset, then $\Xi \subset C_{m_s} = C_{m_s} \cap \mathfrak{a}_T^\Gamma$ if and only if $\Xi \subset C_{m_s}$. The set of s satisfying the latter is open, so the former is likewise. ☐

Corollary 5.17 (cf. [CL10, Proposition 6.1.4]). *The stack* M^ξ *is an open substack of* M.

Proof. This follows from the above by Noetherian approximation. Since $\widetilde{\mathscr{M}}$ is locally of finite type, we may check functor-of-points criteria on affine Noetherian S. Moreover, any such S can be written as a limit of schemes $\varprojlim_i S_i$ finite-type over k, and for M locally of finite type, we have

$$\mathsf{M}(S) = \varinjlim_i \mathsf{M}(S_i)$$

by [Sta22, Tag. 01ZC]. (To be precise, we could replace M with a smooth cover by a scheme). So every $m \in \mathsf{M}(S)$ descends to some $m_i \in \mathsf{M}(S_i)$, in the sense that $\mathsf{M}^\xi \times_{\mathsf{M},m} S \to S$ arises from base change along $\mathsf{M}^\xi \times_{\mathsf{M},m_i} S_i \to S_i$. The latter is an open immersion by the previous proposition, so the latter is also. ☐

CHAPTER 6

Adelic characterization of truncated Hitchin fibers

In this section, k always denotes an algebraically closed field.

6.1. Preliminaries on local triviality and uniformization.

In order to simply between torsors and Beauville–Laszlo descent data, Chaudouard–Laumon make frequent use of the following fundamental local triviality results.

THEOREM 6.1 ([DS95]). *Let $X \to S$ a curve over an arbitrary $\mathbb{Z}[n^{-1}]$-scheme, and let $G \to S$ a reductive group scheme over the base. Let $\mathscr{E}$ a G-bundle on X.*

1. *(Zariski-triviality) After an étale base change $S' \to S$, the torsor $\mathscr{E}|_{S'}$ is Zariski-locally trivial on $X_{S'}$.*
2. *(One-Point Uniformization) Assume G is semisimple. Let $D \subseteq X$ any closed subscheme of X which maps isomorphically to S, and write $U = X \setminus D$. After an étale base change $S' \to S$, $\mathscr{E}|_{U_{S'}}$ is trivial.*

The proof strategy of Drinfeld–Simpson is to reduce to the case of split G by an étale change of the base, then to show B-reductions exist. Since for split groups B admits a filtration by $\mathbb{G}_{a,S}$ and $\mathbb{G}_{m,S}$,[1] we may induct on dimension to show that every B-torsor is Zariski-locally trivial, with base cases given by Hilbert 90 and the comparison isomorphism $H^1_{Zar}(X, \mathscr{O}_X) \cong H^1_{\acute{e}t}(X, \mathbb{G}_a)$.

We require analogous local triviality results in the case when G is defined over X, not constant. The simplest such result, over an algebraically closed field of constants, follows from Steinberg's theorem.

THEOREM 6.2 (Steinberg, cf. [Ser94, III.2.3, Thm 1' and Remarks]). *Let $X \to \operatorname{Spec} k$ be a curve over an algebraically closed field $k = \overline{k}$ with generic point η and let $G \to X$ be a reductive group scheme. Then $H^1(\eta, G_\eta) = 0$, i.e., all G-torsors on X are trivial at the generic point (hence on some Zariski open).*

This implies that adelic uniformization holds for G-torsors on X, in the sense of Weil and Beauville–Laszlo.

THEOREM 6.3 (Weil). *Let $G \to X$ be a reductive group scheme on X a curve over an algebraically closed field k with function field F. Then we have an isomorphism of groupoids*

$$\mathrm{Bun}_G(X)(k) = \left[G(F) \Big\backslash G(\mathbb{A}_X) \Big/ G\left(\prod_{v \in X} \mathscr{O}_v\right)\right].$$

PROOF. This is well known in the case that the group G is split constant. The Beauville–Laszlo patching argument extends directly to our level of generality, using Steinberg vanishing. Instead we found it illuminating to reduce to the split case, an argument we reproduce here.

Let $X' \to X$ an étale Γ-cover with function fields F'/F such that $G' = G \times_X X'$ is split (in particular constant). Étale descent then furnishes us with a closed embedding

$$\mathrm{Bun}_G(X) \cong \left(\mathrm{Bun}_{\mathrm{Res}_{X'/X} G'}(X)\right)^\Gamma \hookrightarrow \mathrm{Bun}_{\mathrm{Res}_{X'/X} G'}(X) = \mathrm{Bun}_{G'}(X').$$

So, by uniformization in the split case, we have

$$\mathrm{Bun}_G(X)(k) \cong \left[G(F') \Big\backslash G(\mathbb{A}_{X'}) \Big/ G\left(\prod_{v' \in X'} \mathscr{O}_{v'}\right)\right]^\Gamma,$$

[1] See the remarks before Thm. 5.3.5 in [Con14], for example.

where the fixed points should be taken in the stacky sense below.

In general, suppose we are given $\mathscr{X}$ a set and H an abstract group acting on it, so that we have the action groupoid $[H\backslash\mathscr{X}]$, along with Γ a finite group acting on H and $\mathscr{X}$ compatibly with the action $H \curvearrowright \mathscr{X}$. Then the *groupoid of fixed points* is

$$[H\backslash\mathscr{X}]^\Gamma = \underline{\mathrm{Hom}}(B\Gamma, [H\backslash\mathscr{X}]).$$

Concretely, this is the action groupoid with object set

$$\{(x, (h_\gamma)_{\gamma\in\Gamma}) \in \mathscr{X} \times Z^1(\Gamma, H) : h_\gamma.(\gamma(x)) = x\}.$$

(the cocycle condition is $h_{\gamma\gamma'} = h_\gamma(\gamma h_{\gamma'})$, as usual), and with the H-action given by

$$h.(x, (h_\gamma)_\gamma) = (hx, (hh_\gamma\gamma(h)^{-1})_\gamma).$$

Viewing the adelic double quotient above as the action groupoid for the action of $H = G(F')$ on the set $\mathscr{X} = G(\mathbb{A}_{X'})/G(\prod_{v'} \mathscr{O}_{v'})$, we can apply this formalism.

By Steinberg's theorem, the cohomology $H^1(\Gamma, G)$ vanishes, so the cocycle h_γ is a coboundary, i.e., there exists $h \in G(F')$ such that

$$h_\gamma = h\gamma(h^{-1})$$

for all γ.[2] The element h is well-defined up to right multiplication by an element of $G(F')^\Gamma = G(F)$. For a pair (x, h), the fixed point condition becomes

$$h\gamma(h^{-1})\gamma(x) = x$$

or equivalently, $h^{-1}x$ is fixed by Γ. So our groupoid is isomorphic to one with objects $(h^{-1}x, h)$, a left $G(F')$-action given by

$$g.(h^{-1}x, h) = (h^{-1}x, gh),$$

and another $G(F)$-action given by

$$g.(h^{-1}x, h) = (gh^{-1}x, hg^{-1}).$$

We can always use the first $G(F')$-action to reduce to $h = e$ the identity (with no residual automorphisms), so this reduces to the groupoid of Γ-fixed adelic families up to the left-multiplication $G(F)$-action.

The action of Γ on an adelic family $x = (g_{v'})_{v'\in X'} \in \prod'_{v'} \mathrm{Gr}_{v',G'}$ is to permute the v' lying over a fixed v, so an adelic family $h^{-1}x$ on X' is fixed exactly if it arises by pullback from an adelic family $(g_v)_{v\in X}$. Since the local extensions are all trivial, $\mathrm{Gr}_{v',G'} \cong \mathrm{Gr}_{v,G}$ by pullback. We have computed

$$\left[G(F')\Big\backslash G(\mathbb{A}_{X'})\Big/ G\left(\prod_{v'\in X'} \mathscr{O}_{v'}\right)\right]^\Gamma \cong \left[G(F)\Big\backslash G(\mathbb{A}_X)\Big/ G\left(\prod_{v\in X} \mathscr{O}_v\right)\right].$$

□

Example 6.4. Consider the case of $G = U_{X'/X}(1)$ associated to étale double cover $X' \to X$ with involution σ and extension of function fields F'/F. A G-torsor is a line bundle $\mathscr{L}$ on X' equipped with a skew-symmetric structure $\sigma^\star\mathscr{L} \cong \mathscr{L}^\vee$. Beauville–Laszlo data, that is, an adelic uniformization, for this G reduces to a divisor for the line bundle. The theorem above asserts that we can find a divisor D on X' representing $\mathscr{L}$ which is skew-symmetric on the nose, i.e., with $D + \sigma^\star D = 0$, rather than up to linear equivalence. Concretely, given any rational section s for $\mathscr{L}$, then the skew-symmetry isomorphism identifies $s\sigma(s)$ with a rational function $c \in (F')^\times)^\sigma = F^\times$. Steinberg's theorem in this setting is the observation that the norm map $N : (F')^\times \to F^\times$ is surjective (cf. [Ser68, X.§7, Prop. 11]), so we can write $c = d\sigma(d)$. Then $D = \div(sd^{-1})$ is a skew-symmetric divisor for $\mathscr{L}$.

The corresponding claim fails to hold for curves over finite fields because Steinberg's theorem requires algebraically closed constants, but the obstruction lies purely in the field of constants, in the following sense.

[2] The equivalence relation on cocycles in nonabelian H^1 is $(h_\gamma)_\gamma \sim (h'_\gamma)_\gamma$ if there exists h such that $hh'_\gamma = h_\gamma\gamma(h)$. Plugging in the trivial cocycle $h'_\gamma = 1$ gives our formula.

Corollary 6.5. *Let k_0 a finite field with algebraic closure k. If $G_0 \to X_0$ is a reductive group scheme over smooth geometrically connected curve $X_0 \to \operatorname{Spec} k_0$, then writing $G \to X$ for its pullback to $k = \overline{k_0}$ and $\tau = 1 \times \tau$ the arithmetic Frobenius on $X = X_0 \times_{k_0} k$, we have*

$$\mathrm{Bun}_{G_0}(k_0) = \left[G(F)\Big\backslash G(\mathbb{A}_X)\Big/ G\left(\prod_{v \in X} \mathscr{O}_{v'}\right)\right]^{\tau},$$

where the fixed locus is taken in the stacky sense.

Proof. Proof is as above. The cocycle $(h_\gamma)_\gamma$ from the previous proof necessarily splits over a finite extension of constants, by Steinberg's theorem, so factors as a cocycle for the Galois group of k_0. $\square$

6.2. One-point uniformization.

In the semisimple case, adelic uniformization is vastly strengthened by the following one-point uniformization theorem of Heinloth.

Theorem 6.6 ([Hei10]). *Let $G \to X$ be a semisimple group scheme whose algebraic fundamental group $\pi_1(G)$ is not a multiple of the characteristic p.*[3] *Fix a closed point $x \in X$ and $X^\circ = X \setminus \{x\}$. Let S Noetherian and $\mathscr{E} \in \mathrm{Bun}_G(S)$ a G-torsor on $X \times S$. Then there exists an étale cover $S' \to S$ such that $\mathscr{E}|_{X^\circ \times S'}$ is trivial.*

Proof. [Hei10, Thm. 4] is exactly this result in the case that $G \to X$ is also simply connected. For general G, the étale cover $S' \to S$ allows us to take finite extensions of k, so we reduce to the case that k is algebraically closed. Writing

$$0 \to Z \to G^{sc} \to G \to 0$$

for the simply connected cover of G, the obstruction to lifting $\mathscr{E}|_{X^\circ \times S}$ to a G^{sc}-torsor is a locally constant class in $H^2(X^\circ, Z)$, as in [Hei10, Lemma 14]. Here $Z \to X$ is a finite flat commutative group scheme whose (constant) fiberwise-order n is not a multiple of p, so is étale by the fibral smoothness criterion, even locally constant. Since X° is affine, [Sta22, Tag 03SC] implies that $H^2(X^\circ, Z)$ vanishes.

So there exists a G^{sc}-torsor $\mathscr{E}^{\circ,sc}$ on $X^\circ \times S$ such that $\mathscr{E}^{sc,\circ} \times^{G^{sc}}_{X^\circ \times S} G \cong \mathscr{E}|_{X^\circ \times S}$. By Steinberg's theorem, this G^{sc}-torsor is trivial on an open. Choose such a trivialization and use it to glue $\mathscr{E}'$ with the trivial G^{sc}-bundle on a formal neighborhood of x. The result is a G^{sc}-torsor $\mathscr{E}^{sc}$ on $X \times S$ extending $\mathscr{E}^{\circ,sc}$:

$$\mathscr{E}^{sc}|_{X^\circ \times S} \cong \mathscr{E}^{\circ,sc}.$$

By the simply connected case, there is an étale cover $S' \to S$ such that

$$\mathscr{E}^{sc}|_{X^\circ \times S'} \cong \mathscr{E}^{\circ,sc}|_{X^\circ \times S'}$$

is trivial. Forming this associated product is compatible with flat/étale pullback, so we deduce the desired triviality of

$$(\mathscr{E}^{sc}|_{X^\circ \times S'}) \times^{G^{sc}}_{X^\circ \times S'} G \cong (\mathscr{E}^{\circ,sc} \times^{G^{sc}}_{X^\circ \times S} G)|_{X^\circ \times S'} \cong \mathscr{E}|_{X^\circ \times S'}.$$

$\square$

This result applied to a DVR is very useful in checking valuative criteria.

Corollary 6.7. *Let $G \to X$ be a semisimple group scheme whose algebraic fundamental group $\pi_1(G)$ is not a multiple of the characteristic, and R be a DVR k-algebra. Then for each torsor $\mathscr{E}$ on X_R, there is an étale cover $R \to R'$ such that $\mathscr{E}|_{R'}$ is Zariski-locally trivial on X_R.*

[3]By an examination of types, this follows from assuming p does not divide the order of the Weyl group.

6.3. Truncated Hitchin fibers and Beauville–Laszlo descent.

In the remainder of this section we describe the points of a truncated Hitchin fiber adelically. With our eye on point counting, these notions will be defined only *X-rationally* and only on the compatible locus of Definition 3.9. Recall that k is algebraically closed.

DEFINITION 6.8. For each semistandard parabolic $P \supseteq T$ with Levi M, any $w \in W_G(\infty)$, and any place $v \in X(k)$, we have a map $H_{P,v} : \mathrm{Gr}_{G,v} \to \mathfrak{a}_M$ defined by its pairing with *globally rational* characters: for all $\mu \in X^\star(M) = \underline{X}^\star(M)^\Gamma$ and any $g \in G(F_v)$,

$$\mu(H_{P,v}(g)) = -\operatorname{val}_v(\mu(p))$$

where $g = pk \in P(F_v)G(\mathscr{O}_v)$ is the local Iwasawa decomposition.

Given an adelic family $(g_v)_v$, we define

$$H_P((g_v)_v) = \sum_v H_{P,v}(g_v).$$

For semistandard Levi M, we write $H_M : M(\mathbb{A}) \to \mathfrak{a}_M$ for the restriction of H_P to Levi factor M, where P is any semistandard parabolic for M. This is independent of P, by construction.

REMARK 6.9. Note that $H_P(g_v)$ is defined by virtue of the Iwasawa decomposition in $G_v = G_{F_v}$ relative to P_v. The local groups G_v can be more split than G, so note that our definitions of $H_{P,v}$ and H_P include an implicit projection onto $\mathfrak{a}_M = \underline{\mathfrak{a}}_M^\Gamma = \underline{\mathfrak{a}}_{M,\Gamma}$: we have evaluated only on *globally* defined characters.

PROPOSITION 6.10. *Suppose that $\mathscr{E}$ is a G-torsor on X which corresponds to an adelic family $(g_v)_v \in \prod'_v \mathrm{Gr}_{G,v}$ as in Theorem 6.3. Fix a parabolic P and suppose that we have Iwasawa decompositions $g_v = p_v G(\mathscr{O}_v)$. Then the family $(p_v)_v$ patch together to give a P-reduction $\mathscr{E}_P$ of $\mathscr{E}$, and its degree is given by*

$$\deg(\mathscr{E}_P) = H_P((g_v)_v) \in \mathfrak{a}_M.$$

PROOF. First, Beauville–Laszlo patching is compatible with pushforward in the obvious sense. That is, given $\phi : G \to H$ an X-homomorphism of affine, smooth, finite type group schemes on X and a G-torsor $\mathscr{E}$ on X with B–L gluing data $(g_v)_v$, the pushforward torsor $\mathscr{E} \times^G H$ corresponds to B–L data $(\phi(g_v))_v$. This follows from the construction.

Now Beauville–Laszlo gluing for P (a smooth, affine group scheme over X) implies that the family $(p_v)_v$ do combine to give a P-torsor $\mathscr{E}_P$ on X. Since $p_v \equiv g_v \in \mathrm{Gr}_{G,v}$, the same family of elements viewed as transition functions for a G-torsor gives a torsor isomorphic to $\mathscr{E}$, so by the compatibility observed above, $\mathscr{E}_P \times^P G \cong \mathscr{E}$.

To confirm the claim about degrees, let $\mu \in X^\star(P)^\Gamma$ be an X-rational character. The compatibilities above imply that $\mathscr{E}_P \times^{P,\mu} \mathbb{G}_m$ is the $\mathbb{G}_m$-torsor corresponding to B–L data $(\mu(p_v))_v$, and by the elementary case of $\mathbb{G}_m$ we have

$$\mu(\deg(\mathscr{E}_P)) = \deg(\mathscr{E}_P \times_X^{P,\mu} \mathbb{G}_m) = -\sum_v \operatorname{val}_v(\mu(p_v)) = \mu(H_P((g_v)_v)).$$

$\square$

DEFINITION 6.11. Let $(a,t) \in \mathsf{A}_{M\text{-ell}}(k)$ be a point in the compatible base. Then $\mathscr{X}^\xi_{(a,t)}$ is the set of pairs $(X, (g_v)_{v\in X(k)})$ as follows:

1. $X \in \mathfrak{m}(F)$ is a G-regular semisimple M-elliptic element *which is conjugate by an element of $M(F_\infty)$ to $w(t_a)$.* (In particular, $\chi_G(X) = a|_\eta$).
2. $(g_v)_{v\in X(k)}$ is a family of affine Grassmannian elements $g_v \in \mathrm{Gr}_{v,G}(k)$ such that
   - (a) g_v is trivial cofinitely often,
   - (b) (D-integrality) $\mathrm{Ad}(g_v^{-1})X \in z_v^{-d_v} L^+_{\mathfrak{g},v}(k)$ (in particular, at ∞ we have $\mathrm{Ad}(g_\infty^{-1})X \in L^+_{\mathfrak{g},v}$ and so $\chi_G(X) \in \mathfrak{c}(\mathscr{O}_\infty)$), and
   - (c) (weighted stability) ξ lands within C_g, the convex defined by the intersection of the cones
   $$-(H_P((g_v)_v)) - {}^+\mathfrak{a}_P + \mathfrak{a}_T^P$$
   over all the F-rational semistandard parabolics P in G.

Note that there is a natural action of $M(F)$ on this set given by

$$m.(X,(g_v)_v) = (\mathrm{Ad}(m)X, (mg_v)_v);$$

it preserves the weighted stability condition: normality of N_P implies that if $g_v = u_v m'_v k_v$ is the Iwasawa decomposition of g_v then $mg_v = mu_v m'_v k_v = u'_v m m'_v k_v$ is an Iwasawa decomposition of mg_v, and so

$$\mu(H_P((mg_v)_v)) = -\sum_v (\mathrm{val}_v(\mu(m)) + \mathrm{val}_v(\mu(m'_v))) = \mu(H_P((g_v)_v)) - \sum_{v\in V} \mathrm{val}_v(\mu(m)),$$

and the product formula ensures cancellation. We denote $[M(F)\backslash \mathscr{X}^{\xi}_{(a,t)}] = [M(F)\backslash \mathscr{X}^{\xi}_{(a,t)}](\overline{k})$ the action groupoid.

Proposition 6.12 (cf. [CL10, §7]). *Let $(a,t) \in \mathsf{A}_{M\text{-}ell}(k)$ (recall $k = \overline{k}$). There is an equivalence of categories between the truncated Hitchin fiber $f^{\xi-1}(a,t)$ and $[M(F)\backslash \mathscr{X}^{\xi}_{(a,t)}]$.*

Proof. Given a Higgs bundle $(\mathscr{E},\theta,t)$ lying over (a,t), the bundle $\mathscr{E}$ is generically trivial by Steinberg's Theorem 6.2. Picking such a trivialization, the generic fiber θ_η corresponds to an element $Y \in \mathfrak{g}(F)$ with $\chi_G(Y) = a_\eta$. By Proposition 3.12, there exists $Z \in \mathfrak{m}(F)$ such that $\chi_G(Z) = a_\eta$ and Z is conjugate by $M(F_\infty)$ to $w(t_a)$. Since Y and Z have the same characteristic polynomial, they are conjugate in $G(F)$, so we may assume the trivialization of $\mathscr{E}$ sends θ_η to Z and we have condition (1).

Choosing trivializations of $\mathscr{E}$ also on the formal neighborhoods $\mathscr{O}_v$, the transition data $g_v \in \mathrm{Gr}_{G,v}(k)$ between these neighborhoods and the generic one gives a well-defined family of affine Grassmannian elements $(g_v)_v$. The condition that $\theta \in H^0(X, \mathrm{Ad}_D(\mathscr{E}))$ corresponds exactly to the D-integrality condition (2)(b).

It remains to prove that the weighted stability condition translates to (2).(c). We claim that $\deg(m_P) = \sum_{v\in X(k)} H_P(g_v)$. That is, if we write $g_v \in p_v L^+G(k)$ for the Iwasawa decomposition of g_v, then $\chi_P(Z) = \chi_P(w(t_a))$.

We claim that the family $(Z,(p_v)_v,t)$ define a (w,P)-reduction of m. By Beauville–Laszlo patching we can glue the $(p_v)_v$ with the trivial generic torsor to obtain a P-reduction $\mathscr{E}_P$ of $\mathscr{E}$. The D-integrality conditions for $(g_v)_v$ apply equally to the $(p_v)_v$, so $Z \in \mathfrak{m}(F)$ corresponds to a section of $\mathrm{Ad}_D(\mathscr{E}_P)$. So we have found a P-reduction of m which is in relative position w. By Proposition 6.10, the degree of this P-reduction is given by the formula $\deg(m_P) = H_P((g_v)_v)$, so (2).(c) corresponds to ξ-stability. □

We will find a reduced version of this description useful in our arithmetic calculations. Let $(a,t) \in bAeM(k)$ with preimage $a_M \in \mathsf{A}_M(k)$. Fix $X_{a_M} \in \mathfrak{m}(F)$ a G-regular semisimple element with characteristic polynomial $a_{M,\eta}$, and write J_X for its centralizer. Consider the set $\mathscr{X}^{\xi,\mathrm{red}}_{(a,t)}$ of families $(g_v)_v$ satisfying condition (2) above with respect to $X = X_{a_M}$. This set has an action of $J_X(F)$

Proposition 6.13. *Under the notations as above, there is an equivalence of categories between the truncated Hitchin fiber $f^{\xi-1}(a,t)$ and the action groupoid $[J_X(F)\backslash \mathscr{X}^{\xi,\mathrm{red}}_{(a,t)}]$.*

Proof. The natural functor

$$[J_X(F)\backslash \mathscr{X}^{\xi,\mathrm{red}}_{(a,t)}] \to [M(F)\backslash \mathscr{X}^{\xi}_{(a,t)}]$$

is clearly fully faithful. Any two choices of X are rationally conjugate by [CL10, Lemme 3.8.1], so it is also essentially surjective. □

Part 2

Geometric properties of ξ-stability

CHAPTER 7

ξ-instability, canonical reductions, and destabilizing filtrations

The first goal of this section is to show that M^ξ is the open part of a stratification of M which generalizes, in the case $\xi = 0$, the Harder–Narasimhan–Shatz stratification of Bun_G and $\mathscr{M}$. We begin in the first section by defining canonical "Harder–Narasimhan" P-reductions and a degree of instability for compatible Hitchin triples in the style of [CL10] and Behrend's thesis [Beh91], proving the basic semicontinuity and extension results about these invariants. In the second and third sections, we provide an alternate characterization of canonical reductions, as those arising from the unique filtration which optimizes an instability functional. This situates our constructions within the GIT-style formalism of Halpern–Leistner, itself inspired by the Kempf–Ness stratification for GIT quotients (cf. [Kem78]). For purposes of exposition and because our arguments depend on it, we recall in parallel the corresponding story on Bun_G.

In the fourth section we use this alternate characterization to show that we have defined a Θ-stratification in the sense of Halpern–Leistner [Hal22]. This allows us to apply a general semistable reduction theorem of Alper–Halpern-Leistner–Heinloth [AHH23] for such stratifications: since the untruncated Hitchin fibration satisfies the existence half of the valuative criterion, its open ξ-semistable part does likewise. While the introduction of the GIT and Θ-filtration machinery is a minor detour from the formalism for ξ-stability introduced thus far, it ultimately avoids the necessity of replicating Chaudouard–Laumon's rather technical inductive semistable reduction argument (based on the method of Langton [Lan75], which is also the heart of [AHH23]).

7.1. The ξ-canonical reduction.

Given $\xi \in \mathfrak{a}_T$ (relative cocharacter space), the rational stability convexes allow us to define measures of ξ-instability and ξ-canonical parabolic reductions for each compatible Hitchin triple.

Define d to be the distance in $\mathfrak{a}_T$ induced by the canonical Weyl-invariant bilinear form

$$(v, w) := \sum_{\alpha\in\Phi} \langle v, \alpha\rangle\langle w, \alpha\rangle.$$

This is a nondegenerate inner product on $\mathfrak{a}_T^G$, but sees only the projection into this space. In fact, this defines by restriction a family of compatible inner products on the various $\mathfrak{a}_M^G$:

$$(-, -)|_{\mathfrak{a}_M\times\mathfrak{a}_M} = \sum_{\alpha\in\Phi\setminus\Phi_M} \langle -, \alpha\rangle\langle -, \alpha\rangle.$$

We write also d to denote the minimum distance between compact subsets of these spaces.

Proposition 7.1. *Given an inclusion of semistandard Levis $M \subset L$, the decomposition $\mathfrak{a}_M = \mathfrak{a}_L \oplus \mathfrak{a}_M^L$ is orthogonal with respect to the inner product above.*

Proof. Let $x \in \mathfrak{a}_L$ and $y \in \mathfrak{a}_M^L = \ker(\mathfrak{a}_M \to \mathfrak{a}_L)$. Both pair trivially with all roots in ϕ_M, so the inner product simplifies as we claimed above. We have (rational) Weyl groups $W_M \subset W_L$. The element x is fixed by W_L, and both are fixed by W_M. Decompose the set of roots $\Phi \setminus \Phi_M$ into W_L-orbits. For each such orbit $\mathscr{O}$, the contribution to (x, y) is

$$\sum_{\alpha\in\mathscr{O}} \langle x, \alpha\rangle\langle y, \alpha\rangle.$$

Since x is W_L-invariant, its pairing with each $\alpha \in \mathscr{O}$ is constant, and we can write this sum as

$$\langle x, \alpha_0\rangle \sum_{\alpha\in\mathscr{O}} \langle y, \alpha\rangle = \langle x, \alpha_0\rangle \left\langle y, \sum_{\alpha\in\mathscr{O}} \alpha \right\rangle$$

for any representative of the orbit. By definition of $\mathfrak{a}_M^L$, the element y vanishes when paired with any W_L-invariant character (which would be an element of $\mathfrak{a}_L^\star$), so this expression is 0 for any orbit of roots. We conclude $(x, y) = 0$. $\square$

We can now define a measure of the ξ-instability of a compatible Hitchin triple m.

DEFINITION 7.2. Given $\xi \in \mathfrak{a}_T$ and $m \in \mathsf{M}(K)$ for a field K/k, define the *ξ-degree of instability* $\deg_{i,\xi}(m)$ (or $\deg_i(m)$ if ξ is understood) to be the distance $d(\xi, \overline{C}_m)$ from ξ to $\overline{C}_m$. Define the *ξ-Harder–Narasimhan point* of m to be the unique closest point in $\overline{C}_m$ to ξ.

REMARK 7.3. One could define such a distance for the geometric stability convex and arbitrary $\xi \in \mathfrak{a}_T$ outside the compatible locus, at the cost of having to track additional w-twists. This would extend the definition above up to normalization by factors of $|\Gamma|$.

REMARK 7.4. Note that even when $\xi = 0$, this is superficially different from the standard definition due to Behrend, which is

$$\max_{\mathscr{E}_P}\{\deg(\mathscr{E}_P)(\det_P)\}$$

for $\mathscr{E}_P$ ranging over all parabolic reductions (θ-stable ones for a Higgs bundle). They encode basically the same information. Though ours is a real number, it takes discrete values (since $\deg(\mathscr{E}_P) \in X^\star(P)$ always lies at a lattice point) and jumps semicontinuously by Proposition 5.16.

Next, we wish to define a ξ-canonical semistandard parabolic and ξ-canonical reduction for a compatible Higgs triple. In terms of our stability convex, the condition of the canonical reduction being "maximally ξ-destabilizing" translates to the following condition. Compare also [Beh95, Prop. 3.13].

PROPOSITION 7.5 (cf. [CL10, Prop. 6.3.2]). *Let $m \in \mathsf{M}(K)$ have $f(m) \in \mathsf{A}_{M\text{-}ell}$, and write $\varrho \in \overline{C}_m$ for the ξ-HN point of m. Then there exists a unique parabolic $P \supseteq M$ such that*

(1) ϱ lies in the affine space

$$-\deg(m_P) + \mathfrak{a}_T^P + \mathfrak{a}_G$$

spanned by the facet corresponding to P, and

(2) $\xi - \varrho \in \mathfrak{a}_P^+ \cap \mathfrak{a}_T^G$, where

$$\mathfrak{a}_P^+ := \{p \in \mathfrak{a}_P : \alpha(p) > 0 \ \forall \alpha \in \Delta_P\}$$

is the narrow cone, i.e., the Weyl facet corresponding to P.

PROOF. Identical to the proof of *loc. cit.* or [Beh91, Lemma 5.3.12], performed in the relative root system. Note in particular that the rays in $\mathfrak{a}_T$ from ϱ are stratified by semistandard parabolics, since

$$\mathfrak{a}_T = \bigcup_{P \supseteq T} \mathfrak{a}_P^+.$$

So the vector $\xi - \varrho \in \mathfrak{a}_P^+$ for some semistandard P. Because of the M-ellipticity of m, $\overline{C}_m$ is invariant under translation in the $\mathfrak{a}_T^M$-direction, and so the nearest point ϱ will satisfy $\xi - \varrho \in \mathfrak{a}_M$. In particular, the direction $\xi - \varrho$ will necessarily correspond to a parabolic P which contains M. $\square$

The stratification of rays from ϱ is illustrated for SL_3 in Figure 1, which depicts two stability convexes, a bounded one associated with a T-elliptic Hitchin triple and an unbounded one associated with an intermediate Levi $T \subsetneq M \subsetneq G$. The decomposition of the rays from ϱ into narrow cones is denoted with dashed lines. ϱ is the ξ-HN point in the bounded convex for both ξ_1 and ξ_2, while the ξ_2-HN point of the unbounded convex is different (unmarked).

DEFINITION 7.6. For $\xi \in \mathfrak{a}_T$ and $m \in \mathsf{M}(K)$ with $f(m) \in \mathsf{A}_{M\text{-ell}}$, define the *$\xi$-canonical parabolic* P (resp. the *ξ-canonical reduction* m_P) to be the unique parabolic $P \supseteq M$ (resp. the reduction m_P of m to that parabolic) satisfying conditions (1), (2) of Proposition 7.5.

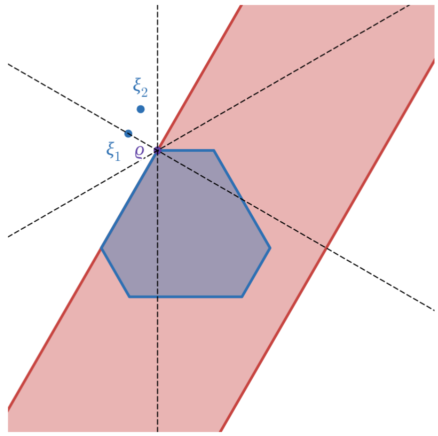

Figure 1. Two stability convexes and ξ-HN points for SL_3

Example 7.7. The strict positivity in the second condition tells us which, among the set of Q-reductions corresponding to closed facets containing the ξ-HN-point, actually has ξ-positive numerical invariants. In particular, the canonical parabolic is *not* necessarily the parabolic such that ϱ lies in the interior of the face corresponding to P, cf. the direction condition in [Beh91, Prop. 5.3.14]. For example, consider in Figure 1 the bounded convex as a stability convex $\overline{C}_m$ for SL_3. The Higgs bundle m has the same ξ-HN-point for $\xi = \xi_1, \xi_2$ and the same degree of instability. However, the ξ_2-canonical reduction is a Borel reduction, while the ξ_1-canonical reduction is to an intermediate parabolic: writing $\xi_1 = (\xi_1^1, \xi_1^2, \xi_1^3)$ with respect to the standard basis of cocharacters, the arrangement in the Figure corresponds to a vector bundle which admits a full flag $0 \subset \mathscr{E}_1 \subset \mathscr{E}_2 \subset \mathscr{E}_3 = \mathscr{E}$ such that

$$\deg(\mathscr{E}_1) - \xi_1^1 = \deg(\mathscr{E}_2/\mathscr{E}_1) - \xi_1^2 > \deg(\mathscr{E}_3/\mathscr{E}_2) - \xi_1^3;$$

the first equality implies that the numerical invariant corresponding to root $\alpha = e^1 - e^2$ is equal to zero, not positive, and so the ξ_1-canonical reduction is the partial flag $0 \subset \mathscr{E}_2 \subset \mathscr{E}_3$.

It follows from Proposition 5.16 that the degree of ξ-instability is upper-semicontinuous in families. We have a partial converse (whose proof follows [Beh91, 7.1.3]):

Proposition 7.8. *Let $m \in \mathsf{M}(R)$ be a compatible Hitchin triple over a DVR R with generic point η and special point s. Assume* $\deg_i(m_s) = \deg_i(m_\eta)$.

(a) The ξ-canonical parabolics P_η, P_s satisfy $P_\eta \supset P_s$ and the P_η-reduction of m_η extends to a P_η-reduction over all of R.
(b) If in addition $f(m) \in \mathsf{A}_{M\text{-}ell}(R)$, that is, the special and generic fiber are elliptic for the same Levi M, then $P_\eta = P_s$ and the canonical reductions glue to a corresponding reduction of m.

Proof. Let $\varrho \in \mathfrak{a}_T$ denote the ξ-HN point of m_η. Lower-semcontinuity implies that $P_\eta \supset P_s$. The only way we can have $\deg_i(m_s) = \deg_i(m_\eta)$ is if $\varrho \in C_{m_s}$ is the ξ-HN point for m_s also. Conditions (1) and (2) are intrinsic to the point ϱ, so under the hypothesis of (b) the canonical parabolics are equal, $P = P_\eta = P_s$.

Now writing $P = P_\eta$, it suffices to show that the canonical reduction of m_η and the unique P-reduction of m_s glue to a reduction of m over R. Write $m_\eta = (\mathscr{E}_\eta, \theta_\eta, t)$ and likewise for m_s. It suffices to show the gluing condition for bare G-bundles (cf. [Beh91, Prop. 7.1.3] applied to the group schemes $G^{\mathscr{E}}$, or [HS10, Lemma 4.4.2]), as the condition of θ-stability can be checked fiberwise. As in *loc. cit.*, consider the short exact sequence of Lie algebras associated to the canonical reduction $\mathscr{E}_{P,\eta}$:

$$0 \longrightarrow \mathrm{Ad}(\mathscr{E}_{P,\eta}) \longrightarrow \mathrm{Ad}(\mathscr{E}_\eta) \longrightarrow \mathfrak{q}_\eta \longrightarrow 0\ .$$

We can by properness of relative Quot schemes extend this across the special fiber to a short exact sequence of coherent sheaves

$$0 \longrightarrow \mathfrak{r} \longrightarrow \mathrm{Ad}(\mathscr{E}) \longrightarrow \mathfrak{q} \longrightarrow 0$$

with $\mathfrak{q}$ coherent, flat over S and $\mathfrak{r}$ defined to be the kernel. Behrend shows that $\mathfrak{q}$ is locally free at the generic point of the special fiber, so, writing $U \subset X$ for the maximal open on which $\mathfrak{q}$ is locally free (i.e. on which $\mathfrak{r}$ is a subbundle), it contains X_η and $V := U \cap X_s$ is nonempty.

By [Beh91, Lemma 7.1.2], there is a parabolic reduction $\mathscr{E}_{P,U} \subset \mathscr{E}$ whose Lie algebra (adjoint subbundle) is $\mathfrak{r}_U$. Properness implies that we can extend $\mathscr{E}_{P,V} = \mathscr{E}_{P,U}|_V$ to a P-reduction $\widetilde{\mathscr{E}}_{P,s}$ of $\mathscr{E}_s$ on the entire special fiber, whose adjoint bundle is the saturation of the subsheaf $\mathfrak{r}_s \subseteq \mathrm{Ad}(\mathscr{E})$. This reduction need not glue with the reduction on the generic fiber in general, but note that it is θ-stable generically, hence everywhere: since $\mathrm{Ad}_D(\widetilde{\mathscr{E}}_{P,s}) \subset \mathrm{Ad}_D(\mathscr{E})$ is a subbundle, the condition of a global section θ factoring through the subbundle is closed. We conclude that $(\mathscr{E}_{P,s}, \theta, t)$ is a P-reduction of m_s. Indeed, from working generically we can see that it is in the same relative position, i.e. still an (e, P)-reduction. Thus, it must be equal to the unique (e, P)-reduction of m_s, (which under hypothesis (b) is the canonical reduction of m_s).

Degree increases under saturation, so

$$\deg(\mathscr{E}_{P,\eta})(\det_P) = \deg(\mathfrak{r}_\eta) = \deg(\mathfrak{r}_s) \leq \deg(\mathrm{Ad}(\widetilde{\mathscr{E}}_{P,s})) = \deg(\mathscr{E}_{P,s})(\det_P)$$

We assumed degree of instability was constant, so the HN-point ϱ lies in $-\deg(\mathscr{E}_{P,\eta}) + \mathfrak{a}_T^P + \mathfrak{a}_G$ *and* in $-\deg(\mathscr{E}_{P,s}) + \mathfrak{a}_T^P + \mathfrak{a}_G$. So the two degrees are equal as elements of $\mathfrak{a}_P^G$. In particular, $\deg(\mathscr{E}_{P,\eta})(\det_P) = \deg(\mathscr{E}_{P,s})(\det_P)$, so $\mathfrak{r}_s$ has the same degree as its saturation $\mathrm{Ad}(\widetilde{\mathscr{E}}_{P,s})$. This implies they are equal, so $\mathfrak{q}$ was locally free everywhere and $\mathscr{E}_{P,s}, \mathscr{E}_{P,\eta}$ glue to a P-reduction of $\mathscr{E}$ over R which is θ-stable everywhere. ☐

We summarize the behavior of our invariants in families across M.

Corollary 7.9. *Given a family $m \in \mathsf{M}(S)$,*

- *The Levi M_s for which $f(m_s)$ is M-elliptic is lower-semicontinuous on S with respect to containment,*
- *the degree of instability $\deg_i(m_s)$ is upper-semicontinuous on S, and*
- *the ξ-canonical parabolic $P_s \supseteq M_s$ is continuous (that is, locally constant) on S for families $m \in \mathsf{M}(S)$ which are constant with respect to both of the invariants above.*

Proof. These have all been established (5.16, 3.9, 3.15) except the last claim. Constructibility of the canonical parabolic is clear from Proposition 5.16, and equality over a DVR is the previous proposition.

These imply the result as in [Beh91, Theorem 7.2.5]. Note that as discussed there, since we are in good characteristic there is no concern about purely inseparable extensions. Concretely, as remarked above, the stability convex C_m and ξ-canonical reduction are insensitive to extensions of L. ☐

7.2. Rees constructions and equivariant degenerations.

In the previous section, we defined the degree of instability $\deg_{i,\xi}(m)$ in the spirit of [Beh91] or [CL10], as a condition on $\deg(\mathscr{E}_P)(\det_P)$. As in Example 7.7, maximizing instability (minimizing distance to the polygon) alone does not uniquely determine the canonical reduction, as ties must be broken by the direction of the vector $\xi - \varrho$, or in Behrend's formalism, by taking the largest parabolic with a given degree of instability. In the formalism of Halpern-Leistner, inspired by Kempf's results in GIT [Kem78], one ranges over a space of *filtered objects*, which in our setting includes the data of the cocharacter $\lambda \in X_\star(T)$ with $P_\lambda = P$ acting on the associated graded M-bundle as well as the reduction $\mathscr{E}_\lambda$. The canonical degeneration is then defined as the unique one maximizing a piecewise-linear functional $\lambda \mapsto \frac{\mathrm{wt}_{\mathscr{E}_\lambda}}{\|\lambda\|}$ with respect to the inner pairing on the cocharacter space.

Write $\Theta = [\mathbb{A}^1/\mathbb{G}_m]$, a k-stack. We first describe the stack of filtered objects with respect to $\mathsf{M} \to \mathsf{A}$:

$$\mathrm{Filt}_{\mathsf{A}}(\mathsf{M}) := \underline{\mathrm{Map}}_{\mathsf{A}}(\Theta_{\mathsf{A}}, \mathsf{M}).$$

Note that all relevant filtration stacks are algebraic by [Hal22, Prop. 1.1.2].

We begin with recalling the structure of filtered objects in Bun_G. We can describe maps $\Theta \to \mathrm{Bun}_G$ using a version of the Rees construction, cf. [Hei17, §1.F.b]. Let $S \subset G$ a maximal split torus over the entire base X. We begin with the following data:

- a G-torsor $\mathscr{E}$ on X,
- a conjugacy class of cocharacter $\lambda \in X_\star(S)$, and
- a reduction $\mathscr{E}_P$ of $\mathscr{E}$ to $P = P_G(\lambda) \subset G$.

One then obtains a morphism $\Theta \to \mathrm{Bun}_G$ as follows. The λ-conjugation map

$$c_\lambda : P \times \mathbb{G}_m \to P \times \mathbb{G}_m$$
$$(p, t) \mapsto (\lambda(t)p\lambda(t)^{-1}, t)$$

by the dynamic definition of $P := P_G(\lambda)$ extends to $P \times \mathbb{A}^1 \to P \times \mathbb{A}^1$. We denote this extension also by c_λ. We can then define the Rees bundle, a P-torsor on $X \times \mathbb{A}^1$ as the associated product

$$\mathrm{Rees}(\mathscr{E}_P, \lambda) := (\mathscr{E}_P \times \mathbb{A}^1) \times_{\mathbb{A}^1}^{P\times\mathbb{A}^1, c_\lambda} (P \times \mathbb{A}^1).$$

This has $\mathbb{G}_m$-equivariant structure given by acting on the equivalence class of a triple (e, p, t) as

$$\alpha.(e, p, t) = (e, \lambda(\alpha)p, \alpha t),$$

so descends to a P-torsor on $X \times [\mathbb{A}^1/\mathbb{G}_m]$. Extending scalars to G gives a G-torsor on $X \times [\mathbb{A}^1/\mathbb{G}_m]$. The converse is the content of the following lemma.

Lemma 7.10 ([Hei17]). *Recall that k is algebraically closed and that G is quasi-split over X with X-Borel and maximal torus (T, B) containing maximal split torus S. We have an isomorphism*

$$\mathrm{Filt}_k(\mathrm{Bun}_G) = \bigsqcup_{\lambda \in X_\star(S)/W(S)} \mathrm{Bun}_{P_\lambda}.$$

Proof. The argument is as in [Hei17, Prop. 1.13, Lemma 3.9], with the simplification of using our global maximal torus to extend from generic cocharacters to cocharacters over X. We recall the core ideas. The restriction to $[0/\mathbb{G}_m]$ gives a G_R-torsor $\mathscr{E}_0$ on $X \times R$ with a $\mathbb{G}_m$-action over $X \times R$:

$$\lambda : \mathscr{E}_0 \times \mathbb{G}_m \to \mathscr{E}_0.$$

Such an action induces a section $\lambda' \in \mathrm{Hom}_{X\times R}(\mathbb{G}_{m,X\times R}, \mathrm{Aut}_G(\mathscr{E}_0))$, whose conjugacy class is locally constant on $X \times R$ by Lemma 2.12 above. Since X is geometrically connected, $\pi_0(X \times R) \cong \pi_0(R)$ and the conjugacy class is locally constant over R.

The group $G' = \mathrm{Aut}_G(\mathscr{E}_0)$ is an inner form of G over X, and the cocharacter $\lambda^{\mathscr{E}_0}$ determines an X-parabolic subgroup $P'_\lambda \subset G'$. The scheme $\mathscr{E}' = \mathrm{Isom}_G(\mathscr{E}, (\mathscr{E}_0)_{X\times R\times[\mathbb{A}^1/\mathbb{G}_m]})$ is a G'-torsor on $X \times R \times [\mathbb{A}^1/\mathbb{G}_m]$ and has a canonical section (the identity) over $X \times R \times [0/\mathbb{G}_m]$. This section induces a canonical P'_λ-reduction of $\mathscr{E}'$ over the same locus, i.e., a section

$$s : X \times R \times [0/\mathbb{G}_m] \to \mathscr{E}'/P'_\lambda$$

mapping to the identity coset.

Since $\pi : \mathscr{E}'/P'_\lambda \to X \times R \times [\mathbb{A}^1/\mathbb{G}_m]$ is smooth, its relative tangent complex T_π is a vector bundle in degree 0, dual to the relative cotangent complex. The pullback $s^\star T_\pi$ is a vector bundle on $X \times R \times [0/\mathbb{G}_m]$, i.e., a family of $\mathbb{G}_m$-representations. The weights of these representations are locally constant over $X \times R$. The deformation theory for lifting s equivariantly along $k[x]/(x^{m+1}) \to k[x]/(x^m)$ is controlled by

$$R\Gamma(B\mathbb{G}_m, s^\star T_\pi \otimes I_m),$$

where $I_m = (x^m)/(x^{m+1})$ is a one-dimensional representation with weight $-m$. Since $\mathbb{G}_m$ is linearly reductive in all characteristics, this complex has no higher cohomology, so there is no obstruction to lifting, and the $X \times R$-scheme parameterizing the fiberwise set of lifts is a torsor for

$$H^0(B\mathbb{G}_m, s^\star T_\pi \otimes I_m) = (s^\star T_\pi \otimes I_m)^{\mathbb{G}_m}.$$

But this vanishes: the identity point in the flag variety has tangent space $\mathrm{Lie}(G')/\mathrm{Lie}(P'_\lambda)$, which has only negative weights, and so the tensor product has no $\mathbb{G}_m$-invariants and the section lifts uniquely. Intuitively, the point eP'_λ is purely repelling for the $\mathbb{G}_m$-action, and so the only $\mathbb{G}_m$-invariant tangent vector is the

0-vector. We conclude that we can find a unique compatible family of sections of $\mathscr{E}'/P'_\lambda$ over Artinian neighborhoods. This family algebraizes by the results of [AHR20] to an $X \times R$-family of sections

$$X \times R \times [\mathbb{A}^1/\mathbb{G}_m] \to \mathscr{E}'/P'_\lambda,$$

that is, a P'_λ-reduction $\mathscr{E}'_\lambda$ of $\mathscr{E}'$. Bialynicki–Birula implies that the λ-twisted action of $\mathbb{G}_m$ on $\mathscr{E}'_\lambda$ extends to a map $\mathbb{A}^1 \times \mathscr{E}'_\lambda \to \mathscr{E}'_\lambda$, and so induces an isomorphism of P'_λ-torsors

$$\mathrm{Rees}(\mathscr{E}'_\lambda|_1, \lambda') \to \mathscr{E}'_\lambda.$$

In order to give the decomposition as above, we need to transfer the cocharacter data λ' and the parabolic reduction $\mathscr{E}'_\lambda$ from the inner form $G' = \mathrm{Aut}_G(\mathscr{E}_0)$ to G. As in [Kot84, Lemma 1.1.3], since G is quasi-split there is a well-defined injective transfer of conjugacy classes of cocharacters

$$X_\star(G^{\mathscr{E}_0})/G^{\mathscr{E}_0}(F) \hookrightarrow X_\star(G)/G(F)$$

given by lifting the conjugacy class to F^s and transferring along an F^s-isomorphism $G^{\mathscr{E}_0}|_{F^s} \cong G|_{F^s}$. The result is a Galois-stable conjugacy class of cocharacters. Conjugating them to lie within fixed maximal torus T with maximal split subtorus S, the isomorphism $X_\star(S)/W(S) \cong (X_\star(T)/W(T))^{\mathrm{Gal}}$ furnishes a lift to an F-conjugacy class in $X_\star(G)$, locally constant in the R-direction. Since we are working over an algebraically closed field, G and G' are isomorphic over a Zariski-open in X, and so by the valuative criterion we can transfer the parabolic $P'_\lambda \subset G'$ to a parabolic $P_\lambda \subset G$ over all of X, and the reduction $\mathscr{E}'_\lambda$ transfers to a P_λ-reduction $\mathscr{E}_\lambda$ of $\mathscr{E}$.[1] □

Proposition 7.11. *Let R a $\mathscr{A}$-scheme. Then* $\mathrm{Filt}_{\mathsf{A}}(\mathsf{M})$ *classifies tuples* $(m, \lambda, \mathscr{E}_\lambda)$*, where*

- $m = (\mathscr{E}, \theta, t) \in \mathsf{M}$,
- $\lambda \in X_\star(S)$*, defining a parabolic* $P_\lambda \supset T$*, and*
- $\mathscr{E}_\lambda$ *a* P_λ*-reduction of* m *(in trivial relative position, i.e., an* (e, P_λ)*-reduction).*

Proof. Since the map $\mathscr{M} \to \mathrm{Bun}_G$ is affine and representable, [Hal22, Prop. 1.3.2] implies that $\mathrm{Filt}_k(\mathscr{M}) \hookrightarrow \mathrm{Filt}_k(\mathrm{Bun}_G) \times_{\mathrm{Bun}_G} \mathscr{M}$ is a closed immersion. Its image consists of the set of parabolic reductions compatible with the Higgs field, as in [AHH23, §6.3], which splits as a disjoint union across $\lambda \in X_\star(S)/W(S)$ by the previous Lemma. We have a formal 2-Cartesian diagram of mapping stacks

$$\begin{tikzcd} \mathrm{Filt}_{\mathscr{A}}(\mathscr{M}) \arrow[r] \arrow[d] & \mathscr{A} \arrow[d] \\ \mathrm{Filt}_k(\mathscr{M}) \arrow[r] & \mathrm{Filt}_k(\mathscr{A}) \end{tikzcd},$$

where the right vertical map corresponds to the trivial $\mathscr{A}$-family $\pi_2 : \Theta_{\mathscr{A}} \to \mathscr{A}$. Since $\mathscr{A} \to k$ is representable, it is easy to see $\mathrm{Filt}_k(\mathscr{A}) \cong \mathscr{A}$ along the "evaluation at 1" map, a one-sided inverse to π_2, so we see that $\mathrm{Filt}_{\mathscr{A}}(\mathscr{M}) \cong \mathrm{Filt}_k(\mathscr{M})$, where the latter is an $\mathscr{A}$-scheme along the map $m \mapsto f(m|_1)$. Compatibility with base change [Hal22, Lemma 1.1.1] implies $\mathrm{Filt}_{\mathsf{A}}(\mathsf{M}) \cong \mathrm{Filt}_{\mathscr{A}}(\mathscr{M}) \times_{\mathscr{A}} \mathsf{A}$. In particular, the disjoint decomposition across $\lambda \in X_\star(S)/W(S)$ rigidifies over A into a decomposition over the relative position (w, P_λ) between t and $\theta|_{\mathscr{E}_{P_\lambda}}$ as in Definition 4.7: the class $[(w, P_\lambda)]$ is a locally constant class on $\mathrm{Filt}_{\mathsf{A}}(\mathsf{M})$, which we can always present as (e, P') since we are over the compatible locus. That is, $\mathrm{Filt}_{\mathsf{A}}(\mathsf{M})$ decomposes first over the conjugacy class $[\lambda]$, then over semistandard P conjugate to P_λ. Choosing the standard $P_\lambda \supset B$ as base-point, the set of semistandard conjugates $P' \sim P_\lambda$ is canonically in bijection with the Weyl conjugacy class of λ, so

$$\mathrm{Filt}_{\mathsf{A}}(\mathsf{M}) = \bigsqcup_{\lambda \in X_\star(S)} \mathrm{Filt}^\lambda_{\mathsf{A}}(\mathsf{M}),$$

where $\mathrm{Filt}^\lambda_{\mathsf{A}}(\mathsf{M})$ parameterizes $(m, \mathscr{E}_\lambda)$ consisting of a Hitchin triple and a P_λ-reduction. This is the stated decomposition. □

[1] Recall we have assumed a maximal torus T exists globally, so a maximal split torus does likewise and the cocharacters extend from the generic point to all of X. Note the contrast with the case of G a parahoric group scheme, as in *loc. cit.*, where a maximal split torus may only exist generically.

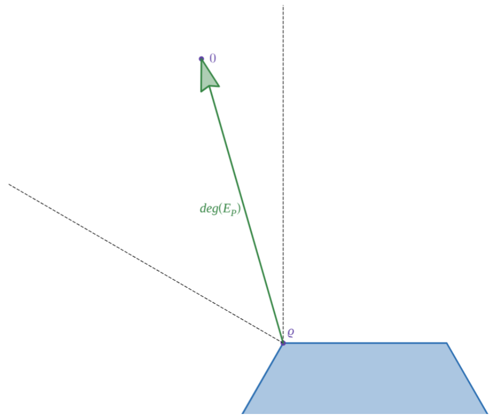

FIGURE 2. The Harder–Narasimhan ray from a Behrend complementary polygon for SL_3

7.3. Maximally destabilizing filtrations.

In this section we define a maximally ξ-destabilizing filtered object associated to each $m \in (\mathscr{E}, \theta, t)$ and prove that its parabolic reduction is indeed the ξ-canonical parabolic of Definition 7.6.

To set notation and outline this proof first in a simpler setting, we first recall the corresponding construction for Bun_G, cf. [Hei17]. For each cocharacter $\lambda \in X_\star(P)$, there is a dual character under the inner products $(-,-)_{\mathfrak{a}_M \times \mathfrak{a}_M}$ defined above:

$$\chi_\lambda := \sum_{\alpha \in \Phi \setminus \Phi_M} \langle \lambda, \alpha \rangle \alpha \in X^\star(P).$$

DEFINITION 7.12. Given a filtered G-bundle $(\lambda, \mathscr{E}_{P_\lambda}) \in \mathrm{Filt}(\mathrm{Bun}_G)$, define its *destabilizing weight* to be

$$\mathrm{wt}_{\mathscr{E}_{P_\lambda}}(\lambda) := \frac{(\deg(\mathscr{E}_{P_\lambda}), \lambda)}{\|\lambda\|} = \frac{\deg(\mathscr{E}_{P_\lambda})(\chi_\lambda)}{\|\lambda\|}.$$

Define the *maximally destabilizing filtration* of $\mathscr{E} \in \mathrm{Bun}_G$ to be the filtered object $(\lambda, \mathscr{E}_\lambda)$ living over $\mathscr{E}$ which maximizes this function.

It is shown in *loc. cit.* that this weight is bounded and maximized by a unique filtration. There also a more conceptual definition for the destabilizing weight is described, by pulling back the determinant line bundle on Bun_G along $\Theta \to \mathrm{Bun}_G$ and applying the classification of line bundles on Θ.

Indeed, the following proposition confirms that the maximally destabilizing filtration gives a construction of the usual Harder–Narasimhan parabolic reduction, in the sense that it is the reduction to the largest parabolic under containment among those $\mathscr{E}_P$ which maximize the instability degree $\deg_i(\mathscr{E}_P) := \deg(\mathscr{E}_P)(\det_P)$ (this is [Beh91, Thm. 6.4.4]), or equivalently that it is the unique parabolic reduction with positive numerical invariants and semistable associated graded ([Beh91, Def. 6.2.6]). We can moreover recover the maximally destabilizing cocharacter λ as the ray $0 - \varrho$. This equivalence is well-known and implicit in [Hei17], [AHH23], but we couldn't find a proof in the literature. We offer a proof which will translate immediately to the case of the Hitchin moduli space below.

PROPOSITION 7.13. *A filtered bundle $(\lambda, \mathscr{E}_{P_\lambda})$ over $\mathscr{E} \in \mathrm{Bun}_G$ maximizes $\mathrm{wt}_{\mathscr{E}_{P_\lambda}}(\lambda)$ if and only if $\mathscr{E}_{P_\lambda}$ is the Harder–Narasimhan reduction of $\mathscr{E}$ and λ generates the ray $0 - \varrho$, where ϱ is the point closest to the origin in the Behrend complementary convex corresponding to a generic T-reduction of $\mathscr{E}_{P_\lambda}$*

PROOF. Writing $\delta_P := \det(\mathrm{Ad}\,|_{\mathrm{Lie}(P)}) = \det(\mathrm{Ad}\,|_{\mathrm{Lie}(U_P)}) \in X^\star(P)$, we have by definition

$$\deg_i(\mathscr{E}_P) = \deg(\mathscr{E}_P)(\delta_P) = \sum_{\alpha \in \Phi} \deg(\mathscr{E}_P)(\alpha).$$

Note the comparison to the definition of the destabilizing weight: up to a factor of 2, they compute the same sum $\deg(\mathscr{E}_P)(\alpha)$ over relative roots α which do not lie in $\mathrm{Lie}(M)$, but the destabilizing weight does so with multiplicities which are allowed to range as λ runs through a facet of the Weyl fan.

The classical Bruhat decomposition applied to parabolic subgroups of $G_\eta^{\mathscr{E}}$ (cf. [SGAIII, Exp. XX, Lemme 4.1.1]) implies that any two parabolic reductions share a common reduction to a maximal torus, $\mathscr{E}_{T',\eta}$, over the generic point. Given such a generic T'-reduction, we obtain a P-reduction $\mathscr{E}_{T',\eta} \times^{T'} P$ for any $P \supset T'$: first generically and then globally by properness. The intersection of the cones $-\deg(\mathscr{E}_P) - {}^+\mathfrak{a}_P + (\mathfrak{a}_T^M)$, equal to the convex hull $\mathrm{cvx}\{-\deg(\mathscr{E}_B)\}$ defines a relative complementary convex $C_{\mathscr{E}_{T',\eta}}$ as in Definition 5.7.

Applying this construction to the Harder–Narasimhan reduction and the maximally destabilizing reduction (which share a torus-reduction generically), we reduce to checking that among the finite set of parabolic reductions induced by that generic torus reduction, the H–N parabolic and the maximally destabilizing parabolic coincide.

Write ϱ for the unique point in $C_{\mathscr{E}_{T',\eta}}$ closest to 0. Then [Beh91, Prop. 5.3.14] proves that the Harder–Narasimhan parabolic is the unique parabolic such that the ray from ϱ to 0 lies in the facet of the Weyl fan centered at ϱ corresponding to $P \supset T$. On the other hand, the maximally destabilizing filtration is that which maximizes the function

$$\mathrm{wt}_{\mathscr{E}_P}(\lambda) = \frac{(\deg(\mathscr{E}_P), \lambda)}{\|\lambda\|}$$

over the finite collection of $\mathscr{E}_P$ and over $\lambda \in \mathfrak{a}_P$ such that $P_\lambda = P$, i.e., over λ which lie in the P-facet of the Weyl fan. We will show that the Harder–Narasimhan parabolic satisfies this latter condition.

Assume P is the Harder–Narasimhan parabolic. Then the above characterization says exactly that $\lambda = \deg(\mathscr{E}_P)$ satisfies $P_\lambda = P$, and it clearly maximizes the functional within that facet, as

$$\mathrm{wt}_{\mathscr{E}_P}(\lambda) = \|\deg(\mathscr{E}_P)\|.$$

It suffices now to show that no other facet contains the unique global maximum, i.e., for any other parabolic Q and any $\lambda \in \mathfrak{a}_Q^+$, there exists $Q', \lambda' \in \mathfrak{a}_{Q'}^+$ such that

$$\mathrm{wt}_{\mathscr{E}_Q}(\lambda) < \mathrm{wt}_{\mathscr{E}_{Q'}}(\lambda').$$

Given our fixed Behrend convex, we can decompose all of $\mathfrak{a}_T$ into regions using conditions (1), (2) of Prop. 7.5. That is, $\mathfrak{a}_T = \bigsqcup_{Q \supset T} C_Q^+$, where C_Q^+ consists of i.e., "outward dominant facet for Q," the set of points ξ such that

1. the closest point ϱ in the closed convex to ξ lies in the affine space $-\deg(\mathscr{E}_Q) + \mathfrak{a}_T^Q + \mathfrak{a}_G$, and
2. $\xi - \varrho$ lies in the facet of the Weyl fan corresponding to Q. That is, $\alpha(\xi - \varrho) > 0$ for all $\alpha \in \Delta_Q$ and $\alpha(\xi - \varrho) = 0$ for all $\alpha \in \Phi_M$, where M is the semistandard Levi factor of Q.

Note the H–N parabolic P is characterized as the unique which *does* satisfy these conditions for the point $\xi = 0$. Moreover, if Q is a parabolic such that

$$0 - (-\deg(\mathscr{E}_Q)) = \deg(\mathscr{E}_Q) \in \mathfrak{a}_Q$$

is in the Weyl facet $\mathfrak{a}_Q^+$, then both (1), (2) hold for that Q and so $Q = P$: (1) holds because, by shifting Lemma 7.14 below, the point $-\deg(\mathscr{E}_Q)$ is the point closest to 0 in the cone $-\deg(\mathscr{E}_Q) - {}^+\mathfrak{a}_Q$, and the Behrend convex is the intersection of such cones over all parabolics. Then once (1) is known, (2) is our assumption. In summary, the H–N parabolic P is the unique parabolic such that $0 + \deg(\mathscr{E}_P) \in \mathfrak{a}_Q^+$.

For any $Q \neq P$, then, defining simple positive roots Δ_Q relative to some $B \subset Q$, one of the following must be true.

(a) there exists $\alpha \in \Delta_Q$ such that $\alpha(\deg(\mathscr{E}_Q)) = \alpha(0 - \varrho) < 0$,
(b) there exists $\alpha \in \Delta_Q$ such that $\alpha(\deg(\mathscr{E}_Q)) = \alpha(0 - \varrho) = 0$, or
(c) there exists $\alpha \in \Phi_M$ such that $\alpha(0 - \varrho) > 0$ (if negative, replace α with $-\alpha$).

If we are in case (a), then let Q' be the adjacent parabolic such that $-\alpha \in \Delta_{Q'}$ and $\Delta_{Q'} \setminus \{-\alpha\} = \Delta_Q \setminus \{\alpha\}$. Consider the Weyl reflection

$$\lambda' = s_\alpha(\lambda) = \lambda - 2\langle \lambda, \alpha \rangle \alpha^\vee.$$

Then $\|\lambda'\| = \|\lambda\|$, the difference lies in the span of $\alpha^\vee$, and we claim that

$$(\deg(\mathscr{E}_Q), \lambda) = \deg(\mathscr{E}_Q)(\chi_\lambda) < \deg(\mathscr{E}_{Q'})(\chi_{\overline{\lambda}}) = (\deg(\mathscr{E}_{Q'}), \overline{\lambda}),$$

viewed either in terms of the inner product on $\mathfrak{a}_M$ or the pairing between $\mathfrak{a}_M$ and $\mathfrak{a}_M^\star$.

As in Proposition 5.12, cf. condition (B2) in the definition of Behrend's convex [Beh91], we have

$$\deg(\mathscr{E}_{Q'}) - \deg(\mathscr{E}_Q) \in \mathbb{R}_{\geq 0}\alpha^\vee.$$

As in the proof, we can reduce the condition to semisimple rank 1: writing M for the Levi of Q and Q' and $L \supset M$ for the Levi of the minimal parabolic containing both, then $\dim \mathfrak{a}_M^L = 1$ and

$$\mathfrak{a}_M^G = \mathfrak{a}_M^L \oplus \mathfrak{a}_L^G$$

is an orthogonal decomposition with respect to the inner products we have defined on each. Since λ and λ' (resp. $\deg(\mathscr{E}_Q)$ and $\deg(\mathscr{E}_{Q'})$ agree in their $\mathfrak{a}_L^G$-components, we reduce to checking the inequality on the line $\mathfrak{a}_M^L$, on which λ points in the same direction as $\alpha^\vee$, but the opposite direction to $\deg(\mathscr{E}_Q)$ by construction.

If we are in case (b), let $Q' \supset Q$ be the minimal parabolic containing Q whose Levi $M' \supset M$ contains $\pm\alpha$, and let

$$\lambda' = \lambda - \langle \lambda, \alpha \rangle \alpha^\vee$$

be the projection of λ into the positive space for Q', whence $\|\lambda'\| < \|\lambda\|$. Degree is compatible with extension of scalars, in the sense that $\deg(\mathscr{E}_Q \times^Q Q')$ is the orthogonal projection onto $\mathfrak{a}_{M'}$ of $\deg(\mathscr{E}_Q) \in \mathfrak{a}_M$. We therefore have

$$(\deg(\mathscr{E}_Q), \lambda) = (\deg(\mathscr{E}_{Q'}), \lambda'),$$

but the denominator of the weight function is smaller for (Q', λ'), so the overall function is larger.

If we are in case (c), then we shrink to the largest smaller parabolic $Q' \subset Q$ whose Lie algebra contains α but not $-\alpha$. Define $\lambda' = \lambda + \epsilon\alpha^\vee$ for ϵ a small positive value to be specified later. Then $\|\lambda'\| = \sqrt{\|\lambda\|^2 + \epsilon^2\|\alpha^\vee\|^2}$, in particular, $\frac{d}{d\epsilon}|_{\epsilon=0}\|\lambda'\| = 0$. On the other hand, an orthogonal decomposition as in the previous cases shows that $(\deg(\mathscr{E}_{Q'}), \lambda')$ is larger than $(\deg(\mathscr{E}_Q), \lambda)$ by a factor growing linearly in ϵ. For ϵ a sufficiently small positive value, the normalized weight is therefore larger. □

Lemma 7.14. *Let $P \supset T$ be a parabolic. For any $x \in \mathfrak{a}_P^+$, the point 0 is the point in the closure of the negative obtuse cone $-\overline{{}^+\mathfrak{a}_P}$ closest to x.*

Proof. Write $\{\omega_i\} = \widehat{\Delta}_P^\vee$ for the image of the fundamental coweights, the basis for $\mathfrak{a}_P$ dual to Δ_P. On the other hand we have $\{\alpha_j^\vee\} = \Delta_P^\vee$, the basis of simple positive coroots for P, dual to the fundamental weights. By definition the narrow cone is

$$\mathfrak{a}_P^+ = \mathbb{R}_{>0}\langle \widehat{\Delta}_P^\vee \rangle,$$

and the closure of the negative obtuse cone is

$$-\overline{{}^+\mathfrak{a}_P} = \mathbb{R}_{\leq 0}\langle \Delta_P^\vee \rangle.$$

It follows from our definition of the inner product that $(\omega_i, \alpha_j^\vee) = 0$ if $i \neq j$, and $(\omega_i, \alpha_i^\vee) > 0$. In particular, we can see from this that for any $x \in \mathfrak{a}_P^+$ and any $y \in -\overline{{}^+\mathfrak{a}_P}$, the inner product satisfies $(x, y) \leq 0$, with equality only if $y = 0$. In particular, then

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2(x, y) \geq \|x\|^2,$$

with equality only if $y = 0$. So $y = 0$ uniquely minimizes the distance. □

Example 7.15. An example of case (a) of the argument above for SL_3 is illustrated in Figure 3. The corner of the convex ϱ is $-\deg(\mathscr{E}_B) - {}^+\mathfrak{a}_B$ for some $B \supset T$, and the narrow cone outlined is the dominant chamber for $P = B$, the Harder–Narasimhan parabolic. Any ray from ϱ in that chamber satisfies $B = B_\lambda$. Since $\deg(\mathscr{E}_B)$ lies in that chamber, we can take $\lambda = \deg(\mathscr{E}_B)$ (which is an integral vector) to optimize the destabilizing weight over valid λ for the reduction $\mathscr{E}_B$. On the other hand, given any λ in the chamber for Q, we have

$$(\deg(\mathscr{E}_Q), \lambda) < (\deg(\mathscr{E}_P), \lambda').$$

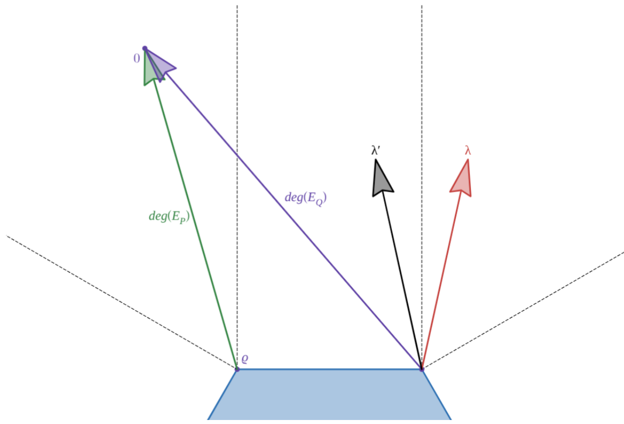

Figure 3. A non-H–N chamber cannot maximize the weight, case (a).

The reduction in the proof to the case of rank 1 here reflects that only the x-components in the inner products differ, and in that component the inner product for P is positive while the inner product for Q is negative.

Now we make the corresponding ξ-weighted definitions for Hitchin triples.

Definition 7.16. Given a filtered (compatible) Hitchin triple $(m, \lambda, \mathscr{E}_\lambda) \in \mathrm{Filt}_{\mathsf{A}}(\mathsf{M})$, we define its ξ-*destabilizing weight* to be

$$\mathrm{wt}^{\xi}_{\mathscr{E}_{P_\lambda}}(\lambda) := \frac{(\deg(\mathscr{E}_{P_\lambda}) - \xi, \lambda)}{\|\lambda\|}.$$

Define the ξ-*maximally destabilizing filtration* of $m \in \mathsf{M}$ to be the filtered object $(m, \lambda, \mathscr{E}_\lambda)$ living over m which maximizes this function.

Since a filtration of a Hitchin triple is in particular a filtration of its underlying G-torsor, it is clear from the case of Bun_G that this function is bounded. Here too, we can check that the ξ-canonical and ξ-maximally destabilizing parabolic coincide.

Lemma 7.17. *Let $(\lambda, \mathscr{E}_\lambda)$ be the ξ-maximally destabilizing filtration of m. Then $\mathscr{E}_\lambda$ is the ξ-canonical parabolic reduction (Definition 7.6), and λ generates the ray $\xi - \varrho$, where ϱ is the ξ-Harder–Narasimhan point of m (Definition 7.2).*

Proof. The proof proceeds as in Prop. 7.13, replacing the role of 0 with ξ. Note however that the stability convex of Chaudouard–Laumon is not, strictly speaking, a Behrend convex unless $f(m)$ is T-elliptic. For a general m with L-elliptic $f(m)$, the stability convex, which naturally lives in $\mathfrak{a}_L^G$, has faces which range only over parabolic reductions for $P \supset L$. These are, after all, the only parabolic reductions possible compatible with the Higgs field. However, the necessary properties of Behrend's convex hold also in our setting by Propositions 5.12 and 7.5. In particular, it is still the case, by the same argument, that only the ξ-canonical parabolic has $\xi + \deg(\mathscr{E}_P) \in \mathfrak{a}_P^+$, and for any $Q \neq P$ the same modifications $(Q, \lambda) \rightsquigarrow (Q', \lambda)$ show that no parabolic Q except the ξ-canonical parabolic can maximize the ξ-weight. Note in particular that all of our constructions, interpreted in $\mathfrak{a}_L^G$, remain within the set of parabolics containing L. So the two canonical parabolics must coincide. Moreover, λ on the ray $\xi - \varrho$ clearly optimizes the ξ-weight within the ξ-canonical facet. $\square$

7.4. Θ-stratification and semistable reduction.

Recall the notion of Θ-stratification introduced by Halpern-Leistner.

Definition 7.18. Let $\mathscr{X}$ an algebraic stack locally of finite type over Noetherian algebraic space B. Recall that restriction to 1 defines a morphism

$$\mathrm{ev}_1 : \mathrm{Filt}(\mathscr{X}) \to \mathscr{X}.$$

Likewise, restriction along $B\mathbb{G}_m = [0/\mathbb{G}_m] \to \Theta$ defines a map to "graded objects"

$$\mathrm{gr} : \mathrm{Filt}(\mathscr{X}) \to \mathrm{Gr}(\mathscr{X}) := \underline{\mathrm{Map}}_S(B\mathbb{G}_m, \mathscr{X}).$$

We then define the notion of a Θ-stratification.

(1) A Θ-*stratum* in $\mathscr{X}$ consists of a union of connected components $\mathscr{Z}^+ \subset \mathrm{Filt}(\mathscr{X})$ such that $\mathrm{ev}_1 : \mathscr{Z}^+ \to \mathscr{X}$ is a closed immersion.

(2) A *well-ordered* Θ-*stratification* indexed by a totally ordered set I with minimal element 0 is a cover of $\mathscr{X}$ by open substacks $\mathscr{X}_{\leq i}$ for $i \in I$ such that $\mathscr{X}_{\leq i} \subset \mathscr{X}_{\leq i'}$ for $i < i'$, along with a Θ-stratum $\mathscr{Z}_i^+ \subset \mathrm{Filt}(\mathscr{X}_{\leq i})$ whose complement is the union of proper smaller strata. We require that for every $x \in |\mathscr{X}|$,
   - the set $\{i \in I : x \in \mathscr{X}_{\leq i}\}$ has a minimal element, and
   - the (totally ordered) set $\{i \in I : \mathscr{X}_{\leq i} \cap \overline{\{x\}} \neq \emptyset\}$ is well-ordered.

In [AHH23, §8], the authors show that canonical (that is, HN) reductions of G-bundles define a Θ-stratification on Bun_G. We now collect the properties proven in this section to show that the ξ-canonical reduction defines a Θ-stratification of M over A.

Proposition 7.19. *For arbitrary ξ, the ξ-canonical reductions of Definition 7.6 define a Θ-stratification of $f : \mathsf{M} \to \mathsf{A}$.*

Proof. We follow the strategy applied to Bun_G in [AHH23, §8]. Let $\mathscr{S} \subset \mathrm{Irred}(\mathrm{Filt}_{\mathsf{A}}(\mathsf{M}))$ the set of irreducible components whose connected component contains a ξ-HN filtration, and define $\mu : \mathscr{S} \to \mathbb{R}$ by

$$\mu : (m, \lambda, \mathscr{E}_\lambda) \mapsto \mathrm{wt}^{\xi}_{\mathscr{E}_{P_\lambda}}(\lambda).$$

Since the degree of a reduction to a fixed parabolic is locally constant, this μ extends by 0 to a locally constant function on $\mathrm{Filt}_{\mathsf{A}}(\mathsf{M})$.[2]

By the criteria of [Hal22], Theorem 2.2.2 and the simplifications which follow, it suffices to show the data $(\mathscr{S}, \mu)$ satisfy the following.

(1) Existence and uniqueness of the canonical reduction: for any finite type unstable point $m \in \mathsf{M}(K)$, there is a unique filtration. This is immediate from the well-definedness of the polyhedral convex $\overline{C}_m$ and Proposition 7.5.

(2) Specialization of canonical reductions: for any family over a DVR, the degree of instability is upper-semicontinuous and if equality holds then the generic canonical reduction extends to the special fiber. The semicontinuity is Proposition 5.16 and the extension is Proposition 7.8.

(3) Openness: if we have a parabolic reduction defined everywhere over a DVR (essentially of finite type) whose special point is canonical, then its generic point is canonical also.

   Proof. Say the reduction of $m \in \mathsf{M}(R)$ in question is to semistandard parabolic P with Levi factor M. Write s for the special point and η for the generic point of R. By local constancy of the degree of a P-reduction, we can see that $\deg(\mathscr{E}_{P,s}) = \deg(\mathscr{E}_{P,\eta})$ as elements of $\mathfrak{a}_M$; that is, the cones $C_{m_{P_\eta}}$ and $C_{m_{P_s}}$ are equal. From this, we can see that the ξ-HN points of the two fibers $\varrho_s \in \overline{C}_{m_s}, \varrho_\eta \in \overline{C}_{m_\eta}$ satisfy $\varrho_s - \varrho_\eta \in (\mathfrak{a}_T^P) + \mathfrak{a}_G$, and so by Proposition 7.5, P is also the canonical parabolic on the generic fiber (and m_{P_η} the unique P-reduction of m_η). □

[2]Two points x, y in the same connected (resp. irreducible) component of a locally Noetherian algebraic stack with quasi-compact diagonal can be connected by a scheme-indexed (resp. integral scheme-indexed) family: realize a chain of generizations and specializations connecting the two points (resp. a single roof $x \leftsquigarrow \eta \rightsquigarrow y$) with a sequence of maps from valuation rings, then glue along open points and pushout along closed points.

(4) Local finiteness: for any family of Hitchin triples over a finite-type scheme S, the canonical filtrations for $s \in S$ land within finitely many connected components of $\underline{\mathrm{Map}}(\Theta, \mathsf{M})$. By constructibility/Noetherian induction as in Proposition 5.16, the convex $\overline{C}_{m_s}$ will take on only finitely many values over S, so in particular, the canonical parabolics and their numerical invariants will take on only finitely many values.
(5) Consistency: if $f : \Theta \to \mathsf{M}$ is the ξ-canonical reduction of $f|_1$, then it is also the canonical reduction of the associated graded $f|_0$. This follows immediately from the characterization of Proposition 7.5, which is equivalent to the ξ-maximally destabilizing parabolic by Lemma 7.17.

$\square$

In Section 10, this will allow us to use a powerful semistable reduction theorem for stacks equipped with a Θ-stratification in order to prove existence in the valuative criterion.

CHAPTER 8

The ξ-stable Higgs moduli stack is of finite type

In this section, we prove finite-typeness of $\widetilde{\mathscr{M}}^{\xi}$ for G/X a quasisplit reductive group with anisotropic center, $A_G^{\mathrm{spl}} = 1$. The key new result outside the split case is a the following lemma guaranteeing the existence of Borel reductions of bounded stability for $\mathscr{E}$-torsors. In the split case it is due to Harder (cf. [Har68], [Har69]).

LEMMA 8.1 (cf. [CL10, Prop. 6.1.6]). *Let $B \supset T$ be a maximal Borel-torus pair inside quasisplit reductive G. There exists a constant $c > 0$ depending only on G such that every G-torsor $\mathscr{E}$ on X admits a Borel reduction $\mathscr{E}_B$ whose degree lies in the corresponding shift of the relative dominant chamber, i.e.*

$$\mathscr{C}(c) = \{H \in \mathfrak{a}_T : \alpha(H) \geq -c \ \forall \alpha \in \Delta_B\}.$$

PROOF. Assume first that G is semisimple adjoint. The condition on the degree states that the "first numerical invariants" $n_i := \deg(\mathscr{E}_B)(\alpha_i) \geq -c$ for all simple roots α_i. We imitate the proof in Harder [Har68, Satz 2.2.6]. In particular, we need as base case the result for groups of split rank 1. It is immediate that the inequalities $\deg(\mathscr{E}_B)(\alpha_i) > -c$ are independent of central isogeny, so we assume G is adjoint. Indeed this suffices: given a central isogeny $G_0 \to G$ sending Borel $B_0 \mapsto B$ and $\mathscr{E}_0$ a G_0-torsor, we have $\mathscr{E}_0/B_0 \cong (\mathscr{E}_0 \times^{G_0} G)/B$, so the Borel reductions found in the adjoint case imply the existence of corresponding Borel reductions for isogeneous G_0.

By [Con14, Prop. 6.4.4], such a G is necessarily of the form $\mathrm{Res}_{X'/X} G'$ for some $X' \to X$ finite étale and some $G' \to X'$ adjoint absolutely simple and quasisplit. By [Tit59], there are only two such possibilities for such a G' with G of split rank 1: adjoint type ${}^1A_1$ (that is, split PGL_2) or ${}^2A_2$ (that is, quasisplit PU_3). Moreover, in order that G still have split rank 1, the étale cover X'/X must be connected. Moreover, since a $\mathrm{Res}_{X'/X} G'$ torsor on X is simply the same data as a G'-torsor on X', Borel reductions correspond, and the degrees are equal up to a factor of $\deg X'/X$, the problem reduces to the absolutely simple case.

That is, we have reduced to 2 base cases: $\mathrm{PGL}_{2,X}$, and $\mathrm{PU}_{3,X'/X}$ associated to a double cover. The former is done by Harder, and we recall the argument. A rank 2 vector bundle $\mathscr{E}$ can be twisted by a line bundle $\mathscr{O}(p)$ until it has degree $2g-1$ or $2g$ (recall that k is algebraically closed, so that points of degree 1 exist on X, and each twist increases the degree of $\mathscr{E}$ by 2). The simple form of Riemann–Roch then applies, and $\mathscr{E}$ has a nonzero global section. The closure of the image of this section provides a sub-line bundle $\mathscr{L} \subset \mathscr{E}$ which admits a global section and therefore is of nonnegative degree. This flag is a B-reduction such that

$$\deg(\mathscr{E}_B) = \deg(\mathscr{L}) - \deg(\mathscr{E}/\mathscr{L}) = 2\deg(\mathscr{L}) - \deg(\mathscr{E}) \geq -\deg(\mathscr{E}) \geq -2g,$$

so we have found that a bound of $c = 2g(X')$ suffices for the degree of the pullback of $\mathscr{E}_B$ to X', where $X' \to X$ is any cover that realizes G as $\mathrm{Res}_{X'/X} G'$ for absolutely simple G'. By Riemann–Hurwitz,

$$g(X') \leq g(X)\deg \pi,$$

but the same factor of $\deg \pi$ appears when relating the degree of a $\mathrm{Res}_{X'/X} G'$-torsor to the degree of its underlying G'-torsor on X', so we obtain a uniform bound of $c = 2g(X)$.

We also need to handle the other base case of quasisplit PU_3. A PU_3-torsor is represented by a rank 3 vector bundle $\mathscr{E}$ on X' equipped with an isomorphism $h : \mathscr{E} \cong \sigma^\star(\mathscr{E}^\vee)$. A Borel reduction corresponds to an isotropic line bundle $\mathscr{L} \subset \mathscr{E}$, in the sense that its complement $\mathscr{P} = h^{-1}(\mathscr{L}^\vee)$ should satisfy $\mathscr{L} \subset \mathscr{P}$, and its degree is necessarily a triple of the form $\left(\frac{d}{2}, 0, -\frac{d}{2}\right)$ for $d = \deg(\mathscr{L})$. We seek a uniform bound c such that every $\mathscr{E}$ has a Borel reduction with $\frac{d}{2} > -c$. Write $g' = g(X')$, and fix a divisor D on X' of degree $2g' - 1$.

Then $W := H^0(X', \mathscr{E}(D))$ admits a quadratic form

$$q_D : W \to H^0(X', \mathscr{O}_{X'}(D + \sigma(D))).$$

By Riemann–Roch, we have $h^0(X', \mathscr{E}(D)) \geq 3g'$, and $h^0(X', \mathscr{O}_{X'}(D + \sigma(D))) = 3g' - 1$. The map q_D is then defined by $3g' - 1$ quadratic forms on a vector space of dimension $3g'$. Projectivizing, the isotropic locus is the intersection of $3g' - 1$ quadric hypersurfaces in $\mathbb{P}(W)$ and so is nonempty. Let $s \in W$ be a nonzero isotropic section. The closure of its image is a line subbundle $\mathscr{F} \subset \mathscr{E}(D)$ which is still isotropic, and twisting down $\mathscr{L} = \mathscr{F}(-D) \subset \mathscr{E}$ is an isotropic line subbundle. Since $\deg(\mathscr{F}) \geq 0$, we have $\deg(\mathscr{L}) > -2g'$. As above, we have $g(X') \leq 2g(X)$, and the same normalization factor of 2 appears in the degree pairing on PU(3), so the uniform bound $c = 2g(X)$ works here as well.

Now let G arbitrary semisimple adjoint. Given a G-torsor $\mathscr{E}$, define a partial order on the (nonempty) set of Borel reductions $\mathscr{E}_B$ via the partial order on $\mathbb{Z}^n$ applied to the "second numerical invariants"

$$(p_1, \dots, p_n) := (\deg(\mathscr{E}_B)(\omega_1), \dots, \deg(\mathscr{E}_B)(\omega_n)) \in \mathbb{Z}^n.$$

Here the ω_i are the fundamental weights, dual to the coroots.[1] Let $\mathscr{E}_B$ be any B-reduction which is maximal for this partial order. Then we claim that $n_i \geq c$ for all simple roots α_i for $c = 2g(X)$.

Assume it did not, so $\deg(\mathscr{E}_B)(\alpha_i) < -c$. We reduce to a split rank 1 computation as follows: let P_i be the minimal standard parabolic whose Lie algebra contains $\mathfrak{g}_{-\alpha_i}$, and write M_i for its associated Levi quotient. Write $\mathscr{E}_{P_i} = \mathscr{E}_B \times^B P_i$, and $\mathscr{E}_{M_i} = \mathscr{E}_{P_i} \times^{P_i} P_i/R_u(P_i)$ for the pushed-forward M_i-torsor. Write $\overline{B}_i$ for the image of B inside the Levi, which is a Borel subgroup.

By the third isomorphism theorem for groups, we have a natural isomorphism

$$P_i/B \xrightarrow{\sim} (P_i/R_u(P_i))/(B/R_u(P_i)) = M_i/\overline{B}_i.$$

This globalizes to a corresponding isomorphism for $\mathscr{E}_{P_i}$:

$$\mathscr{E}_{P_i}/B \xrightarrow{\sim} (\mathscr{E}_{P_i}/R_u(P_i))/(B/R_u(P_i)) = \mathscr{E}_{M_i}/\overline{B}_i.$$

Thus, B-reductions $\mathscr{F}_B$ of $\mathscr{E}_{P_i}$ are in bijection with $\overline{B}_i$-reductions $\mathscr{F}_{B,M}$ of $\mathscr{E}_{M_i}$. From the construction, it is immediate that n_i corresponds to $\deg(\mathscr{F}_{B,M})(\alpha)$, where α is the unique positive relative root of M. In fact, we can further descend everything to the adjoint semisimple form.

Since the result is already proven for semisimple groups of split rank 1 and $\mathscr{E}_{B,M} = \mathscr{E}_B \times^B B/R_u(P_i)$ by assumption fails to meet the bound in this case, we can find a different reduction $\mathscr{F}_{B,M}$ which does satisfy the bound. We can lift this to a different B-reduction $\mathscr{F}_B$ of $\mathscr{E}_{P_i}$ (and therefore of $\mathscr{E}$). The "second numerical invariants" $p_j = \deg(\mathscr{E}_B)(\omega_j)$ can be computed in terms of $\mathscr{E}_B \times^B P^j$, where P^j is the *maximal* proper parabolic whose Lie algebra contains weight spaces for the negation of all simple roots *except* α_j (cf. [Har68, §2.1]). Since $\mathscr{E}_B$ and $\mathscr{F}_B$ agree upon pushforward to $P_i \subset P^j$, it is clear then that $\deg(\mathscr{F}_B)(\omega_j) = \deg(\mathscr{E}_B)(\omega_j)$ for all $j \neq i$.

Since $p_j(\mathscr{E}_B) = p_j(\mathscr{F}_B)$ for all $j \neq i$ but $n_i(\mathscr{E}_B) < n_i(\mathscr{F}_B)$, we show it follows that $p_i(\mathscr{E}_B) < p_i(\mathscr{F}_B)$. The fundamental weights form a basis and ω_j is orthogonal to α_i, so one can write

$$\omega_i = \sum_{j \neq i} c_j \omega_j + c_i \alpha_i. \tag{8.0.1}$$

Pairing with $\alpha_i^\vee$ gives

$$1 = c_i \langle \alpha_i^\vee, \alpha_i \rangle.$$

In particular, $c_i > 0$ and so, pairing the identity (8.0.1) with $\deg(\mathscr{E}_B)$ gives

$$p_i(\mathscr{E}_B) = \sum_{j \neq i} c_j p_j(\mathscr{E}_B) + c_i n_i(\mathscr{E}_B)$$

and likewise for $\mathscr{F}_B$. All the terms are equal except that $n_i(\mathscr{E}_B) < n_i(\mathscr{F}_B)$, so $p_i(\mathscr{E}_B) < p_i(\mathscr{F}_B)$, violating the maximality assumption on $\mathscr{E}_B$ and giving the desired contradiction.

[1] There is no subtlety about non-reduced root systems here, as a non-reduced root system dualizes to a reduced coroot system, so the ω_i are a basis for the character space.

Finally, let G be arbitrary reductive. Writing $\overline{G} = G/A_G$ for its adjoint quotient and likewise for $\overline{B}, \overline{T}$, the root systems are canonically identified. Every G-torsor $\mathscr{E}$ pushes forward to a $\overline{G}$-torsor $\overline{\mathscr{E}}$, and one has $G/B \cong \overline{G}/\overline{B}$ so B-reductions can be canonically identified,

$$\mathscr{E}_B \longleftrightarrow (\mathscr{E}_B \times^B \overline{B}) =: \overline{\mathscr{E}}_{\overline{B}}.$$

For any $\alpha \in \Phi$ one has

$$\deg(\mathscr{E}_B)(\alpha) = \deg(\overline{\mathscr{E}}_{\overline{B}})(\alpha),$$

so the conclusions lift. ☐

In order to apply this result about bare G-torsors to Hitchin triples $(\mathscr{E}, \theta, t)$, we need the following purely degree-theoretic characterization of when reductions of the bundle automatically induce reductions of θ, and thus P-reductions of the Hitchin pair $(\mathscr{E}, \theta)$.

Lemma 8.2 (cf. [CL10, Lemme 6.1.7]). *Let $c > 0$ be the constant of the previous lemma. Let P a standard parabolic and $\Delta_P \subset \Delta_B$ the set of simple relative roots in N_P. There exists a $d > 0$ such that for any $(\mathscr{E}, \theta, t) \in \widetilde{\mathscr{M}}^\xi(k)$, every B-reduction $\mathscr{E}_B$ of $\mathscr{E}$ whose degree satisfies*

$$\deg(\mathscr{E}_B)(\alpha) \geq -c \;\; \forall \alpha \in \Delta_B$$

and

$$\begin{cases} \deg(\mathscr{E}_B)(\alpha) \geq d & \forall \alpha \in \Delta_P \\ \deg(\mathscr{E}_B)(\alpha) \leq d & \forall \alpha \in \Delta_B \setminus \Delta_P \end{cases} \tag{8.0.2}$$

necessarily admits a lift of θ along $\mathrm{Ad}_D(\mathscr{E}_P) = \mathscr{E}_B \times^{B,\mathrm{Ad}} \mathfrak{p}_D \hookrightarrow \mathrm{Ad}_D(\mathscr{E})$.

Proof. As at the end of the previous proof, replacing G with $\overline{G} = G/A_G$, it is easy to reduce to the semisimple case. Now let $X' \to X$ be a splitting Γ-cover with split pullback $G' \to X'$. Likewise denote all objects arising over X' by pullback with an apostrophe, e.g. $P', \mathfrak{g}'$, etc. We wish to reduce to the split case. Note that since degree $\deg(\mathscr{E}_B)$ is compatible with pullback up to a $|\Gamma|$ scalar and the rational roots are the image of the geometric roots under the canonical projection onto the rational space, the conditions of Equation (8.0.2) are equivalent after pullback, i.e., whether considered in the relative space $\underline{\mathfrak{a}}_T(X)$ or the geometric space $\underline{\mathfrak{a}}_T(X')$. Consider now the following diagram

$$\begin{tikzcd} H^0(X, \mathrm{Ad}_D(\mathscr{E}_P)) \arrow[r] \arrow[d] & H^0(X, \mathrm{Ad}_D(\mathscr{E})) \arrow[d] \\ H^0(X', \mathrm{Ad}_D(\mathscr{E}'_P)) \arrow[r] & H^0(X', \mathrm{Ad}_D(\mathscr{E}')) \end{tikzcd}$$

in which the top row is the Γ-invariants of the bottom row by descent, and both horizontal maps are injective because they are induced by the inclusion of a subbundle. The diagram is therefore Cartesian, i.e., if the pullback θ' lifts to some θ'_P, then that θ'_P descends and so θ likewise lifts to θ_P. So we reduce to the split semisimple case, with the corresponding d, which is *loc. cit.*. ☐

Lemma 8.3 (cf. [CL10, Lemme 6.1.6]). *Assume now that G has anisotropic center. Let $B \supset T$ be an X-maximal Borel-torus pair. There exists a finite subset $S \subset X_\star(T)$ such that for every $(\mathscr{E}, \theta, t) \in \widetilde{\mathscr{M}}^\xi(k)$, the torsor $\mathscr{E}$ admits a B-reduction $\mathscr{E}_B$ with $\deg(\mathscr{E}_B) \in S$.*

Proof. By the above, there exists a $c > 0$ such that every $\mathscr{E}$ admits a Borel reduction $\mathscr{E}_B$ whose degree lies in

$$\mathscr{C}(c) = \{H \in \mathfrak{a}_T : \alpha(H) \geq -c \;\; \forall \alpha \in \Delta_B\}.$$

We can cover the relative space $\underline{\mathfrak{a}}_T(X)$ with cones $\mathscr{C}_P(d)$ defined by (5) above for different choices of P, so it suffices to find a finite set S_P such that every Borel reduction whose degree lies in $\mathscr{C}(c) \cap \mathscr{C}_P(d)$ actually lies in S_P. Then $S = \bigcup_P S_P$ suffices. Since $A_G^{\mathrm{spl}} = 1$, we have $\mathfrak{a}_G = 0$, so the simple relative roots span $\mathfrak{a}_T^\star$. In particular, a subset of $\mathfrak{a}_T$ on which every $\alpha \in \Delta_B$ is bounded above and below is compact.

If $P = G$, then $\mathscr{C}(c) \cap \mathscr{C}_P(d)$ is already compact by definition, so we assume $P \neq G$. By the previous lemma, $\deg(\mathscr{E}_B) \in \mathscr{C}(c) \cap \mathscr{C}_P(d)$ implies that $\mathscr{E}_P = \mathscr{E}_B \times^B P$ is compatible with θ, so we obtain a reduction of the Hitchin pair $(\mathscr{E}_P, \theta_P)$. This may not be compatible with the original t, but it certainly gives a reduction of $(\mathscr{E}, \theta, w(t)) \in \widetilde{\mathscr{M}}^{w(\xi)}$ for some $w \in W_G(\infty)$. The definition of ξ-stability implies that $\deg(\mathscr{E}_P) \in -w(\xi) - {}^+\mathfrak{a}_P + \mathfrak{a}_T^P$. Evaluating on $\alpha \in \Delta_P$, we see that in particular we have an upper bound on $\deg(\mathscr{E}_B)(\alpha)$, and so

$$S_P = \mathscr{C}(c) \cap \mathscr{C}_P(d) \cap \bigcup_{w \in W_G(\infty)} (-w(\xi) - {}^+\mathfrak{a}_P + \mathfrak{a}_T^P)$$

bounds the desired finite set. □

THEOREM 8.4 (cf. [CL10, Proposition 6.1.5]). *If G/X is quasisplit reductive with anisotropic center, then $\widetilde{\mathscr{M}}^\xi$ and M^ξ are of finite type over k.*

PROOF. Let (B, T) a Borel-torus pair in G/X. The map $\mathrm{Bun}_B \to \mathrm{Bun}_T$ is a bijection on components with fibers that look like affine spaces. By [Hei10, Theorem 6], we have $(\pi_0(\mathrm{Bun}_T) \cong (\underline{X}_\star(T)_\infty)_\Gamma$, and a well-defined degree map

$$(\underline{X}_\star(T)_\infty)_\Gamma \to ((\underline{X}_\star(T)_\infty)_\Gamma)_{tf}.$$

Its kernel is the torsion subgroup, necessarily finite; we write Bun_B^δ and Bun_T^δ for the finite union of components of fixed numerical degree $\delta \in ((\underline{X}_\star(T)_\infty)_\Gamma)_{tf}$.

Each component Bun_T^δ is of finite type, as follows: When T is split this moduli stack is a product of Jacobians hence clearly of finite type. For general T, splitting it over an étale Γ-cover $X' \to X$ gives

$$\mathrm{Bun}_T \cong \mathrm{Bun}_{T_{X'}}^\Gamma,$$

so there is a closed immersion $\mathrm{Bun}_T^\delta \hookrightarrow \mathrm{Bun}_{T_{X'}}$. The image lands within finitely many connected components, and we know those are of finite type. The fibers of $\mathrm{Bun}_B \to \mathrm{Bun}_T$ are affine spaces, so the Bun_B^δ are likewise of finite type. We have shown above the image of $\widetilde{\mathscr{M}}^\xi$ under the finite-type map $\widetilde{\mathscr{M}} \to \mathrm{Bun}_G$ is covered by the images of finitely many Bun_B^δ. Hence $\widetilde{\mathscr{M}}^\xi$ is of finite type, and its open M^ξ likewise. □

CHAPTER 9

The valuative criterion of properness: uniqueness

In this section, we adapt the proof of [CL10, §9] to confirm the following theorem.

THEOREM 9.1 (Separation of the truncated Hitchin morphism). *Let G/X be a quasisplit reductive group with anisotropic center. For any $\xi \in \mathfrak{a}_T$, the morphism $f^\xi : \widetilde{\mathscr{M}}^\xi \to \widetilde{\mathscr{A}}$ is separated. That is, let R a complete DVR k-algebra with field of fractions K, residue field κ, and uniformizer π. Let $m = (\mathscr{E}, \theta, t), m' = (\mathscr{E}', \theta', t)$ be two elements of $\mathscr{M}^\xi(R)$ such that (1) $f(m) = f(m')$ and (2) there exists an isomorphism $\phi_K : m'_K \xrightarrow{\sim} m_K$ of their generic fibers. Then there exists a unique isomorphism $\phi : m' \to m$ extending ϕ_K.*

Note that since $\widetilde{\mathscr{A}}$ is a scheme, no 2-morphism compatibilities appear in the valuative criterion [Sta22, Tag. 0CLD].

REMARK 9.2. The anisotropic central condition on G is essential. For example, a non-separated Hitchin fiber for GL_2 (and the ξ-truncation) can be found in [Cha14, §5, 6].

The proof will occupy the remainder of this chapter.

9.1. Preliminary reductions and constructions

PROOF. When checking separatedness via the valuative criterion, we can reduce to the case that R is a complete DVR and κ is algebraically closed because $\widetilde{\mathscr{A}}$ is locally Noetherian and the morphism f^ξ is locally of finite type ([LM00, Prop. 7.8]).[1]

A crucial role in the proof is played by the *generic trait*: let B be the local ring in X_R of the *generic point* of the *special fiber* X_κ. So $\operatorname{Spec} B \to X_R$ is a DVR supported on the generic points, with residue field $\kappa(X)$ and fraction field $F = K(X)$. We write η for the generic point of the generic fiber and s for the generic point of the special fiber.

The isomorphism ϕ_K consists of the data of an isomorphism $\mathscr{E}'_K \to \mathscr{E}_K$, also denoted ϕ_K, such that the induced map $\mathrm{Ad}_D(\phi_K)$ sends θ'_K to θ_K. We observe that it is necessary and sufficient to extend ϕ_K over $\operatorname{Spec} B$.

PROPOSITION 9.3. *The isomorphism ϕ_K extends to X_R if and only if $(\phi_K)_\eta$ extends to* $\operatorname{Spec} B$. *These extensions are unique if they exist.*

PROOF. If $(\phi_K)_\eta$ extends to $\operatorname{Spec} B$, then it extends to an open $U \subset X_R$ which contains all the points of codimension ≤ 1 in X_R. Such an isomorphism is the same as a section of $\mathrm{Isom}_G(\mathscr{E}', \mathscr{E})$, which is an affine scheme over X_R (indeed, a torsor for the affine group $\mathrm{Aut}_G(\mathscr{E})$). By algebraic Hartogs' lemma ([DG67, IV, Cor. 20.4.12]), the section over U extends uniquely to all of X_R. The induced isomorphism $\mathrm{Ad}_G(\phi_K)$ sends θ' to a section of the adjoint bundle which agrees with θ on a dense, so must be equal to θ everywhere (because the base is reduced and the total space is separated over the base, so morphisms are determined by their behavior on dense opens). The same reasoning implies that the isomorphism is determined by its restriction to X_K, and so the extension is unique if it exists. □

Moreover, we can then reduce from B to its completion.

[1]All of our stacks are indeed algebraic stacks in the sense of [LM00], in particular, their diagonals are separated and quasi-compact, so this version of the valuative criterion is sufficient to check separatedness. This follows for $\widetilde{\mathscr{M}}$ from the corresponding result for Bun_G, which is well-known.

Lemma 9.4. *Write $\widehat{B}$ for the completion of B, and $\hat{F} = \mathrm{Frac}(\widehat{B})$ for its fraction field, with point $\hat{\eta}$. The isomorphism $(\phi_K)_\eta$ extends to $\operatorname{Spec} B$ if and only if its base change $(\phi_K)_{\hat{\eta}}$ extends to $\operatorname{Spec} \widehat{B}$.*

Proof. Since these isomorphisms are morphisms from their domain into the affine scheme $\mathrm{Isom}(\mathscr{E}', \mathscr{E})$, this is immediate from the fact that $B = \widehat{B} \cap F$ inside $\hat{F}$ induces a pushout in the category of affine schemes. $\square$

The rest of the proof will occur over $\widehat{B}$. To alleviate notation, we abuse notation and write B for $\widehat{B}$ (and F for $\hat{F}$) – it is now a complete DVR with residue field $\kappa(X)$.

Assume that m is (w, L)-elliptic on the generic fiber and (w, M)-elliptic on the special fiber, $M \subseteq L$ (see Definition 3.5). In particular, $f(m) = (a, t) \in \widetilde{\mathscr{A}}_{(v,L)}(R)$, i.e., there is a preimage $(a_L, w(t)) \in \widetilde{\mathscr{A}}_L$, and $[(w, L)] \geq [(w, M)]$.

By a Kostant section over B, we can find $Y \in \mathfrak{l}(B)$ whose L-characteristic polynomial is $a|_{\operatorname{Spec} B}$. Moreover, Proposition 3.11 applied with ambient group L and twisted subgroup (e, M) implies that we can find an element $\overline{Z} \in \mathfrak{m}(\kappa(X))$ and $m \in M(\kappa(X)_\infty)$ such that

$$\chi_L(\overline{Z}) = a_s \quad \text{and} \quad \mathrm{Ad}(m)\overline{Z} = w(t_a)|_{X_\kappa}.$$

Since $\overline{Y}, \overline{Z} \in \mathfrak{l}(\kappa(X))$ have the same L-characteristic polynomial, they are $L(\kappa(X))$-conjugate by [CL10, Lemme 3.8.1] (the proof of which does not require L to be defined over k, only $\kappa(X)$). Since $L(B) \to L(\kappa(X))$ is surjective, we have shown we can replace Y by a conjugate such that $\overline{Y}$ is $M(\kappa(X)_\infty)$-conjugate to $w(t_a)|_{X_\kappa}$.

By Steinberg's theorem, we know that the torsors $\mathscr{E}$ and $\mathscr{E}'$ are trivial on the special fiber of B. These are smooth torsors on a Henselian DVR, so the trivializations lift to B. Under the induced trivializations of $\mathrm{Ad}_D(\mathscr{E}), \mathrm{Ad}_D(\mathscr{E}')$, the sections θ, θ' correspond to regular elements of $\mathfrak{l}(B)$ with the same characteristic polynomial as $Y \in \mathfrak{l}(B)$.

The transporter from θ to Y is a smooth torsor under T_Y, and has a point on $\kappa(X)$ by Steinberg's theorem, so by Henselian lifting it has a section on B. The same argument holds for $\mathscr{E}'$. In other words, we may and do fix trivializations of $\mathscr{E}, \mathscr{E}'$ over B such that θ and θ' both correspond to Y.

Under these trivializations, $(\phi_K)|_\eta : \mathscr{E}'|_K \to \mathscr{E}|_K$ corresponds to left multiplication by an element $\delta \in G(F)$ such that

$$\mathrm{Ad}(\delta^{-1})Y = Y.$$

The isomorphism ϕ_K extends across B, and thus across all of C_R, if and only if $\delta \in G(B)$.

Since our Hitchin base consists only of polynomials which are regular semisimple, $f(m) = a$ is generically regular on both the generic and special fiber and so Y is regular semisimple over B, with centralizer given by a B-torus T_Y. A torus over an irreducible normal scheme is isotrivial, and so splits after a finite étale cover of B.[2]

By construction $\delta \in T_Y(F)$. They claim that there exists a unique $\lambda \in X_\star(T_Y)$ and $h \in T_Y(F) \cap G(B) = T_Y(B)$ such that

$$\delta = \lambda(\pi)h.$$

Certainly, this is true in an unramified extension splitting the torus, since π is still a uniformizer. But by uniqueness, $\lambda(\pi)$ and h are necessarily Galois-invariant, so in fact λ is a rational cocharacter in $X_\star(T)^\Gamma = X_\star(S)$ and $h \in T(B)$.

Since $h \in G(B)$ centralizes Y we can shift the trivialization of $\mathscr{E}$ by h without affecting our setup. So we may assume $h = 1$. It remains to prove that $\lambda = 0$, so that $\delta = 1$ extends across B.

9.2. Proof that $\lambda = 0$: construction of parabolic reductions.

The element $(a, t)|_{X_K} \in \widetilde{\mathscr{A}}_{(w,L)\text{-ell}}$, so Y is necessarily L-elliptic in the function field F with $\chi(X)|_\infty = w(t)$ by Proposition 3.12. That is, $X_\star(Z_{G_F}(Y))^{\mathrm{Gal}(F^s/F)} = X_\star(A_L^{\mathrm{spl}})$. We can assume $L \subset G$ is proper, as otherwise our anisotropic center assumption implies $\lambda \in \underline{X}_\star(A_G^{\mathrm{spl}}) = 0$.

[2] Note that this cover is purely a cover of B, and in particular does not necessarily come from an extension of the scalar ring R. So we cannot absorb this extension using a formal base change argument.

Assume for the sake of contradiction that $\lambda \neq 0$ and define semistandard parabolic P (resp. $\overline{P}$) to be the one containing L whose relative roots $\alpha \in (\Delta_P^\star)^{\mathrm{rel}} \subset \Phi(G,S)$ (i.e., the roots of the associated Levi) are those satisfying $\alpha(\lambda) \geq 0$ (resp. $\alpha(\lambda) \leq 0$). These are proper parabolics because $\lambda \neq 0$ and the split center is trivial, and their intersection is a Levi subgroup containing L, hence M.

Let $r = \max_{\alpha \in \Phi(G,S)} \alpha(\lambda)$. Consider the Lie algebra $\mathfrak{g}/X$, and decompose it as a sum of subbundles into *relative* weight spaces

$$\mathfrak{g} = \bigoplus_{\alpha \in X^\star(S)} \mathfrak{g}_\alpha.$$

Collect these weight spaces according to the value of $\alpha(\lambda)$:

$$\mathfrak{g}_i = \bigoplus_{\substack{\alpha \in X^\star(S) \\ \alpha(\lambda)=i}} \mathfrak{g}_\alpha.$$

Consider the complement of the lowest weights relative to $\mathfrak{a}$:

$$\mathfrak{g}_+ = \bigoplus_{-r<i\leq r} \mathfrak{g}_i.$$

We have $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_{-r}$. Note that $\mathfrak{g}_+$ is stable under P and $\mathfrak{g}_{-r}$ is stable under $\overline{P}$.

Since we have bounded how negative the weights $\lambda(\pi)$ can be, we have an inclusion

$$\pi^r \operatorname{Ad}(\lambda(\pi))\mathfrak{g}(B) \subset \mathfrak{g}(B)$$

of lattices inside $\mathfrak{g}(F)$. This inclusion corresponds to a morphism between two different extensions of the vector bundle $\operatorname{Ad}_D(\mathscr{E}_K)$ to X_R:

$$V \to \operatorname{Ad}_D(\mathscr{E}).$$

Let W and Q denote the kernel and the image of the morphism over the special fiber X_κ.

We claim that W is actually a subbundle of V_κ, while Q is simply a subsheaf of $\operatorname{Ad}_D(\mathscr{E}_\kappa)$. We have $\deg(V_\kappa) = 0$ because by flatness $\deg(V_\kappa) = \deg(V_K) = \deg(\operatorname{Ad}_D(\mathscr{E}))$. Adjoint bundles have degree zero because root systems are symmetric around 0.

So we have $\deg(W) = -\deg(Q)$ from additivity in short exact sequences. Let $\operatorname{Ad}_D(\mathscr{E}') \to V$ be the unique isomorphism defined on the generic fiber by

$$\operatorname{Ad}_D(\mathscr{E}'_K) \xrightarrow{\operatorname{Ad}_D(\phi_K)} \operatorname{Ad}_D(\mathscr{E}_K) \xrightarrow{\pi^r} V$$

and on B by

$$\mathfrak{g}(B) \xrightarrow{\pi^r \operatorname{Ad}(\lambda(\pi))} \pi^r \operatorname{Ad}(\lambda(\pi))\mathfrak{g}(B).$$

This isomorphism extends uniquely to all of X_R by Hartog's lemma, as above. Composing the isomorphism with the homomorphism $V \to \operatorname{Ad}_D(\mathscr{E})$ gives a morphism

$$\operatorname{Ad}_D(\mathscr{E}') \to \operatorname{Ad}_D(\mathscr{E})$$

which has on the special fiber a kernel W'. We have $W' = \mathscr{E}'_{P,\kappa} \times^P_{X_\kappa} \mathfrak{g}_+$ because on X_κ, $\pi^r \operatorname{Ad}(\lambda(\pi))$ kills everything except $\mathfrak{g}_{-r}$. By construction, $W' \cong W$ and the image of the morphism is Q.

We fixed trivializations $\mathscr{E} \cong G$ and $\mathscr{E}'$ on $\operatorname{Spec} B$. Under these trivializations, the parabolic subgroups $P, \overline{P}$ correspond to generic parabolic reductions of $\mathscr{E}, \mathscr{E}'$ on X_κ, which extend by properness to parabolic reductions $\mathscr{E}_{P,\kappa}, \mathscr{E}_{\overline{P},\kappa}, \mathscr{E}'_{P,\kappa}, \mathscr{E}'_{\overline{P},\kappa}$ on the entire special fiber. The argument will conclude by finding incompatible inequalities between the degrees of these P-reductions, one from ξ-stability and the other coming from the relation between their constructions.

9.3. Conclusion from ξ-stability.

In order to apply ξ-stability, we need to know that these reductions extend to m_κ, m'_κ, and indeed that they are reductions in the same relative position. This is an exercise in unpacking the definitions: the subgroups correspond to parabolic reductions of bundles trivialized such that θ_κ corresponds to $\overline{Y}$. The condition of the reductions being θ_κ-stable, i.e., the existence of $\theta_{P,\kappa}$, corresponds to $\overline{Y} \in L(\kappa(X))$. The condition of being in relative position w follows from the condition above that $\overline{Y}$ is $M(\kappa(X)_\infty)$-conjugate to $w(t_a)$.

So we have shown that $\mathscr{E}'_{P,\kappa}$ extends to a (w, P)-reduction of $m'_{P,\kappa}$ and that $\mathscr{E}_{\overline{P},\kappa}$ extends to a $(w, \overline{P})$-reduction of m_κ. Since these are both ξ-stable by assumption, we obtain inequalities in opposite directions:

$$w^{-1}(\xi) \in -\deg(m'_{P,\kappa}) + {}^+\mathfrak{a}_P + \mathfrak{a}_T^M \qquad \text{and} \qquad w^{-1}(\xi) \in -\deg(m_{\overline{P},\kappa}) + {}^+\mathfrak{a}_{\overline{P}} + \mathfrak{a}_T^M$$

In order to contract these positivity conditions in the geometric cocharacter space $\mathfrak{a}_T$ into a single linear inequality, we make use of a relative character which, viewed as a functional on $\mathfrak{a}_T$, has gradient in a direction positive for P and negative for $\overline{P}$. The appropriate character is μ, the determinant of the action of P on $\mathfrak{g}_+$. Viewing μ as a relative or an absolute character, it is a highest weight in $\wedge^{\dim \mathfrak{g}_+}\mathfrak{g}$, so is a nonzero nonnegative linear combination of the weights $\hat{\Delta}_P$. In particular, then, applying μ to the semistability conditions induces opposite inequalities:

$$\deg(\mu(\mathscr{E}'_{P,\kappa})) < -\mu(w^{-1}(\xi)) < \deg(\mu(\mathscr{E}_{\overline{P},\kappa})).$$

On the other hand, since $W' = \mathscr{E}'_{P,\kappa} \times^P_{X_\kappa} \mathfrak{g}_+$ we have $\deg(W') = \deg(\mu(\mathscr{E}'_{P,\kappa}))$. There is an injection $Q \hookrightarrow \mathscr{E}_{\overline{P},\kappa} \times^{\overline{P},\mathrm{Ad}}_{X_\kappa} \mathfrak{g}_{-r}$, as can be checked at the generic point because $\mathscr{E}_{\overline{P},\kappa} \times^{\overline{P},\mathrm{Ad}}_{X_\kappa} \mathfrak{g}_{-r}$ is a subbundle, and under the generic trivializations above the morphism $\mathrm{Ad}_D(\mathscr{E}'_\kappa) \to \mathrm{Ad}_D(\mathscr{E}_\kappa)$ corresponds to multiplication by $\pi^r \mathrm{Ad}(\lambda(\pi))$, which is projection onto $\mathfrak{g}_{-r}$ mod π. The injection therefore gives

$$-\deg(\mu(\mathscr{E}_{\overline{P},\kappa})) = -\deg(W') = \deg Q \leq \deg\left(\mathscr{E}_{\overline{P},\kappa} \times^{\overline{P},\mathrm{Ad}} \mathfrak{g}_{-r}\right) = -\deg \mu(\mathscr{E}_{\overline{P},\kappa}),$$

giving the other direction of inequality and a contradiction. $\square$

CHAPTER 10

The valuative criterion of properness: existence

For simplicity, we continue to work over the compatible locus. Fix an arbitrary element $\xi \in \mathfrak{a}_T$. For R a DVR k-algebra with K its field of fractions, consider the commutative square

$$\begin{tikzcd}
\mathrm{Spec}(K) \arrow[r] \arrow[d] & \mathsf{M}^\xi \arrow[d, "f"] \\
\mathrm{Spec}(R) \arrow[r] & \mathsf{A}
\end{tikzcd}$$

The existence part of the Noetherian valuative criterion can be reduced as follows:

(1) We may assume that the residue field of R is algebraically closed.
(2) We may assume that R is a complete DVR.
(3) It suffices to show that the extended map exists after an arbitrary extension: i.e., that there exists an extension K'/K and a valuation ring $R' \subseteq K'$ dominating R such that we have a lift

$$\begin{tikzcd}
\mathrm{Spec}\, K' \arrow[r] \arrow[d] & \mathrm{Spec}\, K \arrow[r] \arrow[d] & \mathsf{M}^\xi \arrow[d, "f"] \\
\mathrm{Spec}\, R' \arrow[r] \arrow[urr, dashed] & \mathrm{Spec}\, R \arrow[r] & \mathsf{A}
\end{tikzcd}$$

These all follow from [Sta22, Tag. 0H1Y].

Hence, to prove that $\mathsf{M}^\xi \to \mathsf{A}$ is universally closed, it suffices to prove the following result.

Theorem 10.1. *Let G a quasisplit reductive group over X, κ an algebraically closed extension of k and R a complete DVR k-algebra with residue field κ (in particular, R is strictly Henselian). Let $K = \mathrm{Frac}\, R$ and pick an algebraic closure $\overline{K}$. Fix $(a,t) \in \mathsf{A}(R)$ and $m_K \in \mathsf{M}^\xi(K)$ such that $f(m_K) = (a,t)$. There exists a finite extension K'/K and $m \in \mathsf{M}^\xi(R')$, where R' is the integral closure of R in K', such that*

(1) The image of m in $\mathsf{M}^\xi(K')$ is isomorphic to the pullback of m_K.
(2) $f(m) = (a,t)$.

Remark 10.2. By the Cohen structure theorem ([Sta22, Tag 0C0S]), $R \cong \kappa[\![\pi]\!]$ by a choice of uniformizer π.

This semistable reduction result is exactly [CL10, Théorèm 8.1.1] when G is defined (and split) over k. They proceed by a conceptually simple but technically involved variant of Langton's inductive argument: since the untruncated Hitchin fibration is universally closed, choose any integral R-model, then iteratively perform local modifications to make its ξ-instability $\deg_i^\xi(m)$ decrease until it becomes ξ-semistable. The following theorem of Alper–Halpern-Leistner–Heinloth greatly generalizes this algorithm to apply to any reasonable stack with a Θ-stratification.

Theorem 10.3 ([AHH23, Corollary 6.12]). *Let $\mathcal{X}$ be a quasi-separated algebraic stack with affine stabilizers which is locally of finite type over algebraic space S and which satisfies the existence part of the valuative criterion. Assume that $\mathcal{X}$ is equipped with a well-ordered Θ-stratification. Then for any morphism $g : \mathrm{Spec}(R) \to \mathcal{X}$ whose generic fiber $g|_K : \mathrm{Spec}(K) \to \mathcal{X}$ lands inside a stratum, there is (up to a finite extension of R and K) a modification of g which lands entirely within that strata.*

In particular, the minimal stratum $\mathcal{X}^{ss} := \mathcal{X}_{\leq 0}$ satisfies the same existence condition.

Remark 10.4. The framework of §7 can also be used to show the hypotheses of "Θ-reductivity" and "S-completeness" of [AHH23, Theorem 5.4]. These imply that the stack M^ξ admits an adequate moduli space. On the other hand, by [Ngô10, Corollaire 4.11.3] we also know that the stabilizers over $\mathscr{A}^\heartsuit$ are subgroups of a split torus, hence linearly reductive, so in fact M^ξ admits a good moduli space. No such considerations are necessary in this thesis.

10.1. The valuative criterion for $\widetilde{\mathscr{M}}$

Since we have proven in Proposition 7.19 that the ξ-canonical parabolic defines a Θ-stratification of $\mathsf{M} \to \mathsf{A}$, it remains only to recall why the untruncated fibration satisfies the valuative criterion. Here, the restriction to the compatible locus plays no role, so we prove the result for $\widetilde{\mathscr{M}} \to \widetilde{\mathscr{A}}$.

Proof of Theorem 10.1 for $\widetilde{\mathscr{M}}$. Writing $m_K = (\mathscr{E}_K, \theta_K, t)$, we consider the restriction $(\mathscr{E}_K)_\eta$ to the generic point η of X_K. The group G_η is a connected reductive group scheme over the function field F_K of X_K. Writing $G_{\overline{\eta}}$ for the pullback to $X_{\overline{K}}$, its function field $F_{\overline{K}}$ is a function field of a curve over an algebraically closed field, so in particular is of cohomological dimension ≤ 1. It follows from [Ser94, III.2.3, Thm 1' and Remarks] that $H^1(F_{\overline{K}}, G_{\overline{\eta}}) = 0$, and so every G-bundle on $X_{\overline{K}}$ is trivial at the generic point. The isomorphism $\mathscr{E}_{\overline{\eta}} \cong G_{\overline{\eta}}$ descends to a finite extension K'/K, so it follows that, up to replacing K' by a finite extension, we may assume that $\mathscr{E}_K$ is trivial at η.

Fix such an isomorphism $\mathscr{E}_\eta \cong G_\eta$. Under this isomorphism, the Higgs field generically corresponds to an element $Y \in \mathfrak{g}(F_K)$, whose characteristic polynomial $\chi(Y)$ is equal to $a|_\eta$. Now, inside of C_R, consider the local ring of the generic point of C_κ. This is a DVR supported on the two generic points, which we denote B. Its field of fractions is F_K. By the Kostant section, we can choose an $X \in \mathfrak{g}(B)$ whose characteristic polynomial is $a|_{\operatorname{Spec} B}$.

We have now two elements $X, Y \in \mathfrak{g}(F_K)$ whose characteristic polynomials are equal. Viewing the same elements in $\mathfrak{g}(F_{\overline{K}})$, they have the same characteristic polynomial (which is G-regular semisimple at ∞, and thus at the generic point). The transporter functor $T_G(X,Y) = \{g \in G : g.X = Y\}$ is nonempty (it has points over $\overline{F_{\overline{K}}}$), and is a torsor for the torus $Z_G(X)$. In particular $T_G(X,Y)$ is smooth scheme, hence splits over $F^s_{\overline{K}}$. Since $Z_G(X)$ is connected reductive, the same vanishing $H^1(F_{\overline{K}}, Z_G(X)) = 0$ as above implies that $T_G(X,Y)$ is trivial, so there exists an element $g \in G(F_{\overline{K}})$ taking X to Y. Since tensor product commutes with colimits and G is locally of finite presentation over F_K, the element g lies in F_L for some L/K finite. So, up to replacing K by a finite extension, we may assume $g \in G(F_K)$.

By Beauville–Laszlo descent for Hitchin fibers (cf. §6), the element g allows us to glue $(\mathscr{E}_K, \theta)$ on X_K with $(G_R \times_{X_R} \operatorname{Spec} B, X)$ (a trivial bundle with Higgs field X) on $\operatorname{Spec} B$. The result is an extension $(\mathscr{E}_U, \theta_U)$ of $(\mathscr{E}_K, \theta_K)$ to an open $U \subseteq X_R$ which contains all points of codimension ≤ 1. By [CS79, Thm. 6.13], every G-torsor on U extends uniquely to X. By [DG67, IV, Cor. 20.4.12], the section θ_U, defined away from a set of codimension 2, extends to X_R. Thus, we have found an extension $(\mathscr{E}, \theta)$ of $(\mathscr{E}_K, \theta_K)$ to X_R. Since the elements $\chi(\theta)(\infty_R)$ and $\chi(t)$ in $\mathfrak{c}(R)$ agree in $\mathfrak{c}(K)$, they are equal, so $(\mathscr{E}, \theta, t) \in \widetilde{\mathscr{M}}(R)$ is the desired extension. □

Thus, by the theorem of Alper–Halpern-Leistner–Heinloth, Theorem 10.1 is proven.

CHAPTER 11

The ξ-stable Higgs moduli stack is Deligne–Mumford

Here we must assume ξ is in general position, in the sense of Definition 5.10, and restrict to the compatible locus (mainly for convenience).

Theorem 11.1 (cf. [CL10, Théorème 10.1.1]). *Let G/X be quasisplit reductive with anisotropic center and $\xi \in \mathfrak{a}_T$ a parameter in general position. Then M^ξ is a Deligne–Mumford stack.*

Proof. It suffices to show that for every point $m = (\mathscr{E}, \theta, t) \in \mathsf{M}^\xi(k)$, the stabilizer group is unramified. Since it is a representable group scheme locally of finite type, it suffices to show its Lie algebra vanishes. Let M be the Levi such that $a = f(m)$ is M-elliptic. The stabilizer group of m is $\mathrm{Aut}_G(\mathscr{E}, \theta)$, the centralizer of θ in $\mathrm{Aut}_G(\mathscr{E})$. Its tangent space consists of the subspace of $H^0(X, \mathrm{Ad}(\mathscr{E}))$ annihilated by $[\theta, -]$. Note the lack of a twist by D. Let $\phi \in \mathrm{Lie}\,\mathrm{Aut}_G(\mathscr{E}, \theta)$ be such a global section.

By Lemma 3.25 (c) under our characteristic assumptions, and [Ngô10, Corollaire 4.11.3], we have

$$\mathrm{Lie}\,\mathrm{Aut}_G(\mathscr{E}, \theta) \subseteq \mathrm{Lie}(T_\infty^{W_a}) = \mathrm{Lie}(A_M^{\mathrm{spl}})_\infty.$$

Write $Z \in \mathrm{Lie}(A_M^{\mathrm{spl}})_\infty$ for the image of ϕ under this inclusion. Explicitly, the infinitesimal automorphism ϕ gives a global section of the Lie algebra of the smoothened centralizer $I^{\mathrm{lis}}_{(\mathscr{E},\theta)}$, which pushes forward to a global section of the Lie algebra of the Néron model $J_a^\flat$ determined by its value Z at ∞. Since A_M^{spl} is split over proper X (i.e., constant), there is a canonical constant global section, also denoted Z, of $\mathrm{Lie}\,A_M^{\mathrm{spl}}$ restricting to Z at infinity. Since $I^{\mathrm{lis}}_{(\mathscr{E},\theta)} \cong J_a^\flat$ on the regular open and Ngô's inclusion is given by pushing forward, it follows that $Z_\eta \in \mathfrak{g}(F)$ is the generic coordinate of ϕ in any generic trivialization in which θ_η corresponds to a G-regular, M-elliptic element $Y \in \mathfrak{m}(F)$.

Now consider $L := Z_G(Z)$. By 2.1, our characteristic hypotheses ensure L is a Levi. Clearly M centralizes Z, so $L \supseteq M \supseteq T$. Moreover, L is defined over X: after a splitting Γ cover $X' \to X$,

$$\Phi(L_{X'}, T_{X'}) = \{\alpha : d\alpha(Z) = 0\}$$

and this root subset is moreover Γ-invariant, so defines a semistandard Levi over X.

Let Q be a parabolic subgroup with Levi L and denote its unipotent radical N. As in Prop. 6.12, the bundle $\mathscr{E}$ is determined by the generic trivialization above and a family $(g_v)_{v \in X(k)} \in \prod_v' \mathrm{Gr}_{G,v}(k)$ of modifications satisfying conditions as in Definition 6.8. In particular,

$$\mathrm{Ad}(g_v^{-1})Z = \mathrm{Ad}(g_v^{-1})(\phi_\eta) \in L^+_{\mathfrak{g},v}(k)$$

is *integral* at each point because ϕ is a global section of $\mathrm{Ad}(\mathscr{E})$ without D-twist.

We claim that all g_v have representatives in $\mathrm{Gr}_{L,v}(k)$. By the Iwasawa decomposition, we can assume up to disk reparameterization that each g_v can be written $g_v = l_v n_v$ with $l_v \in L(F_v)$ and $n_v \in N(F_v)$. The above argument shows $\mathrm{Ad}(g_v^{-1})Z$ is integral at each place, but Z is constant, so $Z \in L^+_{\mathfrak{g},v}(k)$ also, and so

$$\mathrm{Ad}(n_v^{-1})Z - Z \in L^+_{\mathfrak{g},v}(k) \cap \mathfrak{n}(F_v) = \mathfrak{n}(\mathscr{O}_v).$$

By the Lemma below, this implies the n_v all lie in $N(\mathscr{O}_v)$ and $g_v \sim l_v \in \mathrm{Gr}_{G,v}(k)$. This gives a collection of Beauville–Laszlo data gluing together to an L-reduction $\mathscr{E}_L$ of $\mathscr{E}$. Moreover, writing $Y = \theta_\eta$ for an M-elliptic element in a generic trivialization, we have

$$\mathrm{Ad}(l_v^{-1})Y = \mathrm{Ad}(n_v)\,\mathrm{Ad}(g_v^{-1})Y \in z_v^{-d_v}\mathfrak{g}(\mathscr{O}_v),$$

but also $\mathrm{Ad}(l_v^{-1})Y \in \mathfrak{l}(F_v)$, so

$$\mathrm{Ad}(l_v^{-1})Y \in z_v^{-d_v}\mathfrak{l}(\mathscr{O}_v).$$

The same gluing arguments of §6, applied to the group L, show that Y glues to an L-reduction

$$\theta_L \in H^0(X, \mathrm{Ad}_D(\mathscr{E}_L)) \to H^0(X, \mathrm{Ad}_D(\mathscr{E})).$$

In summary, we have shown that $m = (\mathscr{E}, \theta, t)$ admits a "reduction" to L, in the sense that $(\mathscr{E}, \theta)$ is the extension of scalars of $m_L = (\mathscr{E}_L, \theta_L)$. For each parabolic P having Levi L, the P-reduction of m_L is by uniqueness equal to the P-reduction of m. In particular, this implies that all the P-reductions of m for such P have the same degree, i.e., the projection of C_m into $\mathfrak{a}_L^G$ is a single point. This point must be the image of ξ by the stability assumption, and necessarily lies in the lattice $\mathscr{X}_\star(L)$, contradicting ξ being in general position unless $L = G$. In the latter case $Z \in H^0(X, \mathrm{Lie}(A_G))$ is central. But our characteristic hypotheses imply by the second Lemma below

$$H^0(X, \mathrm{Lie}(A_G)) = \mathrm{Lie}((A_G^{\mathrm{spl}})_\infty) = 0,$$

using the anisotropic center hypothesis. By injectivity, $\phi = 0$.

Lemma 11.2. *Fix any base scheme S and any reductive group scheme $G \to S$ with fiberwise maximal torus T, parabolic Q with unipotent radical N and semistandard Levi factor $L \supset T$. Let $Z \in \mathfrak{g}(S)$ such that its centralizer is exactly L. Then the map*

$$\begin{aligned}\iota : N &\to \mathfrak{n}\\ n &\mapsto \mathrm{Ad}(n^{-1})Z - Z\end{aligned}$$

is an S-isomorphism.

Proof. The map is globally defined, and so can be checked to be an isomorphism on geometric fibers or after a splitting cover. This reduces to the classical statement: since the centralizer of Z is exactly L, the adjoint $\mathrm{ad}(Z)$ acts by nonzero scalar $d\alpha(Z)$ on each weight space $\mathfrak{g}_\alpha \subset \mathfrak{n}$. Both the unipotent radical and its Lie algebra are affine spaces, and filtering both by height of the roots, ι preserves the filtration. The Chevalley commutation relations imply that the morphism ι is a triangular polynomial morphism with invertible diagonal entries, hence is an isomorphism. $\square$

Lemma 11.3. *Let A/X be a torus with tame monodromy, i.e. split by a geometrically connected finite étale cover $X' \to X$ with group Γ such that p does not divide $|\Gamma|$. Then*

$$H^0(X, \mathrm{Lie}\, A) = \mathrm{Lie}(A^{\mathrm{spl}}).$$

In particular, if $A^{\mathrm{spl}} = 1$, then $H^0(X, \mathrm{Lie}\, A) = 0$.

Proof. Since X' is projective and geometrically connected, $H^0(X', \mathscr{O}_{X'}) = k$. By descent, we have

$$H^0(X, \mathrm{Lie}\, A) = (\mathrm{Lie}\, A_{X'})^\Gamma = (\underline{X}_\star(A)(\infty) \otimes_{\mathbb{Z}} k)^\Gamma.$$

The characteristic hypothesis implies taking Γ-invariants is exact, so commutes with the tensor, and

$$H^0(X, \mathrm{Lie}\, A) = \underline{X}_\star(A)(\infty)^\Gamma \otimes_{\mathbb{Z}} k = \mathrm{Lie}(A^{\mathrm{spl}}).$$

$\square$

$\square$

Part 3

Cohomological stabilization

CHAPTER 12

Regular centralizers, Picard stacks, and the s-decomposition

In this section, we recall the necessary ingredients required to define the Picard stack $\mathscr{P} \to \mathscr{A}$ and its action on Hitchin fibers, which is essential to analysis of the fibration (cf. [Ngô10, §4]). In parallel, we adapt the constructions of [CL12, §3, 5] to our generality and define a variant of the degree map on the restriction to the compatible locus $\mathsf{P} \to \mathsf{A}$, whose kernel P^1 preserves ξ-(semi)stability for any ξ.

12.1. Regular Centralizers.

Fix a parabolic P over X with semistandard Levi M. Let I^P denote the universal centralizer subgroup over $\mathfrak{p}$ (recall the latter is a vector bundle over X):

$$I^P = \{(p, Y) \in P \times_k \mathfrak{p} : \mathrm{Ad}(p)Y = Y\}.$$

PROPOSITION 12.1. *The natural closed immersion $I^P \to I^G$ is an isomorphism on the G-regular locus $\mathfrak{p}^{G\text{-}reg}$.*

PROOF. It suffices by the fibral isomorphism criterion to check surjectivity on geometric fibers over X (cf. [Con14], Lemma B.3.1 and the proof of Corollary 5.2.8). This reduces us to the constant geometric setting of [CL12, Lemme 3.1.1]. $\square$

One can then define the *regular centralizer* group scheme J^M over $\mathfrak{c}_M$ either by pullback along a Kostant section or by faithfully flat descent (cf. [Ngô10, 2.1.1]). Recall that $\mathfrak{c}_G$ is equipped with a $\mathbb{G}_m$-action making the Chevalley map equivariant. The group scheme J^G over $\mathfrak{c}_G$ can be equipped with a $\mathbb{G}_m$-equivariant structure for this action (cf. [Ngô10, §2.2.3], [CL12, §5.1]).

PROPOSITION 12.2 ([CL12, 3.2.1]). *The natural isomorphism*

$$(\chi_G^\star J^G)|_{\mathfrak{p}^{G\text{-}reg}} \to I^G|_{\mathfrak{p}^{G\text{-}reg}} = I^P|_{\mathfrak{p}^{G\text{-}reg}}$$

extends to a $(\mathbb{G}_m \times P)$-equivariant homomorphism over $\mathfrak{p}$:

$$(\chi_G^\star J^G)|_{\mathfrak{p}} \to I^P$$

whose composition with $i_P : I^P \to I^G$ is the homomorphism $(\chi_G^\star J^G)|_{\mathfrak{p}} \to I^G|_{\mathfrak{p}}$ from [Ngô10, 2.1.1]. Twisting by D gives a factorization of morphisms over $[\mathfrak{p}_D/P]$:

$$\begin{tikzcd}
{[(\chi_G^\star J_D^G)|_{\mathfrak{p}_D}/P]} \arrow[r] \arrow[dr] & {[I_D^P/P]} \arrow[d, "i_P"] \\
& i_P^\star[I_D^G/G].
\end{tikzcd}$$

PROOF. The complement of the regular locus has codimension at least 2 (as can be counted on geometric fibers), the domain is normal, and the codomain is affine, so we are done by Algebraic Hartog's Lemma. The equivariance is clear on the regular loci, so holds everywhere, and the twisted statement follows immediately. $\square$

12.2. Degree of a regular centralizer.

There is a natural map over the curve X

$$P \to \underline{X}_\star(P) \otimes_{\mathbb{Z}_X} \mathbb{G}_{m,X} = \underline{X}_\star(M) \otimes_{\mathbb{Z}_X} \mathbb{G}_{m,X}$$

(the tensor product taken in abelian sheaves on X) which is fiber-wise dual to the evaluation map $P \times_X \underline{X}^\star(P) \to \mathbb{G}_{m,X}$. The codomain is a torus; in fact,

Proposition 12.3. *The torus $T_M := \underline{X}_\star(M) \otimes_{\mathbb{Z}_X} \mathbb{G}_{m,X}$ satisfies $\underline{X}^\star(T_M) \cong \underline{X}^\star(M)$ canonically. Equivalently, T_M is canonically isomorphic to the maximal torus quotient of M*

$$T_M \cong M^{\mathrm{ab}}.$$

In particular, $X^\star(T_M) = X^\star((M^{\mathrm{ab}})^{\mathrm{spl}})$, the maximal split torus quotient.

Proof. We compute by internal tensor-Hom adjunction

$$\begin{aligned}
\underline{X}^\star(T_M) &= \mathscr{H}om_X(\underline{X}_\star(M) \otimes_{\mathbb{Z}_X} \mathbb{G}_{m,X}, \mathbb{G}_{m,X}) \\
&\cong \mathscr{H}om_X(\underline{X}_\star(M), \mathscr{H}om_X(\mathbb{G}_{m,X}, \mathbb{G}_{m,X})) \\
&= \mathscr{H}om_X(\underline{X}_\star(M), \mathbb{Z}_X) \\
&= \underline{X}^\star(M).
\end{aligned}$$

But every character of M factors through $M/D(M) = M^{\mathrm{ab}}$. □

Pulling the coevaluation map back along $\mathfrak{p} \to X$ and restricting to I^P gives a map

$$I^P \to \underline{X}_\star(P) \otimes_{\mathbb{Z}_X} \mathbb{G}_{m,\mathfrak{p}}.$$

At the level of *regular* centralizers, we can descend this to the Chevalley space:

Proposition 12.4 ([CL12, Proposition 3.3.1]). *There exists a unique morphism of group schemes over $\mathfrak{c}_M$ (itself over X)*

$$(\chi_G^M)^\star J^G \to \underline{X}_\star(M) \otimes_{\mathbb{Z}_X} \mathbb{G}_{m,\mathfrak{c}_M}$$

whose pullback along $\chi_P : \mathfrak{p} \to \mathfrak{c}_M$ is the natural composition

$$\begin{tikzcd}
(\chi_P)^\star(\chi_G^M)^\star J^G = (\chi_G^\star J^G)|_{\mathfrak{p}} \arrow[rr] \arrow[drr] & & I^P \arrow[d] \\
& & \underline{X}_\star(M) \times_{\mathbb{Z}_X} \mathbb{G}_{m,\mathfrak{p}}
\end{tikzcd}.$$

The correseponding results hold after twisting by D, that is, for J_D^G, $\mathfrak{c}_{M,D}$, etc.

Proof. The argument is by faithfully flat descent along χ_P. The required gluing condition can be checked geometric fiberwise over X, where the proof reduces to the one given in [CL12]. All maps in the picture are $\mathbb{G}_m$-equivariant, □

Remark 12.5. Let us emphasize some intuition about this construction when $M = T$. We have shown that there is a $\mathfrak{t}$-morphism $(\chi_G^T)^\star J^G \to \underline{X}_\star(T) \otimes_{\mathbb{Z}_X} \mathbb{G}_{m,\mathfrak{t}} = T \times_X \mathfrak{t}$ which, given $Y \in \mathfrak{b}$, extends the natural projection

$$I_Y^B = Z_B(Y) \subseteq B \to T.$$

The above map says that this projection, extended to the regular centralizer over $\mathfrak{b}$, is well-defined at the level of characteristic polynomials. That is, given any $Y \in \mathfrak{b}^{G\text{-reg}}$ with that characteristic polynomial and any b centralizing Y (which is necessarily in B, see [Ngô10, Lemme 2.4.3]), the image of b in the torus is *independent of* Y under the canonical isomorphisms $Z_G(Y) \cong Z_G(Y')$.

12.3. Description Galoisienne.

Making use of the degree map, we can characterize J^G in terms of data along χ_G^T. Recall that W/X is the finite étale Weyl group scheme. The morphism constructed above when $M = T$,

$$(\chi_G^T)^\star J^G \to X_\star(T) \otimes_{\mathbb{Z}_X} \mathbb{G}_{m,\mathfrak{t}} = T \times_X \mathfrak{t},$$

is W-equivariant for the diagonal action on $T \times_X \mathfrak{t}$, (this can be checked fiberwise over X), and so the adjoint morphism

$$J^G \to (\chi_G^T)_\star(T \times_X \mathfrak{t}),$$

where $(\chi_G^T)_\star$ denotes the Weil restriction, factors through the W-fixed points

$$((\chi_G^T)_\star(T \times_X \mathfrak{t}))^W.$$

It is clear that this is a smooth group scheme, isomorphic to a torus over the regular locus, and in particular commutative everywhere. It follows from the definitions that there is a natural map

$$X_\star(T) \otimes_{\mathbb{Z}_X} (\chi_G^T)_\star \mathbb{G}_{m,\mathfrak{t}} \to (\chi_G^T)_\star (X_\star(T) \otimes_{\mathbb{Z}_X} \mathbb{G}_{m,\mathfrak{t}})$$

which can be checked to be an isomorphism étale-locally, so we have an alternative characterization

$$((\chi_G^T)_\star(T \times_X \mathfrak{t}))^W = (X_\star(T) \otimes_{\mathbb{Z}_X} (\chi_G^T)_\star \mathbb{G}_{m,\mathfrak{t}})^W.$$

Proposition 12.6 ([Ngô10, Corollaire 2.4.8]). *The morphism $J^G \to ((\chi_G^T)_\star(T \times_X \mathfrak{t}))^W$ is an isomorphism over the connected component containing the identity. In particular, we have a infinitesimal characterization*

$$\mathrm{Lie}(J^G) = (X_\star(T) \otimes_{\mathbb{Z}} (\chi_G^T)_\star \mathscr{O}_{\mathfrak{t}})^W.$$

12.4. D-twisting and the degree map on the Picard stack.

We now define the Picard stack associated to our Hitchin fibration and a degree map on its elements.

Definition 12.7. The *Picard stack* for $G \to X$ is the relative stack of J_D-bundles on X over $\mathscr{A}$:

$$\mathscr{P} := \mathrm{Bun}_{J_D/\mathscr{A}}(X) = \underline{\mathrm{Map}}_{X \times_k \mathscr{A}/\mathscr{A}}(X \times_k \mathscr{A}, B_{\mathscr{A}} J_D),$$

a stack over $\mathscr{A}$. Explicitly, it has functor of points associating to $S \to \mathscr{A}$ the groupoid of $(X \times \mathscr{A})$-morphisms

$$X \times_k S \to B_{\mathscr{A}}(J_D)$$

Since J is commutative, the contracted product indeed equips $\mathscr{P}$ with the structure of a Picard stack over $\mathscr{A}$ in the sense of [AGV73, Exp. XVIII, Def. 1.4.2].

Specializing, for any k-scheme R and $a \in \mathscr{A}(R)$ we have by formal base change relations (cf. [Hal22, Lemma 1.1.1]) a Picard R-stack

$$\mathscr{P}_a := a^\star \mathscr{P} = \underline{\mathrm{Map}}_{X_R/R}(X_R, B_{X_R} J_a),$$

where $J_a := a^\star J_D^G$. We write $\mathsf{P} \to \mathsf{A}$ for the restriction of $\mathscr{P}$ to the compatible locus of Definition 3.9.

Now we proceed to define the degree map. The following sheaf, constructed as in [CL12, §6], is its natural target – as in Proposition 2.11, the dual is a space of coinvariants mod torsion, or more canonically, H_0 mod torsion.

Definition 12.8. Denote by $\mathscr{X}_\star$ the constructible sheaf on A whose restriction to $\mathsf{A}_{M\text{-ell}}$ is the constant sheaf of value $H_0(X, \underline{X}_\star(M))_{tf}$ and such that the specialization maps corresponding to $M \subset L$ are those induced from the natural map

$$\underline{X}_\star(M) \to \underline{X}_\star(L).$$

Abusively, we write also $\mathscr{X}_\star(M) = H_0(X, \underline{X}_\star(M))_{tf}$, the dual of the relative character lattice, as in Proposition 2.11.

Given $a \in \mathsf{A}_M(R)$ (compatible) with a preimage $a_M \in \mathsf{A}_M(R)$, the D-twisted map over $\mathfrak{c}_{M,D}$ constructed in Prop. 12.4 pulls back along a_M to a morphism over X_R

$$\begin{aligned} J_a = a^\star J_D^G = a_M^\star (\chi_G^M)^\star J_D^G \longrightarrow\ & a_M^\star \left[(\underline{X}_\star(M) \otimes_{\mathbb{Z}_X} \mathbb{G}_{m,X}) \times_X \mathfrak{c}_{M,D} \right] \\ &= \underline{X}_\star(M) \otimes_{\mathbb{Z}_X} \mathbb{G}_{m,X} \\ &\cong M_R^{\mathrm{ab}}. \end{aligned}$$

Applying $B_{X_R}(-)$ gives a map

$$B_{X_R} J_a \to B_{X_R}(M_R^{\mathrm{ab}}),$$

and taking mapping stacks from X_R this becomes

$$\mathsf{P}_a \to \underline{\mathrm{Map}}_{X_R/R}(X_R, B_{X_R} M_R^{\mathrm{ab}}).$$

Taking degrees of M^{ab}-torsors then defines the degree, which is naturally valued in the dual space $\mathscr{X}_\star(M)$ of the global characters $X^\star(Z)$.

Lemma 12.9. *Given a torus Z over proper X, there is a homomorphism*

$$\deg : (B_X Z)(X) \to \mathrm{Hom}_{\mathbb{Z}}(X^\star(Z), \mathbb{Z}) \cong H_0(X, \underline{X}_\star(Z))_{tf}$$

natural in homomorphisms of tori and compatible with étale pullback.

Proof. Given a Z-torsor $\mathscr{E}$ on X and an X-character $\chi : Z \to \mathbb{G}_m$, one has a line bundle $\mathscr{E} \times^{T,\chi} \mathbb{G}_m$. This defines the first map, whose compatibilities are left to the reader. The isomorphism is Proposition 2.11. □

Remark 12.10. We want to emphasize this subtlety which has first arisen here: the degree of a regular centralizer element or torsor is naturally dual to rational characters, and so lives in the dual $\mathscr{X}_\star(M)$ of $X^\star(M)$ – this is *not* the integral space $X_\star(M)$, but rather Galois coinvariants for the geometric space of cocharacters. In the geometric part of this work, it was more convenient to tensor to $\mathbb{R}$, after which the distinction between $\underline{X}_\star(T)^\Gamma$ and $\underline{X}_\star(T)_{\Gamma,tf}$ vanishes. Here and in the forthcoming arithmetic calculations, it is necessary to work integrally.

Since degree is locally constant in families, we can compose with this lemma point-by-point over R, giving a composition $\deg_M : \mathsf{P}_a \to \underline{\mathrm{Map}}_{X_R/R}(X_R, B_{X_R} J_a) \to H_0(X, \underline{X}_\star(M))_{tf,R} = \mathscr{X}_\star(M)_R$ for each R-point a which lands inside the image of A_M inside A_G.

In particular, applying the map over the universal point inside A_M gives

$$\deg_M : \mathsf{P}|_{\mathsf{A}_M} \to \mathscr{X}_\star(M) \times \mathsf{A}_M. \tag{12.4.1}$$

The degree maps over each M-elliptic locus will be assembled into one map below, using the following compatibility between $\deg_M$ and $\deg_L$.

Proposition 12.11 ([CL12, Prop. 5.3.1]). *Let $M \subseteq L$ be an inclusion of semistandard Levis, and let $a \in \mathsf{A}_M \subseteq \mathsf{A}_L$. Then we have a commutative triangle*

$$\begin{tikzcd} \mathsf{P}_a \arrow[r, "\deg_M"] \arrow[dr, "\deg_L"'] & \mathscr{X}_\star(M) \arrow[d] \\ & \mathscr{X}_\star(L) \end{tikzcd}$$

where the vertical map is the specialization map of 12.8.

Proof. The proof is exactly as in *loc. cit.* By Prop. 2.15 we can choose parabolics P for L and Q for M such that $Q \subseteq P$. The compatibilities between the various Chevalley maps imply that the homomorphisms $\chi_P^\star(\chi_G^L)^\star J^G|_{\mathfrak{p}^{\mathrm{reg}}} \to I^P|_{\mathfrak{p}^{\mathrm{reg}}} \to \underline{X}_\star(L) \otimes_{\mathbb{Z}_X} \mathbb{G}_{m,\mathfrak{c}_L}$ of Prop. 12.4 are compatible with that for M, Q. Applying B preserves that commutativity. Likewise, naturality of the degree evaluation of Lem. 12.9 and compatibility of the maximal torus quotients for $M \subset L$ implies that the maps $(B_X L^{\mathrm{ab}})(X) \to \mathscr{X}_\star(L)$ and $(B_X M^{\mathrm{ab}})(X) \to \mathscr{X}_\star(M)$ commute with the natural projections. □

Proposition 12.12 ([CL12, Prop. 6.4.1]). *The degree maps of (12.4.1) glue to a unique map*

$$\deg : \mathsf{P} \to \mathscr{X}_\star$$

over A.

Proof. Uniqueness is clear. For existence, note that the universal $\deg_M$ from (12.4.1) pulls back along χ_G^M to give

$$(\chi_G^M)^\star \mathsf{P} \to \mathscr{X}_\star(M)_{\mathsf{A}_M},$$

or by adjunction

$$\mathsf{P} \to (\chi_G^M)_\star(\mathscr{X}_\star(M)_{\mathsf{A}_M}).$$

Assembling these over $M \supset T$ gives a map

$$\deg : \mathsf{P} \to \bigoplus_M (\chi_G^M)_\star(\mathscr{X}_\star(M)_{\mathsf{A}_M}).$$

The sheaf $\mathscr{X}_\star$ can be identified with the subsheaf of this direct sum consisting of compatible elements, i.e., the sheaf whose stalk at a point $a \in \mathsf{A}_{M\text{-ell}}$ consists of elements $\bigoplus_{L \supseteq M} p_L$ such that $p_L \mapsto p_{L'}$ under $\mathscr{X}_\star(L) \to \mathscr{X}_\star(L')$. The compatibility of Proposition 12.11 implies that deg lands within this subsheaf. $\square$

Definition 12.13. Denote by $\mathsf{P}^1 = \ker_{\mathsf{A}}(\deg)$ the fiberwise kernel of the degree map. That is, over $a \in \mathsf{A}_{M\text{-ell}}$ it is the substack of BJ_a-torsors whose degree when paired with each character in $X^\star(M)$ is *zero.*

12.5. Action on Hitchin fibers.

There is an action of $\mathscr{P}$ on $\mathscr{M}$ over $\mathscr{A}$, which restricts to an action of P on M over A.

Proposition 12.14 ([Ngô10, §4.3], [CL12, §5.5]). *Let $a \in \mathscr{A}$ and $m \in \mathscr{M}_a$. Then there is a map $J_a \to m^\star[I_D^G/G] = \mathrm{Aut}(m)$, where we have interpreted m as a map $X \to [\mathfrak{g}_D/G]$.*

Proof. The map $\chi^\star J \to I$ is $\mathbb{G}_m \times G$-equivariant, so descends to a homomorphism $[(\chi_{G,D}^\star J_D^G)/G] \to [I_D^G/G]$ over $[\mathfrak{g}_D/G]$. Pulling back along $m : X \to [\mathfrak{g}_D/G]$ gives a map

$$J_a = m^\star[(\chi_{G,D}^\star J_D^G)/G] \to m^\star[I_D^G/G].$$

The latter is the group of automorphisms of $m = (\mathscr{E}, \theta)$: the automorphisms of the torsor $\mathscr{E}$ are given by $[\mathscr{E}]^\star[G/G]$, where $[\mathscr{E}] : X \to BG$ is the classifying map. The condition of commuting with θ restricts to $m^\star[I_D^G/G]$. $\square$

So we have a map $J_a \to \mathrm{Aut}(m)$. This induces an action of $\mathscr{P}_a$ on $\mathscr{M}_a$ by associated product of torsors:

$$\mathscr{F} \in BJ_a : m \mapsto m' := (\mathscr{F} \times^{J_a} m).$$

The new Higgs bundle m' lies in the same Hitchin fiber m because they are isomorphic locally over the curve, so their characteristic polynomials agree everywhere.

12.6. The P-action and ξ-stability.

Let $(m, t) \in \mathsf{M}$ and P a semistandard parabolic subgroup. A P-reduction of (m, t) is exactly the data of a factorization

$$\begin{tikzcd} X \arrow[r, "m_P"] \arrow[dr, "m"'] & {[\mathfrak{p}_D/P]} \arrow[d, "i_P"] \\ & {[\mathfrak{g}_D/G]} \end{tikzcd}$$

such that the composition $a_P : X \xrightarrow{m_P} [\mathfrak{p}_{G,D}/P] \to \mathfrak{c}_P = \mathfrak{c}_M$ satisfies $a_P(\infty) = \chi_M^T(t)$, i.e., lands inside A_M.

Proposition 12.15 (cf. [CL12, 5.5.1]). *Let $(a, t) \in \mathsf{A}_{M\text{-}ell}$ and $(m, t) \in \mathsf{M}_a$. Then for any $p \in \mathsf{P}$, we have*

$$C_{p \cdot m} = C_m + \deg_M(p).$$

In particular, the degree 0 substack P^1 preserves the ξ-semistable substack M^ξ for any ξ.

Proof. Recall that $J_a := a^\star J_D^G$ along $a : C \to \mathfrak{c}_{G,D}$. For any parabolic $P \supset M$ let $m_P : X \to [\mathfrak{p}_D/P]$ the unique P-reduction (that is, (e,P)-reduction) of (m,t).

From 12.2, we have a factorization

$$[(\chi_G^\star J_D^G)|_{\mathfrak{p}_D}/P] \to [I_D^P/P] \to i_P^\star[I_D^G/G]. \tag{12.6.1}$$

Note that $[(\chi_G^\star J_D^G)|_{\mathfrak{p}_D}/P] \cong i_P^\star[\chi_{G,D}]^\star J_D^G$ by the commutativity of

$$\begin{tikzcd}
\mathfrak{p}_D \arrow[r, "i_P"] \arrow[d] & \mathfrak{g}_D \arrow[d] \arrow[dd, bend left, "\chi_G"] \\
{[\mathfrak{p}_D/P]} \arrow[r, "i_P"] \arrow[d, "{[\chi_{P,D}]}"'] & {[\mathfrak{g}_D/G]} \arrow[d, "{[\chi_{G,D}]}"'] \\
\mathfrak{c}_{P,D} \arrow[r] & \mathfrak{c}_{G,D}
\end{tikzcd}.$$

Pulling back (12.6.1) along m_P gives a factorization

$$a^\star J_D^G = a_P^\star[\chi_{G,D}]^\star J_D^G = m_P^\star i_P^\star[\chi_{G,D}]^\star J_D^G \to m_P^\star[I_D^P/P] \to m_P^\star i_P^\star[I_D^G/G] = \mathrm{Aut}(m).$$

The middle term $m_P^\star[I_D^P/P]$ is canonically $\mathrm{Aut}(m_P)$. We have factored $J_a \to \mathrm{Aut}(m)$ through $J_a \to \mathrm{Aut}(m_P)$, so the first map defines an action of J_a-torsors on P-reductions of Hitchin triples compatible with the action on the triples. From this compatibility, the twisted P-Hitchin triple $p \cdot m_P := p \times^{J_a} m_P$ defines a P-reduction of $p \cdot m$. Indeed, it is the unique (e,P)-reduction because of the compatibility of the P-action with the characteristic polynomial map: that is, $p \times^{J_a} m_P$ looks locally like m_P, hence has characteristic polynomial which is equal to a_P over a cover, and so $a_P(\infty) = \chi_M^T(t)$ is preserved.

Finally, we can compute for any $\mu \in X^\star(P)$

$$\deg_P(p \times^{J_a} m_P)(\mu) = \deg(m_P)(\mu) + \deg_M(p).$$

This is equivalent to commutativity of the following diagram, in which all mapping stacks are taken in the sense above:

$$\begin{tikzcd}
\underline{\mathrm{Map}}(X, B_X J_a) \times \underline{\mathrm{Map}}(X, [\mathfrak{p}_D/P]) \arrow[r] \arrow[d] \arrow[rr, bend left, "{(\deg_M, \deg_P)}"] & \underline{\mathrm{Map}}(X, B_X M^{\mathrm{ab}})^2 \arrow[r] \arrow[d] & \mathscr{X}_\star(M)^2 \arrow[d, "\Sigma"] \\
\underline{\mathrm{Map}}(X, [\mathfrak{p}_D/P]) \arrow[r] \arrow[rr, bend right, "\deg_P"'] & \underline{\mathrm{Map}}(X, B_X M^{\mathrm{ab}}) \arrow[r] & \mathscr{X}_\star(M)
\end{tikzcd}.$$

Here $M^{\mathrm{ab}} = M/D(M)$, the horizontal maps are the compositions used to define the degree maps on Hitchin triples and J_a-torsors, and we have used the fact that the degree of an M-torsor factors through the associated M^{ab}-torsor.

The square on the left commutes before taking $\underline{\mathrm{Map}}(X, -)$ by functoriality of the associated product. But now the middle vertical map is the associated product, which *is* the group operation on the Picard stack, so the commutativity of the right square is just the statement that the map of Lemma 12.9 is a homomorphism. $\square$

12.7. The morphism $\mathbb{X}_\star \to \pi_0(\widetilde{\mathscr{P}})$.

The starting point for Ngô's endoscopic decomposition of the Hitchin fibration is an analysis of the fiberwise component group of $\mathscr{P}$, or more explicitly, its pullback $\widetilde{\mathscr{P}}$ to $\widetilde{\mathscr{A}}$.

Proposition 12.16 ([Ngô06, 6.2]). *There exists a constructible sheaf of groups $\pi_0(\mathscr{P})$ over $\mathscr{A}$ whose fiber over $a \in \mathscr{A}$ is $\pi_0(\mathscr{P}_a)$.*

A crucial (and defining) property of $\mathscr{A}^{G\text{-ell}}$ in Ngô's work is that it is exactly the locus on which $\pi_0(\mathscr{P}_a)$ is finite, so the $\mathscr{P}_a$-actions on cohomology of Hitchin fibers factor though finite groups. The restriction to P^1 will allow this to extend to A. Recall the definition of the intrinsic monodromy invariant $W_{\tilde{a}} \subset W(\infty)$ from Definition 3.17. The essential result on $\pi_0(\mathscr{P})$ is the following.

DEFINITION 12.17. Define $\mathbb{X}_\star := \underline{X}_\star(T)(\infty)_{\mathsf{A}}$, the constant group scheme over A. Define $\mathscr{Y}$ to be the quotient of $\mathbb{X}_\star$ over A whose fiber over $\tilde{a} = (a,t)$ is $(\mathbb{X}_\star)_{W_{\tilde{a}}}$.

PROPOSITION 12.18 ([Ngô10, §5.5], [Ngô06, 6.3–6.7]). *There exists a canonical commutative diagram of fiberwise-surjective maps*

$$\begin{tikzcd} \mathbb{X}_\star \arrow[r] \arrow[dr] & \mathscr{Y} \arrow[d] \\ & \pi_0(\widetilde{\mathscr{P}}) \end{tikzcd}.$$

PROOF SKETCH. We recall the construction of the map $(\mathbb{X}_\star)_{\mathbb{W}_{\tilde{a}}} \to \pi_0(\widetilde{\mathscr{P}}_a)$ from various work of Ngô. Assume a is a geometric point with residue field K. Define $\widetilde{\mathscr{P}}'_a$ to be the stack of J_a^0-torsors, where J_a^0 is the connected component of the identity. Generically, a is regular semisimple, so the cokernel $\pi_0(J_a) = \operatorname{coker}(J_a^0 \hookrightarrow J_a)$ is supported on a proper closed subset. We therefore have an exact sequence

$$H^0(X_K, \pi_0(J_a)) \to H^1(X_K, J_a^0) \to H^1(X_K, J_a) \to H^1(X_K, \pi_0(J_a)) = 0.$$

The middle H^1 terms are the groups of isomorphism classes of J_a^0-torsors (resp. J_a-torsors), so the map $\widetilde{\mathscr{P}}'_a \to \widetilde{\mathscr{P}}_a$ is essentially surjective on geometric points, hence surjective on π_0.

We then make a second reduction to the Néron model. Explicitly, write $U_a \subset X_K$ for the open on which a is regular semisimple and V_a its preimage in the cameral cover $\widetilde{X}_a$, so that $J_a^0|_{U_a} = J_a|_{U_a}$ is a torus, canonically isomorphic to T on a complete formal neighborhood $\hat{\mathscr{O}}_{\infty_K}$ of ∞_K. The theory of connected Néron models implies that any torus Z on U_a, i.e., $J_a^0|_{U_a}$, extends to a universal connected group scheme $N^0(Z)$ over X_K. We write $N_a^0 := N^0(J_a|_{U_a})$, so that the Néron mapping property

$$J_a^0 \to N_a^0$$

extending the identity on U_a. Again, this is a generically an isomorphism, so the long exact sequence in cohomology implies a surjection $H^1(X_K, J_a^0) \to H^1(X_K, N_a^0)$, inducing a corresponding surjection on connected components of Picard stacks. But, unlike above, the defect to injectivity $H^0(X_K, N_a^0/J_a^0)$ is an affine part coming from the normalization, in particular, is connected. It therefore maps into $(\widetilde{\mathscr{P}}'_a)^0$, so the map on connected components is also injective.

Finally, Ngô defines a morphism $(\mathbb{X}_\star)_{W_{\tilde{a}}} \to \pi_0(\mathrm{Bun}_{N_a^0})$ using the formalism of Kottwitz invariants. We consider additive functors $F : \mathsf{Tori}_{U_a} \to \mathsf{Ab}$ which are

(1) right-exact and
(2) satisfy $F(Z) \cong \mathbb{Z}$ naturally for induced tori Z.

It is easy to see that

$$A : Z \mapsto [\underline{X}_\star(Z)(\infty_K)]_{\pi_1(U_a, \infty_K)}$$

satisfies these conditions. Kottwitz's lemma says that these properties determine the functor uniquely, in the following sense. The proof proceeds by writing an arbitrary torus as a quotient of induced tori.

LEMMA 12.19 ([Kot85, 2.2], [Ngô06, 6.5]). *Let $B : \mathsf{Tori}_{U_a} \to \mathsf{Ab}$ be any additive functor satisfying (1) and (2) above. Then $A \cong B$, with natural isomorphisms*

$$\mathrm{Hom}(\underline{X}_\star(-)_{\infty_K}, B) \cong \mathrm{Hom}(A, B)$$

and

$$\mathrm{Hom}(\underline{X}_\star(-)_{\infty_K}, B) \cong B(\mathbb{G}_{m,U_a}).$$

Moreover, $B(\mathbb{G}_{m,U_a})$ is free of rank 1, and $\phi : A \to B$ is an isomorphism if and only if it corresponds to a generator of this free group.

Ngô shows that $B : Z \mapsto \pi_0(\mathrm{Bun}_{N^0(Z)})$ satisfies the necessary conditions, so a choice of generators for $\pi_0(\mathrm{Bun}_{N^0(\mathbb{G}_{m,U_a})})$ (that is, a line bundle on X) induces a morphism

$$\underline{X}_\star(J_a)(\infty_K) \to \pi_0(\mathrm{Bun}_{N_a^0}) = \pi_0(\widetilde{\mathscr{P}}_a') \to \pi_0(\widetilde{\mathscr{P}}_a)$$

which factors through the $\pi_1(U_a, \infty)$-coinvariants.

Now, the restricted cameral map $V_a \to U_a$ is the pullback of $\mathfrak{t}_D^{\mathrm{reg}} \to \mathfrak{c}_D^{\mathrm{reg}}$, and the isomorphism 12.2 implies that

$$J_a \times_{U_a} V_a \cong J_D^G \times_{\mathfrak{c}_D^{\mathrm{reg}}} V_a \cong T \times_X V_a.$$

This isomorphism identifies the $\pi_1(U_a, \infty)$-action on J_a with the monodromy action of Definition 3.17, cf. [CL12, p. 1680]. Our choice of point $\widetilde{\infty}$ in the cameral cover (via t) therefore determines an isomorphism $\underline{X}_\star(J_a)|_\infty \cong \underline{X}_\star(T)|_\infty$ such that $\pi_1(U_a, \infty)$ acts on the composition through W_a. So Ngô's map is the same data as a map $(\mathbb{X}_\star)_{W_a} \to \pi_0(\widetilde{\mathscr{P}}_a)$, as desired. ☐

The morphism $\mathbb{X}_\star \to \pi_0(\widetilde{\mathscr{P}})$ constructed above is not very explicit, passing through Kottwitz's yoga of induced tori and also through the surjection $\pi_0(\widetilde{\mathscr{P}}') \to \pi_0(\widetilde{\mathscr{P}})$, whose kernel is inexplicit in general.

Drinfeld proposed a functorial Beauville–Laszlo construction of a refinement $\mathbb{X}_\star \to \widetilde{\mathscr{P}}'$ (or to $\widetilde{\mathscr{P}}$, more weakly) which induces Ngô's map above upon passing to components. Functoriality of this construction also justifies why the maps constructed above geometric-fiber-by-geometric-fiber interpolate over A.

Proposition 12.20 ([CL12, §7.3], [Ngô06, 6.8]). *There exists an $\widetilde{\mathscr{A}}$-morphism $\mathbb{X}_\star \to \widetilde{\mathscr{P}}$ (in fact, to $\widetilde{\mathscr{P}}'$) refining the map above.*

Proof. Let $(a, t) \in \widetilde{\mathscr{A}}(S)$. The data of t determines an isomorphism $J_a|_{\infty_S} \cong T|_{\infty_S}$ by [Ngô10, 2.1.1]. Since χ_G^T is étale on the regular locus and $a(\infty) = \chi_G^T(t)$ is regular, formally-étale lifting extends t to a section of $\mathfrak{t}^{\mathrm{reg}}$ on the complete neighborhood $\mathrm{Spec}(\hat{\mathscr{O}}_{\infty_S})$ of $\infty_{\widetilde{\mathscr{A}}}$ (where $\hat{\mathscr{O}}_{\infty_S} := \hat{\mathscr{O}}_{X_S, \infty_S}$), and thus a canonical isomorphism

$$J_a \times_X \hat{\mathscr{O}}_{X_S, \infty_S} \cong T \times_X \hat{\mathscr{O}}_{X_S, \infty_S}.$$

Since ∞ is a geometric point (though we may choose it to live over a k-point), the restrictions $T|_\infty$ and $T|_{\hat{\mathscr{O}}_\infty}$ are split. Writing z_{∞_S} for a uniformizer at ∞_S, we can thus build for each $\lambda \in X_\star(T)(\infty_{\mathsf{A}})$ a J_a^0-torsor (or indeed a J_a^0-torsor) by modifying the trivial torsor by $z_{\infty_S}^\lambda \in J_a(F_{\infty_S})/J_a(\hat{\mathscr{O}}_{\infty_S})$, that is, by gluing the trivial torsors on

$$\mathrm{Spec}(\hat{\mathscr{O}}_{\infty_S}) \quad \text{and the complement} \quad X_S^\circ = X_S \setminus \{\infty_S\}$$

along the automorphism given by multiplication by $z_{\infty_S}^\lambda \in T(F_{\infty_S})$. As in [Ngô06, 6.8], the induced map into $\pi_0(\widetilde{\mathscr{P}})$ corresponds to the one above, and in particular is independent of d. ☐

Remark 12.21. We used above that the data of t (a point in the cameral cover over ∞) determined the isomorphism $J_a|_\infty \cong T|_\infty$. Likewise, given a Γ-splitting cover $\pi : X' \to X$, a choice of ∞' lying over ∞ (i.e., a *pointed* twisting datum) determines an isomorphism $T_\infty \cong T_0$, the maximal torus in the split form. We obtain a homomorphism $X_\star(T_0) \to \widetilde{\mathscr{P}}$ which depends upon ∞', but the argument of [Ngô06, 6.8] shows that at the level of $\pi_0(\widetilde{\mathscr{P}}')$, the induced map is well-defined. Anyways, it corresponds to the canonically constructed morphism above.

12.8. The component group $\pi_0(\mathsf{P}^1)$ and lattice actions.

The previous section defined a canonical surjective morphism $\mathbb{X}_\star = \underline{X}_\star(T)(\infty)_{\widetilde{\mathscr{A}}} \to \pi_0(\widetilde{\mathscr{P}})$, which we can of course restrict to the compatible locus. On the other hand, on that locus we have previously defined a degree morphism $\mathsf{P} \to \mathscr{X}_\star$, which factors through $\pi_0(\mathsf{P})$ because $\mathscr{X}_\star$ is fiberwise discrete. Note the asymmetry, not visible in the split case, that the former map has as domain the **split** or **geometric** cocharacter group, while the latter has codomain valued in the **rational** data $\mathscr{X}_\star(M) = H_0(X, \underline{X}_\star(M))_{tf}$. The composition $\underline{X}_\star(T)(\infty)_{\mathsf{A}} \to \mathscr{X}_\star$ is easy to understand.

Lemma 12.22. *The composition* $\mathbb{X}_\star \to \pi_0(\mathsf{P}) \to \mathscr{X}_\star$ *over* $\tilde{a} = (a,t) \in \mathsf{A}_{M\text{-}ell}$ *is the natural map*

$$\underline{X}_\star(T)(\infty) \to \underline{X}_\star(M)(\infty) \to \underline{X}_\star(M)(\infty)_{\pi_1(X,\infty),\, tf}$$

dual to $X^\star(M) \to \underline{X}^\star(M)(\infty) \to \underline{X}^\star(T)(\infty)$.

Proof. Fix (a,t) compatible, M-elliptic, and work fiberwise. We can compute the composition as $\mathbb{X}_\star \to \mathsf{P} \to \mathscr{X}_\star$ making use of the explicit Beauville–Laszlo patching construction. This reduces the problem to computing the M-degree of the J_a-torsor j_λ with Beauville–Laszlo data z_∞^λ at ∞ (and trivial data everywhere else on X). That is, for each character $\mu \in X^\star(M)$, $\deg(j_\lambda)(\mu) := \deg(j_\lambda \times^{J_a} \mathbb{G}_{m,X})$ is the pushforward along

$$J_a \to \underline{X}_\star(M) \otimes_{\mathbb{Z}_X} \mathbb{G}_{m,X} \to \mathbb{G}_{m,X}.$$

On the disk $\widehat{\mathscr{O}}_\infty$, $J_a \cong T$ by choice of uniformizer z_∞. As in Definition 6.8 and Prop 6.10, $\deg(j_\lambda)(\mu)$ is given by the valuation of $(\mu \circ \lambda)(z_\infty)$. This is the map in the statement of the Lemma because the coinvariants mod torsion is dual to $X^\star(M)$ as in Proposition 2.11. □

We can now introduce a lattice whose action will control the fine s-decomposition below.

Definition 12.23. Define

$$\mathscr{Y}^1 := \ker(\mathscr{Y} \to \mathscr{X}_\star)$$

and

$$\mathbb{X}_\star^1 := \ker(\mathbb{X}_\star \to \mathscr{X}_\star).$$

Then we have a commutative triangle of surjective maps restricting those previously

$$\begin{tikzcd} \mathbb{X}_\star^1 \arrow[r] \arrow[dr] & \mathscr{Y}^1 \arrow[d] \\ & \pi_0(\mathsf{P}^1) \end{tikzcd}.$$

For each semistandard Levi M, define the *degree-zero lattice*

$$\mathbb{X}_{\star,M}^1 := \ker(\mathbb{X}_\star = \underline{X}_\star(T)(\infty) \to \underline{X}_\star(M)(\infty)_{\pi_1(X,\infty),tf} = \mathscr{X}_\star(M)).$$

This is the fiber of $\mathbb{X}_\star^1$ over each M-elliptic point a.

It is immediate from the definitions that for any pair $M \subset L$, one has

$$\mathbb{X}_{\star,M}^1 \subseteq \mathbb{X}_{\star,L}^1.$$

Proposition 12.24. *For semistandard Levi M, the dual group is*

$$\widehat{\mathbb{X}_{\star,M}^1} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{X}_{\star,M}^1, \overline{\mathbb{Q}}_l^\times) \cong \hat{T}/Z_{\widehat{M}}^{\Gamma,0},$$

where

$$Z_{\widehat{M}}^{\Gamma,0} := (Z_{\widehat{M}}^{\Gamma})^0 = ((Z_{\widehat{M}}^0)^\Gamma)^0.$$

In particular, each finite-order character of $\mathbb{X}_{\star,M}^1$ *is a torsion element of this dual group. These identifications are natural in the Levi: if* $M \subseteq L$ *then the map*

$$\mathbb{X}_{\star,M}^1 \hookrightarrow \mathbb{X}_{\star,L}^1$$

is dual to the natural quotient

$$\hat{T}/Z_{\widehat{L}}^{\Gamma,0} \twoheadrightarrow \hat{T}/Z_{\widehat{M}}^{\Gamma,0}.$$

Proof. One has a short-exact sequence

$$0 \longrightarrow \mathbb{X}_{\star,M}^1 \longrightarrow \mathbb{X}_\star \longrightarrow \mathscr{X}_\star(M) \longrightarrow 0\ .$$

By definition, $\mathscr{X}_\star(M)$ is dual to $X^\star(M)$. The corresponding subtorus is $(Z_{\widehat{M}}^\Gamma)^0 = ((Z_{\widehat{M}}^0)^\Gamma)^0$. The naturality is clear. □

Note the order of operations! $Z^{\Gamma,0}_{\widehat{M}} \neq (Z^0_{\widehat{M}})^\Gamma$.

A sublattice of $\mathbb{X}^1_{\star,M}$ will induce the *coarse* s-decomposition below, which will be central in what follows. We have a short exact sequence of tori

$$1 \longrightarrow T \cap D(M) \longrightarrow T \longrightarrow M^{\mathrm{ab}} \longrightarrow 1$$

and taking $\underline{X}_\star(-)(\infty)$ gives a short exact sequence of $(W_{M,\infty} \rtimes \Gamma)$-modules, which we denote

$$0 \longrightarrow \mathbb{X}_{\star,D(M)} \longrightarrow \mathbb{X}_\star \longrightarrow \mathbb{X}_{\star,M^{\mathrm{ab}}} \longrightarrow 0\ . \tag{12.8.1}$$

Since the M-degree map factors through M^{ab}, it follows that $\mathbb{X}_{\star,D(M)} \subseteq \mathbb{X}^1_{\star,M}$. Direct computation describes this dually.

Proposition 12.25. *The inclusion* $\mathbb{X}_{\star,D(M)} \subseteq \mathbb{X}^1_{\star,M}$ *is dual to the surjection*

$$\hat{T}/(Z^\Gamma_{\widehat{M}})^0 \to \hat{T}/Z^0_{\widehat{M}}.$$

The distinction between these lattices is only visible in the non-split setting, as follows. Since $W_{M,\infty}$ acts trivially on $\mathbb{X}_{\star,M^{\mathrm{ab}}} = \underline{X}_\star(M^{\mathrm{ab}})(\infty)$, the monodromy subgroup $W_a \subseteq W_{M,\infty} \rtimes \Gamma$ of Definition 3.17 acts on it through Γ (onto which W_a surjects, since it is a graph). So the coinvariants by W_a and Γ, and the map defining $\mathbb{X}^1_\star$ is the diagonal map in the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{X}_{\star,D(M)} \arrow[r] \arrow[d, two heads] & \mathbb{X}_\star \arrow[r] \arrow[d, two heads, "q_T"] \arrow[dr, "\psi_a"] & \mathbb{X}_{\star,M^{\mathrm{ab}}} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & (\mathbb{X}_{\star,D(M)})_{W_a,\,tf} \arrow[r] & (\mathbb{X}_\star)_{W_a,\,tf} \arrow[r] & (\mathbb{X}_{\star,M^{\mathrm{ab}}})_{W_a,\,tf} \arrow[r] & 0
\end{tikzcd}\ .$$

By a simple diagram chase on the 3×3 commutative square containing these maps and their kernels, one computes that

$$(\mathbb{X}^1_\star)_a = \ker(\psi_a) = \mathbb{X}_{\star,D(M)} + \ker(q_T),$$

where $\ker(q_T)$ is the set of elements whose coinvariant class is torsion. Of course, in the split case, the W_a-action on $\underline{X}_\star(M^{\mathrm{ab}})(\infty) = X_\star(M^{\mathrm{ab}})$ is trivial, so since the lattice is torsion-free the second term is invisible.

Remark 12.26. In order to justify the choice that the degree map should be valued in *rational* cocharacters while Ngô's map should be defined on the *geometric* cocharacters, as well as to illustrate the discrepancy between the lattices $\mathbb{X}_{\star,D(M)} \subseteq \mathbb{X}^1_{\star,M}$, it may be helpful to recall the case when $G = T$ is an X-torus. Assume G splits over a geometrically irreducible Γ-Galois étale cover X'/X. For example, we could take the non-split rank 1 torus $G = U_{X'/X}(1) = \mathrm{Res}^1_{X'/X}\,\mathbb{G}_m$ along a quadratic cover. By commutativity, the Chevalley map $\chi : \mathfrak{g}_D \to \mathfrak{c}_D$ is an isomorphism and

$$\mathscr{M} \cong \underline{\mathrm{Map}}(X, [\mathfrak{g}_D/G]) = \underline{\mathrm{Map}}(X, \mathfrak{g}_D \times BG) = \underline{\mathrm{Map}}(X, \mathfrak{g}_D) \times \mathrm{Bun}_G,$$

with the Hitchin map the projection onto the first factor. We then have $\mathscr{M}_a \cong \mathscr{P}_a \cong \mathrm{Bun}_G$. The components of Bun_G are easy to compute, e.g. by the sledgehammer of [Hei10, Theorem 6]:

$$\pi_0(\mathrm{Bun}_G) \cong \pi_1(G_{\overline{\eta}})_\Gamma \cong (\underline{X}_\star(T)(\infty))_\Gamma.$$

Note that this can have torsion: for $G = U_{X'/X}(1)$ it is $\mathbb{Z}/2$. By the Beauville–Laszlo construction of Proposition 12.20, Ngô's map is easily seen to induce an isomorphism as above. On the other hand, the natural degree map lands inside the rational cocharacter space and necessarily cannot see this torsion. For $G = U(1)$, the only possible degree map $\mathbb{Z}/2 \to \mathbb{Z}$ is the zero map, reflecting the fact that a purely imaginary line bundle has numerical degree 0.

At the end of the day, this finite number of components causes us no issue. It is only split tori which give rise to a lattice of components, just as in automorphic theory it is only split tori which cause noncompactness issues.

We can describe more fully the fiber of $\mathscr{Y}^1$ at $a \in \mathsf{A}_{M\text{-ell}}$. By definition, it is

$$\mathscr{Y}^1_a = \ker((\mathbb{X}_\star)_{W_a} \to (\mathbb{X}_{\star,M^{\mathrm{ab}}})_{\pi_1(X,\infty),\, tf} = (\mathbb{X}_{\star,M^{\mathrm{ab}}})_{W_a,\, tf}).$$

We have a similar diagram

$$\begin{tikzcd}
 & (\mathbb{X}_{\star,D(M)})_{W_a} \arrow[r, "i_{M,W_a}"] \arrow[d, two heads] & (\mathbb{X}_\star)_{W_a} \arrow[r] \arrow[d, two heads, "q_{T,W_a}"] \arrow[dr, "\psi_a"] & (\mathbb{X}_{\star,M^{\mathrm{ab}}})_{W_a} \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & (\mathbb{X}_{\star,D(M)})_{W_a,\, tf} \arrow[r] & (\mathbb{X}_\star)_{W_a,\, tf} \arrow[r] & (\mathbb{X}_{\star,M^{\mathrm{ab}}})_{W_a,\, tf} \arrow[r] & 0
\end{tikzcd}.$$

Definition 12.27. For $a \in \mathsf{A}_{M\text{-ell}}$, call $\mathscr{Y}^1_a$ the *fine symmetry group*. Define the *coarse*, or *endoscopic*, *symmetry group* to be $\Sigma_a := \mathrm{im}(i_{M,W_a})$, and the *residual torsion group* to be $\Xi_M := (\mathbb{X}_{\star,M^{\mathrm{ab}}})_{\Gamma,tors}$.

The snake lemma then describes $\mathscr{Y}^1_a$ as a canonical extension

$$0 \longrightarrow \Sigma_a \longrightarrow \mathscr{Y}^1_a \longrightarrow \Xi_M \longrightarrow 0\ . \tag{12.8.2}$$

In the split case, W_a acts trivially, so $(\mathbb{X}_{\star,M^{\mathrm{ab}}})_{W_a}$ is torsion-free and the right term vanishes, so $\mathscr{Y}^1_a = \mathrm{im}(i_{M,W_a})$.

We record the dual short exact sequence of (12.8.2), whose description is an elementary computation.

Proposition 12.28. *Taking* $\mathrm{Hom}(-, \overline{\mathbb{Q}}_l^\times)$ *and observing that* W_a *acts on the center of* $\widehat{M}$ *through* Γ*, the dual of (12.8.2) is given by*

$$\begin{tikzcd}
1 \arrow[r] & \widehat{\Xi}_M \arrow[r] \arrow[d, equal] & \widehat{\mathscr{Y}^1_a} \arrow[r] \arrow[d, equal] & \widehat{\Sigma}_a \arrow[r] \arrow[d, equal] & 1 \\
1 \arrow[r] & \pi_0((Z^0_{\widehat{M}})^\Gamma) \arrow[r] & \hat{T}^{W_a}/Z^{\Gamma,0}_{\widehat{M}} \arrow[r] & \hat{T}^{W_a}/(Z^0_{\widehat{M}})^\Gamma \arrow[r] & 1
\end{tikzcd}$$

The dual residual torsion group is equal to

$$\widehat{\Xi}_M = \frac{(Z^0_{\widehat{M}})^\Gamma}{Z^{\Gamma,0}_{\widehat{M}}}$$

measuring the failure of the Γ*-invariants to be connected.*

Ultimately, the structural feature we need from the full $\mathscr{Y}^1$ is that the map $\mathbb{X}^1_\star \to \pi_0(\mathsf{P}^1)$ is surjective and factors through a finite quotient—thus, the restriction from P to P^1 is enough to ensure finite groups of components, so that we can obtain a well-behaved isotypic decomposition for the action of $\pi_0(\mathsf{P}^1)$ beyond the G-elliptic locus.

Lemma 12.29 (cf. [CL10, Lemme 8.3.3]). *For every Levi* M *and* $a \in \mathsf{A}_{M\text{-ell}}$,

(a) $(\mathbb{X}_{\star,D(M)})_{W_a}$ *is finite, and*
(b) the fiber of $\mathscr{Y}^1$ *over* a *is finite.*

Proof. The first claim is (b) of Lemma 3.25. For the second claim, by the description (12.8.2) it suffices to show the sub and quotient are finite. Part (a) implies $\mathrm{im}(i_{M,W_a})$ is finite, and the quotient is the torsion in a finitely generated $\mathbb{Z}$-module, hence is finite. □

The specialization maps between the $\mathscr{Y}^1_a$ above are also natural.

Proposition 12.30. *Let* $a \in \mathsf{A}_{L\text{-ell}}$ *and* $a' \in \mathsf{A}_{M\text{-ell}}$ *be a specialization of* a*. Then* $M \subset L$ *and* $W_{a'} \subset W_a$*. Moreover, the specialization map* $\mathscr{Y}^1_{a'} \to \mathscr{Y}^1_a$ *induces a morphism of short exact sequences*

$$\begin{tikzcd}
0 \arrow[r] & \Sigma_{a'} \arrow[r] \arrow[d] & \mathscr{Y}^1_{a'} \arrow[r] \arrow[d] & \Xi_M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Sigma_a \arrow[r] & \mathscr{Y}^1_a \arrow[r] & \Xi_L \arrow[r] & 0
\end{tikzcd},$$

where the morphism $\Sigma_{a'} \to \Sigma_a$ is induced from the natural map

$$(\mathbb{X}_{\star,D(M)})_{W_{a'}} \to (\mathbb{X}_{\star,D(L)})_{W_a}.$$

Proof. The fact that $M \subset L$ is the content of §3, with $W_{a'} \subset W_a$ given by Proposition 3.24. The last claim is immediate from the definitions and the computation of $\mathscr{Y}^1_a$ above. $\square$

12.9. The coarse and fine s-decompositions.

Definition 12.31. For each semistandard M, define the M-parabolic open

$$U_M = \bigcup_{L \supset M} \mathsf{A}_{L\text{-ell}} \subset \mathsf{A}_G.$$

Denote $f^{\xi, M\text{-par}} : \mathsf{M}^{\xi, M\text{-par}} \to U_M$ the restriction of the Hitchin map to U_M. Of course, if $M = G$ then M-par $= G$-ell.

Collating the properties of $\mathscr{Y}^1$ from the previous section gives the following.

Lemma 12.32. *Over U_M, the homomorphisms of sheaves $(\mathbb{X}_{\star,D(M)})_{U_M} \subseteq \mathbb{X}^1_\star \to \mathscr{Y}^1 \to \pi_0(\mathsf{P}^1)$ induce group homomorphisms*

$$\mathbb{X}_{\star,D(M)} \subseteq \mathbb{X}^1_{\star,M} \to \Gamma(U_M, \mathscr{Y}^1) \to \Gamma(U_M, \pi_0(\mathsf{P}^1)).$$

These maps are natural in the obvious sense given a containment of Levis $M \subseteq L$ and the corresponding open immersion $U_L \subseteq U_M$. Moreover, $\Gamma(U_M, \pi_0(\mathsf{P}^1))$ is a finite abelian group.

Proof. Given $a \in \mathsf{A}_{L\text{-ell}}$ for $M \subseteq L$ we get a composition

$$\mathbb{X}^1_{\star,M} \subseteq \mathbb{X}^1_{\star,L} \to \mathscr{Y}^1_a.$$

It is clear from the definition of $\mathscr{Y}^1$ as a quotient that these combine to a map to the global sections of $\mathscr{Y}^1$, which we can then compose with $\Gamma(U_M, \mathscr{Y}^1) \to \Gamma(U_M, \pi_0(\mathsf{P}^1))$. The finiteness on global sections of the sheaf of components follows from the finiteness on fibers of Lemma 12.29. The naturality follows from that in Proposition 12.30. $\square$

We have assembled all necessary ingredients to generalize the isotypic decomposition of [CL12, 8.5] in the nonsplit setting. In fact, we will obtain two isotypic decompositions corresponding to the two lattices $\mathbb{X}_{\star,D(M)} \subseteq \mathbb{X}^1_{\star,M}$. The fine decomposition provides the most direct generalization of [Ngô10, §6.2] outside the elliptic locus, while the coarse decomposition will appear naturally for the purposes of point-counting.

We have constructed an action of P^1 on M^ξ by 12.15, which we restrict to $f^{\xi, M\text{-par}} : \mathsf{M}^{\xi, M\text{-par}} \to U_M$. Functoriality then induces an action on ${}^p\mathscr{H}^i(Rf^{\xi, M\text{-par}}_\star \overline{Q}_l)$, which factors through its fiberwise component group $\pi_0(\mathsf{P}^1)$ by the *lemme d'homotopie* (cf. [LN08, Lemme 3.2.3]). Thus we have an isotypic decomposition for the action of $\Gamma(U_M, \pi_0(\mathsf{P}^1))$ on ${}^p\mathscr{H}^i(Rf^{\xi, M\text{-par}}_\star \overline{\mathbb{Q}}_l)$:

$${}^p\mathscr{H}^i(Rf^{\xi, M\text{-par}}_\star \overline{\mathbb{Q}}_l) = \bigoplus_{s' \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma(U_M, \pi_0(\mathsf{P}^1)), \overline{\mathbb{Q}}_l^\times)} {}^p\mathscr{H}^i(Rf^{\xi, M\text{-par}}_\star \overline{\mathbb{Q}}_l)_{s'}.$$

But we have maps

$$\mathbb{X}_{\star,D(M)} \subseteq \mathbb{X}^1_{\star,M} \to \Gamma(U_M, \pi_0(\mathsf{P}^1)).$$

The fine s-decomposition is the decomposition which remembers only the action of $\mathbb{X}^1_{\star,M}$, and the coarse s-decomposition is the decomposition which remembers only the action of $\mathbb{X}_{\star,D(M)}$.

Theorem 12.33 (cf. [CL12, Thm. 8.5.1, Cor. 8.5.3]). *For every semistandard M, denote*

$$S^f_M := (\hat{T}/Z^{\Gamma,0}_{\widehat{M}})_{tors}, \qquad S^c_M := (\hat{T}/Z^0_{\widehat{M}})_{tors}.$$

For any $i \in \mathbb{Z}$ and $s \in S^f_M$, let

$${}^p\mathscr{H}^i(Rf^{\xi, M\text{-}par}\overline{\mathbb{Q}}_l)_s$$

denote the sum of all s'-isotypic components where $s' \in \mathrm{Hom}_{\mathbb{Z}}(\Gamma(U_M, \pi_0(\mathsf{P}^1)), \overline{\mathbb{Q}}_l^\times)$ restricts to s on $\mathbb{X}^1_{\star,M}$.

Then one has the fine s-decomposition*, a direct-sum decomposition*

$$ {}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-}par}\overline{\mathbb{Q}}_l) = \bigoplus_{s\in S_M^f} {}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-}par}\overline{\mathbb{Q}}_l)_s $$

with only finitely many nonzero summands. Grouping the summands along $S_M^f \to S_M^c$ gives the coarse s-decomposition

$$ {}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-}par}\overline{\mathbb{Q}}_l) = \bigoplus_{\overline{s}\in S_M^c} {}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-}par}\overline{\mathbb{Q}}_l)_{\overline{s}}, \qquad {}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-}par}\overline{\mathbb{Q}}_l)_{\overline{s}} = \bigoplus_{s\mapsto\overline{s}} {}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-}par}\overline{\mathbb{Q}}_l)_s. $$

If $M\subseteq L$, write $j_M^L : U_L \to U_M$ for the inclusion. Then the fine decomposition is functorial in the sense that for any $s\in S_M^f$,

$$ (j_M^L)^\star({}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-}par}\overline{\mathbb{Q}}_l)_s) = \bigoplus_{s'} {}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-}par}\overline{\mathbb{Q}}_l)_{s'}, $$

where the sum runs over the preimage of s under the natural map

$$ S_L^f \to S_M^f. $$

The coarse decomposition satisfies the analogous functoriality for the fibers over

$$ S_L^c \to S_M^c. $$

Proof. The proof of *loc. cit.* transfers to our setting. We know by Lemma 12.32 that $\Gamma(U_M, \pi_0(\mathsf{P}^1))$ is a finite abelian group, so we have an action of $\mathbb{X}^1_{\star,M}$ on the cohomology which factors through a finite quotient. In particular, the characters of $\Gamma(U_M,\pi_0(\mathsf{P}^1))$ pull back to a finite number of finite-order characters on $\mathbb{X}^1_{\star,M}$, which correspond under Proposition 12.24 to a finite number of elements of S_M^f. This gives the existence and finiteness of the decomposition, and the coarse decomposition follows by restriction.

For the compatibility along $M\subseteq L$, functoriality of $(j_M^L)^\star$ implies that $\Gamma(U_M,\pi_0(\mathsf{P}^1))$ acts on $(j_M^L)^\star({}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-par}}\overline{\mathbb{Q}}_l)_s)$ through the restriction map $\Gamma(U_M,\pi_0(\mathsf{P}^1))\to\Gamma(U_L,\pi_0(\mathsf{P}^1))$. Naturality in 12.32 says this corresponds to the natural map $\mathbb{X}^1_{\star,M}\to\mathbb{X}^1_{\star,L}$. Naturality in Proposition 12.24 implies this dualizes to the claimed decomposition. $\square$

Note that the coarse decomposition and the fine decomposition differ on the M-elliptic locus essentially in whether they capture the residual torsion group Ξ_M. The coarse decomposition is the decomposition appropriate for weighted point-counting, but in order to compare with the results of Ngô, we will need to pass the coarse and the fine decomposition on the elliptic locus. This will ensure our global version of the WFL matches the expected form, as in [MW16b, §VII.2.2]. Over a fiber $a\in\mathsf{A}_{M\text{-ell}}$, of course, the set of isotypic characters $s\in\widehat{\mathscr{Y}_a^1}$ which restrict to a given $\overline{s}$ can be described as a torsor under the dual $\widehat{\Xi}_M$.

Our descriptions of $\widehat{\Sigma}_a$ and $\widehat{\mathscr{Y}_a^1}$ imply a strong bound on the support of the s-isotypic component.

Proposition 12.34 (cf. [CL12, Théorème 8.5.1(1)]). *Let $a\in\mathsf{A}_{M\text{-}ell}$ have monodromic invariant W_a. Then the stalk of the fine s-isotypic component, for $s\in(\hat{T}/Z_{\widehat{M}}^{\Gamma,0})_{tors}$, vanishes unless s admits a lift to $\hat{T}^{W_a}$, i.e. unless*

$$ s\in\widehat{\mathscr{Y}_a^1}\cong\hat{T}^{W_a}/Z_{\widehat{M}}^{\Gamma,0}. $$

Likewise, the stalk of the coarse $\overline{s}$-component vanishes unless

$$ \overline{s}\in\widehat{\Sigma}_a\cong\hat{T}^{W_a}/(Z_{\widehat{M}}^0)^\Gamma. $$

Proof. If the fine s-component is nonzero at a, then by definition there is a character s' of $\Gamma(U_M,\pi_0(\mathsf{P}^1))$ with nonzero isotypic component at a. The action of $\mathbb{X}^1_{\star,M}$ on the stalk ${}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-par}}\overline{\mathbb{Q}}_l)_a$ is defined through the composition

$$ \mathbb{X}^1_{\star,M}\to\Gamma(U_M,\pi_0(\mathsf{P}^1))\to\pi_0(\mathsf{P}_a^1), $$

so considering the (finer) isotypic decomposition by $\pi_0(\mathsf{P}_a^1)$, we conclude at least one character s'' of this last group has nonzero component restricting to s. But the composition $\mathbb{X}^1_{\star,M}\to\pi_0(\mathsf{P}_a^1)$ factors through

$\mathscr{Y}_a^1$; the image of s'' in $\widehat{\mathscr{Y}_a^1}$ is a character which maps to $\widehat{\mathbb{X}^1_{\star,M}}$. It follows from the dual computations above that the dual map to

$$\mathbb{X}^1_{\star,M} \twoheadrightarrow \mathscr{Y}_a^1$$

is

$$\hat{T}^{W_a}/Z_{\widehat{M}}^{\Gamma,0} \hookrightarrow (\hat{T}/Z_{\widehat{M}}^{\Gamma,0})_{tors}.$$

The statement for the coarse decomposition follows from that for the fine decomposition, or by a parallel argument. $\square$

CHAPTER 13

The good locus and δ-invariants

In this section, we recall the definition of the locus $\mathscr{A}^{good}$ over which Ngô's support theorem applies. Applying this to the terms in the s-decomposition of Theorem 12.33, we will in the next section deduce the cohomological version of Arthur's stabilization over $\mathsf{A} \cap \widetilde{\mathscr{A}}^{good}$, see Theorem 14.24 below. In [CL12, §9], the authors prove by a deformation argument a general sufficient criterion (Théorème 9.1.3 in *loc. cit.*) for a point to lie in $\mathscr{A}^{good}$ which is then used in their proof of the WFL: a local polynomial is globalized, and the result is seen to lie in the good locus. While we expect the corresponding result to hold perfectly well in our setting, we choose instead to absorb the argument into our globalization step, and will show by hand that our globalization of a fixed local polynomial lies in the good locus.

13.1. Néron models.

Recall that the Picard stack $\mathscr{P} \to \mathscr{A}$ parameterizes J_a-torsors on the curve. In §12, we have described its group of connected components. The structure of its identity component is also analyzed over $\mathscr{A}^{\heartsuit}$ by Ngô, as follows.[1]

Ngô in [Ngô10, Proposition 4.8.2] shows the fiber $\mathscr{P}_a^0$ of the Picard stack over a geometric point a admits a canonical decomposition

$$1 \to \mathscr{R}_a \to \mathscr{P}_a^0 \to (\mathscr{P}_a^\flat)^0 \to 1, \tag{13.1.1}$$

with $\mathscr{R}_a$ affine and $(\mathscr{P}_a^\flat)^0$ an abelian stack. Here, $\mathscr{P}_a^\flat$ is the moduli space of torsors for the global finite-type Néron model $J_a^\flat$ of the regular centralizer J_a. Recall that by the latter we mean the Néron model whose set of $\mathscr{O}_v$-points at each geometric point on the curve is the maximal bounded subgroup $J_a(F_v)^\flat$ of $J_a(F_v)$; its component group in the special fiber is the torsion subgroup of that for the lft Néron model of [BLR90, Ch. 10] (cf. [PR08, §3.b] and the references therein). This Néron model $J_a \to J_a^\flat$ is the terminal object in the category of smooth (in particular, finite-type) X-schemes Z equipped with maps $J_a \to Z$ which are the identity on $U_a = a^{-1}(\mathfrak{c}_D^{G\text{-reg}})$. It is easier to describe than J_a.

Proposition 13.1 ([Ngô10, Corollaire 4.8.1]). *We have a natural isomorphism*

$$J_a^\flat \cong (\mathrm{Res}_{X_a^\flat/X}(T \times_X X_a^\flat))^W,$$

where $X_a^\flat \to X_a$ is the normalization of the cameral cover associated to a.

Proof. The unit of the adjunction induces a natural map

$$J_a \to \mathrm{Res}_{X_a^\flat/X}(J_a \times_X X_a^\flat),$$

which lands inside the W-fixed points (since W acts on X_a over X, by functoriality of normalization it also acts on $X_a^\flat$). By the Galois description of J in §12.3, $J_a|_{U_a}$ and $T|_{U_a}$ are canonically isomorphic after pullback to $X_a^\flat$, and we know that T is the global Neron model of $T|_{U_a}$, so by its universal property we get a W-equivariant $X_a^\flat$-map

$$J_a \times_X X_a^\flat \to T \times_X X_a^\flat.$$

Applying Weil restriction and composing gives the natural map $\varphi : J_a \to (\mathrm{Res}_{X_a^\flat/X}(J_a \times_X X_a^\flat))^W$. But the same argument applies replacing J_a with any smooth finite-type group X-scheme isomorphic to J_a on U_a, so we find that φ is terminal in the category of above and is the structure map of the global Néron model.

[1]In the commutative case compare also with geometric class field theory as in [Ser88].

Alternatively Beauville–Laszlo gluing and the universal property of the Néron model, we can see that $J_a^\flat$ is defined by gluing local Néron models for $J_a|_{\hat{\mathscr{O}}_v}$ at each $v \in X \setminus U_a$ with $J_a|_{U_a}$. The global map above is an isomorphism on U, so it suffices to show the corresponding natural map in the local setting is an isomorphism at each place $v \in X \setminus U_a$. There, a local version of the same adjunction argument above shows that $(\mathrm{Res}_{X_a^\flat/X} T \times_X X_a^\flat)^W$ satisfies the universal property of the local finite-type Néron model. □

From the description above, it is easy to see that $(\mathscr{P}_a^\flat)^0$ is an *abelian stack*, that is to say, it is of the form $A \times BD$ for A an abelian variety and D a diagonalizable group. Since the universal X-map $J_a \to J_a^\flat$ is an isomorphism on the open U where a is regular semisimple, $\mathscr{R}_a$ is a product of affine, finite-type group schemes $\mathscr{R}_{a,v}$ supported on $v \in X \setminus U$.

13.2. The δ-invariant.

Following Ngô's support theorem, we study the cohomology only over a locus $\mathscr{A}^{good}$ where the abelian variety part of $\mathscr{P}_a$ controls its action on cohomology.

Definition 13.2. For $a \in \mathscr{A}^\heartsuit(k)$ (and thereby for $(a,t) \in \widetilde{\mathscr{A}}$, define the δ-invariant

$$\delta_a := \dim(\mathscr{R}_a).$$

The above characterization of $J_a^\flat$ implies a basic formula for the δ-invariant.

Proposition 13.3 ([Ngô10, Corollaire 4.9.4]). *For any $a \in \mathscr{A}^\heartsuit(k)$, one has*

$$\delta_a = \dim H^0(X, (\mathfrak{t} \otimes_{\mathscr{O}_X} (\pi^\flat_{a\star}\mathscr{O}_{X_a^\flat}/\pi_{a\star}\mathscr{O}_{X_a}))^W),$$

where $\pi_a^\flat : X_a^\flat \to X$ is the composite of the normalization map with $\pi_a : X_a \to X$.

On the other hand, there is also the global version of Bezrukavnikov's dimension formula [Bez96]—note that locally at each place over $k = \overline{k}$ our groups become split, which is the generality of *loc. cit.*

Proposition 13.4 ([Ngô10, Proposition 4.9.5]). *For any $a \in \mathscr{A}^\heartsuit(k)$, one has*

$$\delta_a = \frac{1}{2}\left(\|\Phi\| \deg(D) - \sum_{v \in X \setminus U} c_i(v)\right),$$

where $c_i(v)$ is the difference between $\mathrm{rank}\, T$ *and the toric rank of the special fiber of $J_a^\flat$ at v.*

By pullback, we have a corresponding δ-invariant $(a,t) \mapsto \delta(a)$ on $\widetilde{\mathscr{A}}$.

Proposition 13.5 ([Ngô10, §5.6, Lemme 5.6.3]). *As a function on the base $\widetilde{\mathscr{A}}$, the δ-invariant is upper-semicontinuous.*

Proof. The proof is short and clever, so we recall it. Up to rigidification (by, say, picking a fixed trivialization of our J_a-torsors at ∞) the identity component of the Picard stack, which is all that is needed to compute δ, becomes a smooth group scheme of finite type. So let $P \to S$ be a smooth commutative group scheme of finite type with connected fibers over a base which we can take to be a strictly Henselian trait. For any prime l coprime to the residue characterstic, the l-torsion $P[l] \subset P$ is a quasi-finite étale group scheme over S. Its cardinality over any geometric point $x \in S$ can be computed easily: the fiber P_x decomposes as $T_x \times U_x \times A_x$, a product of a torus, a unipotent group, and an abelian variety; from this we conclude

$$\#P_x[l] = l \dim(T_x) + l^2 \dim(A_x).$$

This implies that abelian varieties can degenerate to tori and tori can degenerate to unipotent groups, but not the other way around, as follows. By étale-lifting, every point in the special fiber extends to an S-section, so writing s for the special point and η for the generic point we have

$$\#P_s[l] \leq \#P_\eta[l].$$

Taking l sufficiently large, this implies $\dim(A_s) \leq \dim(A_\eta)$. Since the abelian variety part of P is lower-semicontinuous, its affine part is upper-semicontinuous. □

13.3. The good locus.

With these preliminaries out of the way, we can define the good locus as the largest open satisfying a semismallness condition for the affine part of the Picard stack.

Definition 13.6. Define $\widetilde{\mathscr{A}}^{good} \subset \widetilde{\mathscr{A}}$ to be the largest open subset consisting of Zariski points such that

$$\operatorname{codim}_{\widetilde{\mathscr{A}}}(a) \geq \delta_a. \tag{13.3.1}$$

That is, $\widetilde{\mathscr{A}}^{good}$ is the topological interior of the subset of such points. Define A^{good} to be the intersection of $\widetilde{\mathscr{A}}^{good}$ with the compatible locus.

Remark 13.7. Note a subtlety of the definition above: $\widetilde{\mathscr{A}}^{good}$ does not necessarily consist of *all* points satisfying (13.3.1), which could be topologically poorly behaved. In Ngô's work, he works over an even (*a priori*) smaller open, defined as follows. Consider the stratification of $\widetilde{\mathscr{A}}$ by δ-value:

$$\widetilde{\mathscr{A}} = \bigsqcup_{\delta} \widetilde{\mathscr{A}}_\delta.$$

Then (13.3.1) is satisfied on a given stratum $\widetilde{\mathscr{A}}_\delta$ if and only if it is satisfied at the generic points of the irreducible components of that stratum. If $\widetilde{\mathscr{A}}^{good} \neq \widetilde{\mathscr{A}}$, write δ_0 for the minimal value such that $\operatorname{codim}_{\widetilde{\mathscr{A}}}(\widetilde{\mathscr{A}}_{\delta_0}) < \delta_0$, that is, that the generic point η of some irreducible component of $\widetilde{\mathscr{A}}_{\delta_0}$ satisfies $\operatorname{codim}_{\widetilde{\mathscr{A}}}(\eta) < \delta_0$. Then the subset

$$\bigsqcup_{\delta > \delta_0} \widetilde{\mathscr{A}}_\delta$$

is clearly closed, and its complement is open. This is the good locus as defined in Ngô; it is more strict than the definition above, as we have thrown away all components of all strata that could potentially lie in the closure of a bad point, regardless of whether they actually do.

Results like [CL12, Théorème 6.1.3] indicate that the good locus is quite large. It is conjectured by Ngô that the good locus is all of $\widetilde{\mathscr{A}}$, and this is known in characteristic 0.

Proposition 13.8 ([Ngô10, Lemme 5.6.4, Proposition 5.7.2]). *If $\deg(D) > 2g$, then the 0-stratum $\widetilde{\mathscr{A}}_0$ is nonempty. In particular, the good locus is nonempty.*

Moreover, for any given δ, there exists a bound N such that if $\deg(D) > N$ then $\operatorname{codim}_{\widetilde{\mathscr{A}}}(\widetilde{\mathscr{A}}_\delta) > \delta$.

13.4. The affine δ-invariant.

While the decomposition of (13.1.1) resembles a Chevalley decomposition, caution is warranted by the fact that group stacks do not have a uniquely defined Chevalley decomposition. By the below, we could rigidify our Picard stack to obtain a scheme. Applying the Chevalley decomposition to such a rigidification, we can define an invariant δ_a^{aff} which is closely related to, but not in general equal to, δ_a. In the next section, we will make use of some results of Ngô for group schemes, which when applied to such a rigidification involve δ_a^{aff}.

Proposition 13.9 ([Ngô10, Proposition 4.5.7]). *For each $a \in \mathscr{A}^\infty$, define P_a^∞ to be the stack classifying J_a-torsors equipped with a trivialization at ∞. Then P_a^∞ is representable by a smooth scheme locally of finite type. By modifying the trivialization we have an action of $J_{a,\infty}$ on P_a^∞. This action induces an canonical isomorphism $[P_a^\infty / J_{a,\infty}] \cong \mathscr{P}_a$.*

Moreover, a choice of t determines an isomorphism $T_\infty \cong J_{a,\infty}$, so $(a,t) \in \widetilde{\mathscr{A}}$ determines a canonical isomorphism $[P_a^\infty / T_\infty] \cong \widetilde{\mathscr{P}}_a$

Let $a \in \widetilde{\mathscr{A}}$ and choose any rigidification of $\mathsf{P}_{0,a}^0$, that is, a presentation of as a cokernel of smooth fiberwise-connected commutative group schemes

$$1 \to Q_{-1,a} \to Q_{0,a} \to \mathsf{P}_a^0 \to 1,$$

with Q_{-1} affine. Consider the Chevalley decomposition of $Q_{0,\overline{a}}$ at a geometric point $\overline{a}$ over a:

$$1 \to Q_{0,\overline{a}}^{\mathrm{aff}} \to Q_{0,\overline{a}} \to Q_{0,\overline{a}}^{\mathrm{ab}} \to 1.$$

As observed at the beginning of [Ngô10, §4.12], since $Q_{-1,\overline{a}}$ is affine the composition $Q_{-1,\overline{a}} \to Q_{0,\overline{a}} \twoheadrightarrow Q^{\mathrm{ab}}_{0,\overline{a}}$ associated to any Chevalley decomposition has image which is proper, affine, smooth, and connected, hence trivial, and we have an induced surjection with kernel which we denote $K_{\overline{a}}$:

$$1 \to K_{\overline{a}} \to \mathsf{P}^0_{\overline{a}} \to Q^{\mathrm{ab}}_{0,\overline{a}} \to 1.$$

Definition 13.10. Given $a \in \mathscr{A}^{\heartsuit}$, define the *affine δ-invariant* to be

$$\delta^{\mathrm{aff}}_a := \dim K_{\overline{a}}.$$

Note that this is independent of choice of geometric point $\overline{a}$ over a. Moreover, $\dim(Q^{\mathrm{ab}}_{0,\overline{a}})$ and thus δ^{aff}_a are also independent of choice of rigidification (as long as $Q_{-1,a}$ is required to be affine). By the same reasoning as the usual δ-invariant, δ^{aff}_a is upper-semicontinuous. The basic comparison between the two invariants is the following computation.

Proposition 13.11 (cf. [CL12, Proof of 10.5.1]). *We have*

$$\delta_a - \delta^{\mathrm{aff}}_a = \dim(\mathbb{T}^{W_a}) - \dim((Z\mathbb{G})^{\pi_1}),$$

where W_a is the monodromy subgroup of Definition 3.17 and $(Z\mathbb{G})^{\pi_1}$ is the set of $\pi_1(X,\infty)$-fixed points in the center of the split form of G. In particular, $\delta_a \geq \delta^{\mathrm{aff}}_a$ with equality exactly on the G-elliptic locus.

Proof. We may assume a is a geometric point. Recall that the usual δ-invariant is defined as the dimension of $\mathscr{R}_a$ in the short exact sequence of stacks (13.1.1). Since the inertia in $\mathscr{P}^0_a$ is $H^0(X, J_a)$ and the inertia in $(\mathscr{P}^\flat_a)^0$ is $H^0(X, J^\flat_a)$, with coarse moduli of dimensions given by the corresponding H^1, one has

$$\delta = H^1(X, J_a) - H^0(X, J_a) - H^1(X, J^\flat_a) + H^0(X, J^\flat_a).$$

Now choose rigidifications for $\mathscr{P}_a$ and $\mathscr{P}^\flat_a$ compatibly, i.e., two-term complexes

$$1 \to Q_{-1} \to Q_0 \to \mathscr{P}^0_a \to 1 \qquad \text{and} \qquad 1 \to S_{-1} \to S_0 \to (\mathscr{P}^\flat_a)^0 \to 1$$

with the kernels affine and equipped with a distinguished triangle

$$R[0] \longrightarrow Q_\bullet \longrightarrow S_\bullet \xrightarrow{+1} \quad .$$

The computation of δ above comes from taking dimensions in the cohomology long exact sequence associated to this triangle:

$$\mathscr{H}^{-1}(Q_\bullet) \longrightarrow \mathscr{H}^{-1}(S_\bullet) \longrightarrow \mathscr{H}^0(R[0]) \longrightarrow \mathscr{H}^0(Q_\bullet) \longrightarrow \mathscr{H}^0(S_\bullet)\ .$$

The map of rigidifications $\mathscr{H}^0(Q_\bullet) \to \mathscr{H}^0(S_\bullet)$ is surjective and $R^{\mathrm{aff}}_a := \ker(\mathscr{H}^0(Q_\bullet) \to \mathscr{H}^0(S_\bullet))$ admits a surjection from R_a, hence is affine. On the other hand, since $(\mathscr{P}^\flat_a)^0$ is an abelian stack, its rigidification $\mathscr{H}^0(S_\bullet)$ is an abelian variety, and so the short exact sequence

$$1 \to R^{\mathrm{aff}}_a \to \mathscr{H}^0(Q_\bullet) \to \mathscr{H}^0(S_\bullet) \to 1$$

is the Chevalley decomposition of its middle term, the rigidification of $\mathscr{P}^0_a$, and $\dim(R^{\mathrm{aff}}_a) = \delta^{\mathrm{aff}}_a$. From the long exact sequence, then, we obtain an exact sequence

$$1 \to \mathscr{H}^{-1}(Q_\bullet) \to \mathscr{H}^{-1}(S_\bullet) \to R_a \to R^{\mathrm{aff}}_a \to 1.$$

Considering the inertia of $\mathscr{P}_a$ and $\mathscr{P}^\flat_a$ at their identities, we can see that this map on $\mathscr{H}^{-1}$ is the injection

$$H^0(X, J_a)^0 \to H^0(X, J^\flat_a)^0.$$

By taking dimensions we find

$$\delta^{\mathrm{aff}}_a = \delta_a - \dim(H^0(X, J^\flat_a)) + \dim(H^0(X, J_a)).$$

Recalling our standing hypothesis of good characteristic, by [Ngô10, Proposition 4.5.5] we have $H^0(X, J_a) = (Z\mathbb{G})^{\pi_1}$. On the other hand,

$$H^0(X, J^\flat_a) = \mathbb{T}^{W_a},$$

which follows from Proposition 13.1, cf. the discussion before Corollaire 4.11.3 in [Ngô10]. In particular, we have $(Z\mathbb{G})^{\pi_1} \subseteq \mathbb{T}^{W_a}$. Identifying their dimensions with split ranks, equality holds if and only if $A_{J_a}^{\mathrm{spl}} = A_G^{\mathrm{spl}}$, which is G-ellipticity. □

13.5. A sufficient criterion for goodness.

A substantial amount of [CL12, §9] is devoted to giving a deformation-theoretic argument for a general criterion under which a point $a \in \widetilde{\mathscr{A}}_G$ lies in the good locus. In this section we adapt their argument.

Recall from [Ngô10, §5.2] the moduli problem of normalized cameral curves.

Definition 13.12. Let $\mathscr{B} = \mathscr{B}_G$ the functor which associates to a k-scheme S the groupoid of triples $(a, X_a^\flat, \nu)$, where

- $a \in \mathscr{A}_G^\heartsuit(S)$,
- $X_a^\flat$ is a proper smooth curve over S,
- $\nu : X_a^\flat \to X_a$ is a W-equivariant normalization in family of the cameral curve X_a (cf. §3.5, [Ngô10, §5.1]).

There is a natural map $\mathscr{B}_G \to \mathscr{A}^\heartsuit$ which is a bijection on geometric points. We define then

$$\widetilde{\mathscr{B}} := \mathscr{B} \times_{\mathscr{A}^\heartsuit} \widetilde{\mathscr{A}}.$$

Both $\mathscr{B}$ and $\widetilde{\mathscr{B}}$ are representable by k-schemes of finite type. Indeed, they admit an equivalent description which is essentially the definition of [CL12].

Lemma 13.13 ([Ngô10, Lemme 5.2.2]). *The functor $\mathscr{B}_G$ is isomorphic to the functor which associates to k-scheme S the set of pairs $(X_a^\flat, \gamma)$, where*

- *$X_a^\flat$ is a smooth proper curve over S equipped with a W-action and a morphism $\pi : X_a^\flat \to X \times S$ which is a W-torsor on a nonempty open which surjects onto S, and*
- *$\gamma : X_a^\flat \to \mathfrak{t}_D \times S$ is a W-equivariant S-morphism such that the preimage of the regular semisimple locus is dense in the fiber over each $s \in S$.*

Since the map $\widetilde{\mathscr{B}} \to \widetilde{\mathscr{A}}$ is a bijection on geometric points, as in [Ngô10, §5.4-5.6] the δ-stratification on $\widetilde{\mathscr{A}}$ induces a *locally constant* δ-stratification on $\widetilde{\mathscr{B}}$.

Definition 13.14 (The cotangent complex on $\mathscr{B}$). Fix a geometric point $b = (a, X_a^\flat, \nu) \in \mathscr{B}(\overline{k})$, identified with $\pi : X_a^\flat \to X$ and $f : X_a^\flat \to \mathfrak{t}_D$ as in Lemma 13.13. Consider the cotangent complex of f over X:

$$L_b := L_{X_a^\flat/\mathfrak{t}_D} = [f^\star\Omega^1_{\mathfrak{t}_D/k} \to \Omega^1_{X_a^\flat/k}] \cong [f^\star\Omega^1_{\mathfrak{t}_D/X} \to \Omega^1_{X_a^\flat/X}]$$

supported in degrees $(-1, 0)$. Note that $\mathscr{H}^{-1}(L_b)$ is a vector bundle of rank $r = \operatorname{rank} G$, and $\mathscr{H}^0(L_b) = \Omega^1_{X_a^\flat/X_a}$ is torsion. The complex L_b is naturally W-equivariant, where W/X is the finite étale Weyl group scheme. Define the deformation complex

$$\mathscr{E}_b := (\pi_\star R\mathscr{H}om_{X_a^\flat}(L_b, \mathscr{O}_{X_a^\flat}))^W,$$

a complex of $\mathscr{O}_X$-modules. Then define $E_b := R\Gamma(X, \mathscr{E}_b)$.

This is the appropriate deformation complex which controls the infinitesimal structure of $\mathscr{B}$ at b. In particular, as in *loc. cit.*,

- $H^0(E_b)$ is the space of infinitesimal automorphisms of $\mathscr{B}$ at b,
- $H^1(E_b) = T_b\mathscr{B}$, and
- $H^2(E_b)$ is the obstruction to smoothness.

Remark 13.15. In fact, $H^0(E_b) = 0$. Choosing the presentation

$$L_b = [f^\star\Omega^1_{\mathfrak{t}_D/k} \to \Omega^1_{X_a^\flat/k}],$$

by smoothness this perfect complex dualizes to the complex of tangent bundles

$$R\mathscr{H}om(L_b, \mathscr{O}_{X_a^\flat}) \cong [T_{X_a^\flat/k} \to f^\star T_{\mathfrak{t}_D/k}].$$

Its 0-th cohomology is the kernel of this map, which in particular lies in the kernel of $T_{X_a^\flat/k} \to T_{X/k}$. But generically $X_a^\flat \to X$ is a W-torsor, hence finite étale, and the map on tangent bundles is generically an isomorphism. So a section of $T_{X_a^\flat/k}$ which vanishes upon pushforward is generically, hence identically, 0. We have shown

$$\mathscr{H}^0(R\mathscr{H}om(L_b, \mathscr{O}_{X_a^\flat})) = 0,$$

and taking $\pi_\star$ and W-invariants are exact, so the same holds for $\mathscr{H}^0(\mathscr{E}_b)$. The hypercohomology spectral sequence shows that the only possible contribution to $H^0(E_b)$ is $H^0(X, \mathscr{H}^0(\mathscr{E}_b)) = 0$.

Remark 13.16. When the group W is constant (étale under our characteristic hypotheses), its invariants are exact and we have

$$R\Gamma(X, (\pi_\star R\mathscr{H}om_{X_a^\flat}(L_b, \mathscr{O}_{X_a^\flat}))^W) = R\Gamma(X_a^\flat, R\mathscr{H}om_{X_a^\flat}(L_b, \mathscr{O}_{X_a^\flat}))^W = R\operatorname{Hom}_{\mathscr{O}_{X_a^\flat}}(L_b, \mathscr{O}_{X_a^\flat})^W,$$

so this generalizes the definition of [CL12]. However, the right side makes no sense for W a group scheme over X. In our generality one could interpret E_b as the equivariant derived Hom

$$R\operatorname{Hom}_W(L_b, \mathscr{O}_{X_a^\flat})$$

but the definition above is more explicit (and equivalent, since W/X is linearly reductive).

In the same way, the deformations of $\mathscr{A}_G$ at a are controlled by

$$R\Gamma(X, \pi_{X_a,\star} R\mathscr{H}om_{X_a}(L_{X_a/\mathfrak{t}_D}, \mathscr{O}_{X_a})^W).$$

This is in fact a complex with cohomology only in degree -1, as befits a smooth scheme. The deformation complex piece has a simple description.

Lemma 13.17 ([CL12, Lemme 9.6.1]). *We have*

$$L_{X_a/\mathfrak{t}_D} \cong \mathscr{H}^{-1}(L_{X_a/\mathfrak{t}_D})[1] = \pi_{X_a}^\star a^\star \Omega^1_{\mathfrak{c}_D/X}[1]$$

for $\pi_{X_a} : X_a \to X$ the natural map.

Proof. The argument given in *loc. cit.* applies verbatim. The constructions in question are étale-local, anyways, so the result transfers by outer twisting from the split case. ☐

Definition 13.18. For any W-equivariant coherent sheaf $\mathscr{L}$ on $X_a^\flat$, define

$$F_L(\mathscr{L}) := (\pi_\star(\mathscr{H}^{-1}(L_b) \otimes \mathscr{L}))^W.$$

This is an exact functor because W is linearly reductive and π is finite. Then define for each $x \in |X|$ the *local invariant obstruction*

$$\kappa_x(b) := \operatorname{length}_{\mathscr{O}_{X,x}} F_L(\Omega^1_{X_a^\flat/X})_x$$

and the *global invariant obstruction*

$$\kappa(b) = \sum_{x \in |X|} \kappa_x(b).$$

For any geometric point $a \in \mathscr{A}_G^\heartsuit$, we write $\kappa_x(a) = \kappa_x(b)$ and $\kappa(a) = \kappa(b)$ for b the unique point of $\mathscr{B}$ corresponding to a under the natural bijection $\mathscr{B} \to \mathscr{A}^\heartsuit$.

We collect the following basic properties of this invariant, whose proofs follow exactly as in [CL12, §9.7].

Lemma 13.19 ([CL12, Remarques 9.7.2, Proposition 9.7.4]). *The function κ is upper-semicontinuous on $\mathscr{B}$. The local invariant κ_x vanishes if either*

- *$X_a^\flat \to X$ is étale over x, or*
- *X_a is smooth above x.*

The basic global bound is the following.

Lemma 13.20 (cf. [CL12, Lemme 9.8.1]). *If $\kappa(b) < \deg(D) - 2g + 2$, then $H^2(E_b) = 0$ and $\mathscr{B}$ is smooth at b.*

Proof. The proof proceeds essentially unchanged until the final calculation, where we pass to a splitting cover and then descend. By Serre duality and exactness of W-invariants (= coinvariants for vector spaces), the obstruction space is dual to

$$H^0(X, F_L(\Omega^1_{X_a^\flat/k})).$$

Applying the exact functor F_L to the short exact sequence

$$0 \longrightarrow \pi^\star\Omega^1_{X/k} \longrightarrow \Omega^1_{X_a^\flat/k} \longrightarrow \Omega^1_{X_a^\flat/X} \longrightarrow 0\ ,$$

we have an inclusion of vector bundles whose cokernel is torsion of length $\kappa(b)$, cf. [CL12, Remarque 9.7.3]. In the local ring at each such point, we have a containment of lattices with cokernel length $\kappa_x(b)$, so we need to contract by at most $\kappa_x(b)$ to reverse the containment. Summing over points, there exists an effective divisor K of degree at most $\kappa(b)$ such that

$$F_L(\Omega^1_{X_a^\flat/k}) \subset F_L(\pi^\star\Omega^1_{X/k})(K).$$

By the projection formula and the injection $\mathscr{H}^{-1}(L_b) \hookrightarrow f^\star\Omega^1_{\mathfrak{t}_D/X}$, it suffices as in *loc. cit.* to show

$$H^0(X_a^\flat, f^\star\Omega^1_{\mathfrak{t}_D/X} \otimes \pi^\star\Omega^1_{X/k}(K)) = 0.$$

The nonsplit version of the computation of this first tensor factor is

$$f^\star\Omega^1_{\mathfrak{t}_D/X} = \pi^\star(\mathfrak{t}^\vee \otimes \mathscr{O}_X(-D)),$$

where $\mathfrak{t}^\vee$ is a vector bundle over X, so the tensor bundle is

$$\pi^\star(\mathfrak{t}^\vee \otimes \Omega^1_{X/k}(K-D)).$$

Pulling back to a finite étale Galois splitting cover renders $\mathfrak{t}^\vee$ trivial, and by our assumption on $\kappa(b)$ the right factor has negative degree. So every summand has negative degree on every connected component of the pulled-back normalization, so there are no global sections. Étale descent for global sections then implies the vanishing on the original $X_a^\flat$. □

The second result we need to extract is the infinitesimal dimension estimate.

Definition 13.21. Define the exact functor

$$\mathscr{L} \mapsto F_\Omega(\mathscr{L}) := (\pi_\star(\mathscr{L} \otimes \Omega^1_{X_a^\flat/k}))^W.$$

Lemma 13.22 (cf. [CL12, Lemme 9.9.1]). *If $H^2(E_b) = 0$, then*

$$\dim T_b(\mathscr{B}) \leq \dim \mathscr{A}_G - \delta_a.$$

Proof. The relative cotangent complexes (which here all have length 2) fit into a distinguished triangle

$$\nu^\star L_{X_a/\mathfrak{t}_D} \longrightarrow L_{X_a^\flat/\mathfrak{t}_D} \longrightarrow \left[\nu^\star\Omega^1_{X_a/X} \to \Omega^1_{X_a^\flat/X}\right] \xrightarrow{+1} \quad .$$

We apply the exact functor F_Ω and take Euler characteristics.

- The first term $F_\Omega(\nu^\star L_{X_a/\mathfrak{t}_D})$ reduces by Lemma 13.17 to its $\mathscr{H}^{-1}$, which can be computed to be

$$F_\Omega(\nu^\star L_{X_a/\mathfrak{t}_D}) = a^\star\Omega^1_{\mathfrak{c}_D/X} \otimes (\pi_\star\Omega^1_{X_a^\flat/k})^W.$$

  By tame descent as in [CL12, Lemme 9.11.1(2)], we have $(\pi_\star\Omega^1_{X_a^\flat/k})^W \cong \Omega^1_{X/k}$, so the Euler characteristic is

$$-\chi(X, a^\star\Omega^1_{\mathfrak{c}_D/X} \otimes \omega_X)$$

  or by Serre duality

$$\chi(X, a^\star T_{\mathfrak{c}_D/X}).$$

  Our degree hypothesis ensures this Euler characteristic is just its H^0, and $\mathscr{A}_G$ is the space of global sections of $\mathfrak{c}_D$ so its tangent space is the space of global sections of the tangent bundle. This first term therefore gives $\dim \mathscr{A}_G$.

- The middle term is
$$F_\Omega(L_{X_a^\flat/\mathfrak{t}_D}) = (\pi_\star(L_{X_a^\flat/\mathfrak{t}_D}\otimes\Omega^1_{X_a^\flat/k}))^W;$$
and by Serre duality on $X_a^\flat$
$$L_{X_a^\flat/\mathfrak{t}_D}\otimes\Omega^1_{X_a^\flat/k}\cong R\mathscr{H}om_X(L_{X_a^\flat/\mathfrak{t}_D},\mathscr{O}_{X_a^\flat})[-1],$$
so $F_\Omega(L_{X_a^\flat/\mathfrak{t}_D}) = \mathscr{E}_b[-1]$ by definition, and $R\Gamma(X, F_\Omega(L_{X_a^\flat/\mathfrak{t}_D}) = E_b[-1]$. Its Euler characteristic is therefore, by the vanishing of $H^0(E_b)$ as in Remark 13.15 and the vanishing of H^2 by assumption, equal to
$$\dim H^1(E_b) = \dim T_b\mathscr{B}.$$
- The right term contributes two torsion terms
$$\chi(F_\Omega(\nu^\star\Omega^1_{X_a/X})) - \chi(F_\Omega(\Omega^1_{X_a^\flat/X})) = \deg F_\Omega(\nu^\star\Omega^1_{X_a/X}) - \deg F_\Omega(\Omega^1_{X_a^\flat/X}).$$
Write p for the first degree. The short exact sequence of relative differentials for $\mathfrak{t}_D\to\mathfrak{c}_D\to X$ gives, after pullback along $f : X_a^\flat\to\mathfrak{t}_D$ and taking F_Ω,
$$p = \deg F_\Omega(f^\star\Omega^1_{\mathfrak{t}_D/X}) - \deg F_\Omega(f^\star\chi^\star\Omega^1_{\mathfrak{c}_D/X}),$$
just as in [CL12, Lemme 9.10.2]. The first term splits across the decomposition $f^\star\Omega^1_{\mathfrak{t}_D/X} = \mathfrak{t}^\vee\otimes\pi^\star\mathscr{O}_X(-D)$ as in [CL12, Lemme 9.10.3], and splitting $\Omega^1_{X_a^\flat}$ across the short exact sequence of relative differentials for $X_a^\flat\to X\to k$, we deduce just as in [CL12, Lemme 9.10.4] from Ngô's formula 13.3 that
$$p = \delta + \deg((\mathfrak{t}^\vee\otimes\pi_\star\Omega^1_{X_a^\flat/X})^W).$$
There is a mild discrepancy between the $\mathfrak{t}^\vee$ which appears here and the $\mathfrak{t}$ in the formula above. Our hypotheses on characteristic are enough to ensure the equality of these two formulae: in *loc. cit.* they fix a global nondegenerate W-invariant bilinear form, but we need not do so globally: étale locally the group splits and we can choose on the split Cartan a Weyl-invariant bilinear form $\mathfrak{t}_0\cong\mathfrak{t}_0^\vee$ and conclude that the local lengths agree. But of course the value of these local lengths descends from an étale cover, though the pairing itself may not.
- It remains to show that
$$\deg((\mathfrak{t}^\vee\otimes\pi_\star\Omega^1_{X_a^\flat/X})^W) - \deg((\pi_\star(\Omega^1_{X_a^\flat/X}\otimes\Omega^1_{X_a^\flat/k}))^W)\geq 0.$$
As in [CL12, Lemme 9.11.1(3)-(4)], we have at a geometric point $\tilde{x}\in X_a^\flat$ over $x\in X$ with inertia $W_{\tilde{x}}\subset W$ that
$$\mathrm{length}_x((\mathfrak{t}^\vee\otimes\pi_\star\Omega^1_{X_a^\flat/X})^W) = \dim(\mathfrak{t}) - \dim(\mathfrak{t}^{W_{\tilde{x}}}),$$
and
$$\mathrm{length}_x((\pi_\star(\Omega^1_{X_a^\flat/X}\otimes\Omega^1_{X_a^\flat/k}))^W) = \begin{cases}0 & \text{if } X_a^\flat \text{ is étale over } x\\ 1 & \text{otherwise}\end{cases}.$$
These are both étale-local computations and reduce to the split case. At étale points both vanish, and at ramified points, since the Weyl group acts faithfully, $\mathfrak{t}^{W_x}\neq\mathfrak{t}$, our desired inequality holds.

This concludes the proof. □

We collect finally the handful of local calculations from [CL12] that we will require in the application. All of the assertions reduce to the split case: for any geometric point $\overline{x}\to X$, after base change to the strict Henselization $\operatorname{Spec}\mathscr{O}^{\mathrm{sh}}_{X,\overline{x}}$ the twisting datum defining G/X is trivial, and the Cartan, Chevalley map, Weyl group scheme, discriminant, and cameral cover identify with those for the corresponding split group. Hence, all assertions concerning geometric properties of the completed local cameral cover, discriminant multiplicities, stabilizers, smoothness, and the local invariants δ_x, κ_x may be checked after this base change.

Lemma 13.23 ([CL12, Lemmes 9.13.2, 9.14.1]).

(1) For any $b \in \mathscr{B}$ living over $a \in \mathscr{A}_G$ and any closed point $x \in X$, one has the inequality

$$\kappa_x(b) \leq \frac{|W_G|}{2} v_x(\mathfrak{D}_G(a)).$$

(2) For any $b \in \mathscr{B}$ living over $a \in \mathscr{A}_G$ and any closed point $x \in X$, if the intersection multiplicity of $\mathfrak{D}_G$ and $a(X)$ at x is 1, then the cameral cover X_a is smooth over x. In particular by Lemma 13.19 $\kappa_x(b)$ vanishes for any $b \in \mathscr{B}$ over a.

We can now state and prove Chaudouard–Laumon's general criterion for goodness of a parameter of the Hitchin base. Let H/X be an arbitrary endoscopic group for G, including, for example, a semistandard Levi or an endoscopic group for a Levi. Recall the discriminant and resultant divisors $\mathfrak{D}_H$, $\mathfrak{R}_H^G$ from §2.11. Write $\Delta = (\mathfrak{D}_H + \mathfrak{R}_H^G)_{\mathrm{red}}$.

Theorem 13.24 ([CL12, Théorème 9.1.3]). *For H/X as above, let $a_H \in \widetilde{\mathscr{A}}_H(k)$ and write $a = \chi_G^{H'}(a_H) \in \mathscr{A}_G$ for its image under the appropriate Chevalley map. Assume that there is a finite set of points $S \subset |X|$ such that the following two conditions hold.*

(1) For every point $x \notin S$, the curve $a_M(X)$ either avoids Δ or intersects exactly one of $\mathfrak{D}_H$, $\mathfrak{R}_H^G$ transversally at a smooth point.

(2) One has

$$\frac{|W_G|}{2} \sum_{x \in S} v_x(\mathfrak{D}_G(a)) < \deg(D) - 2g + 2$$

where $v_x(\mathfrak{D}_G(a))$ is the intersection multiplicity of $a(X)$ and $\mathfrak{D}_G$ at x.

Then a lies in the good locus $\widetilde{\mathscr{A}}_G^{good}$.

Proof. The proof is essentially the same as *loc. cit.*, reducing the local computations to the split case as above. We outline the argument. The functions δ, κ are defined on $\widetilde{\mathscr{A}}_G$ by pullback from the corresponding function on $\mathscr{A}_G$; that is, they are independent of t. Since the good locus is defined as the *interior* of the set of points satisfying

$$\operatorname{codim}_{\mathscr{A}}(a) \geq \delta_a$$

we need to show that the inequality holds not just for a but for any generization of it. It suffices to consider one-parameter generizations

$$\beta : \operatorname{Spec} K[[u]] \to \mathscr{A}_G$$

for some field K with special point a. Write η for the generic point of the DVR and s for its special point. We now calculate $\kappa(\beta_\eta)$.

Let x_η be a point on the generic curve which specializes to $x \notin S$. If $a_H(x) \notin \Delta$ or $a_H(x)$ crosses $\mathfrak{D}_H$ transversely, then at worst the same is true for x_η. In the first case the cameral cover is étale and in the second the G-discriminant multiplicity is 1 and by Lemma 13.23.(2) the cameral cover is smooth. In either case $\kappa_{x_\eta} = 0$.

If, on the other hand, the special point intersects $\mathfrak{R}_H^G$ transversally, then (passing to a splitting étale cover or the strict Henselization) any chosen cameral lift $\tilde{x}_s$ of the special point x_s identifies a unique root hyperplane $\pm\alpha \subset \Phi_G \setminus \Phi_H$ such that

$$\operatorname{Stab}_{W_G}(\tilde{x}_s) = \langle s_\alpha \rangle.$$

By generization, we have $|W_{\tilde{x}_\eta}| \leq 2$, so the ambient discriminant valuation is 0, 1, or 2. In the first two cases we are done. In the last case $v_{x_\eta}(\mathfrak{D}_G(\beta_\eta)) = 2$, for $\tilde{x}'_\eta$ a point in the normalization $X_a^\flat$ over x_η, the Ngô–Bezrukavnikov formula gives

$$2 = v_{x_\eta}(D_G(a_\eta)) = 2\delta_{x_\eta} + \dim \mathfrak{t} - \dim \mathfrak{t}^{W_{\tilde{x}_\eta}}.$$

Since $W_{\tilde{x}'_\eta} \subseteq \langle s_\alpha \rangle$ and $\mathfrak{t}^{\langle s_\alpha \rangle}$ is a root hyperplane, we obtain a parity contradiction unless $W_{\tilde{x}'_\eta}$ is trivial. In that case the normalization is étale over x_η, and again $\kappa_{x_\eta} = 0$ by Lemma 13.19.

So $\kappa(\beta_\eta)$ is supported entirely on points x_η specializing to $x \in S$. By Lemma 13.23.(1),

$$\kappa(\beta_\eta) = \sum_{x_\eta \rightsquigarrow S} \kappa_{x_\eta}(\beta_\eta) \le \frac{|W_G|}{2} \sum_{x_\eta \rightsquigarrow S} v_{x_\eta}(\mathfrak{D}_G(\beta_\eta)).$$

Clearly we have

$$\sum_{x_\eta \rightsquigarrow S} v_{x_\eta}(\mathfrak{D}_G(\beta_\eta)) \le \sum_{x \in S} v_x(\mathfrak{D}_G(a))$$

(in fact equality holds, though we don't need this), so by our hypothesis bound

$$\kappa(\beta_\eta) < \deg(D) - 2g + 2.$$

Since this bound holds for every generic deformation of a and κ is constructible, the locus of points satisfying this inequality on κ contains a Zariski neighborhood U of a. For an irreducible component Z of the $\delta = r$ stratum meeting U, take its generic point and the corresponding normalized cameral curve $b \in \mathscr{B}$. Then

$$\kappa(b) < \deg(D) - 2g + 2,$$

so the deformation obstruction vanishes and $\dim Z \le \dim_b \mathscr{B} \le \dim \mathscr{A}_G - r$ by the dimension estimate of Lemma 13.22. Every δ-stratum meeting U has the required codimension, so $U \subseteq \mathscr{A}_G^{good}$ and $a \in \mathscr{A}_G^{good}$. $\square$

CHAPTER 14

Support and the cohomological WFL

14.1. The Hitchin fibration.

We recall the setup of the Hitchin fibration we will apply. Let X_0/k_0 a smooth geometrically integral curve over a finite field $k_0 = \mathbb{F}_q$, and let $G_0 \to X_0$ a reductive group scheme with maximally split fiberwise-maximal torus $T_0 \to X_0$. Write X, G, T, etc. for the base change of these objects to $k = \overline{k}_0$. Assume now that G has anisotropic center, i.e., $A_G^{\mathrm{spl}} = 1$, and when necessary that $\xi in \mathfrak{a}_T$ is in general position. This is sufficient for all our previous geometric theorems to apply.

Write $\tau : x \mapsto x^q$ for the (arithmetic) Frobenius automorphism of k and the induced Frobenius of X, G, T, etc. Let D an effective τ-stable divisor on X of degree $\deg(D) > 2g$, and assume $D = 2D'$, so that the choice of D' defines a Kostant–Hitchin section $\mathscr{A} \to \mathscr{M}$ as in [BČ22, §4.3.3]. Pick $\infty \in X(k)$ lying over a rational point of X_0. Assume that the fiber of G_0 is split at ∞.

The Hitchin base $\widetilde{\mathscr{A}}_G$, resp. the compatible Hitchin base A_G, is equipped with an action of τ, under which the closed subschemes $\widetilde{\mathscr{A}}_M$, resp. A_M, for semistandard X-Levis M are stable, cf. Corollary 2.21. Likewise $\widetilde{\mathscr{M}}_G$ and M_G have an action of τ, for which the Hitchin morphism is equivariant.

For any $\xi \in \mathfrak{a}_T$, the ξ-stable substack M_G^{ξ} is τ-stable because the stability convex C_m is insensitive to the extension k/k_0, cf. Remark 5.6. Indeed, one can directly define the truncated Hitchin fibration $f_0^{\xi} : \mathsf{M}_{0,G} \to \mathsf{A}_{0,M}$ over k_0, and upon base change to k we would recover our geometric constructions of §5: parabolic reductions of Higgs bundles m for G_0 are determined by the M_0-elliptic type of $f_0(m)$ just as in §4, and the combinatorics of relative position remain unchanged because the group G_0 is already split over ∞.

Write d_f for the relative dimension of $f : \mathsf{M}_G \to \mathsf{A}_G$. This is well-defined: by [Ngô10, Corollaire 4.16.4] the Hitchin morphism f over $\mathscr{A}^{\heartsuit}$ is of constant relative dimension (it is equal to the dimension of P_a, which is locally constant). By [Ngô10, Corollaire 4.16.3] each Hitchin fiber is equidimensional, so the relative dimension remains the same constant after restricting to the compatible truncated Hitchin morphism f^{ξ}. We denote $d_{f^{\xi}} = d_f$. Write $d_{\mathsf{M}^{\xi}}$ for the dimension of M^{ξ} (equal to the dimension of M) and d_{A} for the dimension of the base, so $d_{\mathsf{M}^{\xi}} = d_{\mathsf{A}} + d_{f^{\xi}}$.

14.2. Decomposition and actions of the Tate module.

Now we recall the essential cohomological machinery in our setting—this section is entirely an exposition of [Ngô10, §7] and [CL12, §10.1-10.3], with all results and arguments transferring directly. Many of the results below apply in the general setting of a proper morphism $f_0 : M_0 \to A_0$ of smooth k_0-schemes of constant fiber dimension equipped with a relative action of a smooth commutative group scheme $g_0 : P_0 \to A_0$ satisfying mild hypotheses, cf. [Ngô10, §7] which works in this generality. In particular, the Chevalley decomposition of the fibers of P_0 induces a δ invariant as above, and one requires the good locus hypothesis

$$\mathrm{codim}_A(a) \geq \delta_a$$

of Definition 13.6. Of course, our Hitchin fibrations are maps of Deligne–Mumford stacks, not schemes, but the results apply anyways, either by translating the proofs directly or by rigidifying our torsors with trivializations at ∞, with the only caviat that the arguments below will involve the rigidified δ_a^{aff} instead of the real δ_a. See in particular [Ngô10, 4.5.6, 4.12] for this point. Since our focus in this work is on the application to the Hitchin fibration, we have not usually bothered to state the results generally.

Now assume ξ is in *general position* in the sense of Definition 5.10. The geometric results of §8, §9, §10, and §11 imply that f_0^ξ and f^ξ are proper morphisms (of constant relative dimension d_{f^ξ}) of smooth Deligne–Mumford stacks. As noted above, the results of [Bei+18] extend to this setting (say, by choosing appropriate rigidifications, cf. [Ngô10, §6.2, §7.8]), and in particular Deligne purity implies that over k we have

$$Rf_\star^\xi\overline{\mathbb{Q}}_l[d_{\mathsf{M}^\xi}] \cong \bigoplus_i {}^p\mathscr{H}^i(Rf_\star^\xi\overline{\mathbb{Q}}_l[d_{\mathsf{M}^\xi}])[-i],$$

with each graded piece splitting as a sum of intermediate extensions of local systems on locally closed subsets $Z_\alpha \hookrightarrow \mathsf{A}$:

$${}^p\mathscr{H}^i(Rf_\star^\xi\overline{\mathbb{Q}}_l[d_{\mathsf{M}^\xi}]) = \bigoplus_{i_\alpha : Z_\alpha \hookrightarrow \mathsf{A}} i_{\alpha,!\star}\mathscr{L}_\alpha[\dim(Z_\alpha)].$$

Switching the order of the summation, we can write the full relative cohomology complex as a sum over the Zariski points of A of intermediate extensions of graded local systems. To be precise,

$$\text{(14.2.1)} \qquad {}^p\mathscr{H}^i(Rf_\star^\xi\overline{\mathbb{Q}}_l[d_{\mathsf{M}^\xi}]) = \bigoplus_{a\in\mathsf{A}} j_{a,!\star}\mathscr{L}_a^\bullet[d_a],$$

where $\mathscr{L}_a^\bullet$ is a graded local system on solely the point $\{a\}$ with the property that it extends to some open in $\overline{\{a\}}$, and we have somewhat abusively written $j_{a,!\star}$ for the well-defined functor which performs intermediate extension to A from any such open. The pieces $\mathscr{L}_a^n$ are moreover pure of weight n.

Define the *support* of the sheaf $Rf_\star^\xi\overline{\mathbb{Q}}_l[d_{\mathsf{M}^\xi}]$ to be the set of irreducible closed subsets appearing in this decomposition with nonzero local systems. Define the *socle* to be the finite subset of Zariski points of A which are the generic points of the support. In the decomposition of (14.2.1) it is the finite set of points at which $\mathscr{L}_a^\bullet$ is nonzero. For each $\mathscr{L}_a^\bullet$, define its *amplitude* to be the range of degrees in which it is supported

$$\operatorname{Amp}(\mathscr{L}_a^\bullet) = \max\{n : \mathscr{L}_a^n \neq 0\} - \min\{n : \mathscr{L}_a^n \neq 0\}.$$

A Poincaré duality argument due to Goresky–MacPherson gives a general bound on the cohomological amplitude and the codimension of the socle.

Proposition 14.1 ([Ngô10, Théorème 7.3.1] or [CL12, Lemme 10.2.1]). *Let $a \in \mathsf{A}$ a point in the socle of $Rf_\star^\xi\overline{\mathbb{Q}}_l[d_{\mathsf{M}^\xi}]$. Then*

$$\operatorname{Amp}(\mathscr{L}_a^\bullet) \leq 2(d_{f^\xi} - d_{\mathsf{A}} + d_a),$$

and

$$\operatorname{codim}_{\mathsf{A}}(a) \leq d_{f^\xi},$$

where d_a is the dimension of the closure of a.

Ngô's support theorem is built on alternative bounds for the amplitude and the codimension, found by analyzing an action of the Tate module associated to the k_0-Picard stack $g_0 : \mathsf{P}_0^0 \to \mathsf{A}_0$ (the fiberwise identity component)

$$V_l\,\mathsf{P}_0^0 := R^{2d_f-1}g_{0,!}\overline{\mathbb{Q}}_l(d_f)$$

on the factors appearing in the decomposition (14.2.1). We recall the construction of this action and its properties. Writing $K_0^\bullet = Rf_{0,\star}\overline{\mathbb{Q}}_l$, there is a natural cup product

$$V_l\,\mathsf{P}_0^0 \otimes K_0^\bullet \to K_0^\bullet[-1]$$

defined by convolution along the correspondence $\mathsf{M}_0 \xleftarrow{a} \mathsf{P}_0 \times \mathsf{M} \xrightarrow{\pi_2} \mathsf{M}_0$, i.e.

$$(v \in V_l\,\mathsf{P}_0, x \in R^kf_{0,\star}\overline{\mathbb{Q}}_l) \mapsto \pi_{2,!}(\pi_1^\star v \cup a^\star x),$$

where a is the action map. Here the trace map $\pi_{2,!}$ lowers degree by $2d_f$ and cancels the twist (d_f), giving an output in $R^{k-1}f_{0,\star}\overline{\mathbb{Q}}_l$. Indeed, this is more naturally interpreted as the (-1)-part of the action of

$$\Lambda^\bullet := Rg_{0,!}\overline{\mathbb{Q}}_l[2d_f](d_f)$$

on $K_0^\bullet$ by the same formula. Extending the usual argument for families of abelian varieties, there is likewise a multiplication $\Lambda^\bullet \otimes \Lambda^\bullet \to \Lambda^\bullet$ which equips its cohomology with a graded-commutative algebra structure, and in particular a morphism

$$\bigwedge^i V_l\, \mathsf{P}_0^0 \to H^{-i}(\Lambda^\bullet),$$

which is actually an isomorphism. In fact, $\Lambda^\bullet$ is formal, in the sense that the complex splits canonically as a sum of its cohomology, cf. [Ngô10, §7.4.4]. We obtain an isomorphism

$$\Lambda^\bullet = \bigoplus_{i\geq 0} \bigwedge^i V_l\, \mathsf{P}_0^0[i] = \bigwedge^\bullet V_l\, \mathsf{P}_0^0$$

compatible with the multiplication.

The action of the Tate module (which we can view as a lisse sheaf supported in degree 0) induces an action on perverse cohomology: tensoring with a classical lisse sheaf is manifestly perverse t-exact, so

$$V_l\, \mathsf{P}_0^0 \otimes {}^p\mathscr{H}^i(K_0^\bullet) \cong {}^p\mathscr{H}^i(V_l\, \mathsf{P}_0^0 \otimes K_0^\bullet) \to {}^p\mathscr{H}^i(K_0^\bullet[-1]) = {}^p\mathscr{H}^{i-1}(K_0^\bullet).$$

The action of the Tate module necessarily splits across the decomposition of (14.2.1). It follows from the projection formula that for a point a in the socle, $V_l\, \mathsf{P}_{0,a}^0$ acts on each $\mathscr{L}_a^\bullet$ as an endomorphism of degree -1.

For a point a in the socle, one has a canonical decomposition of rational Tate modules into an affine and abelian part,

$$0 \to (V_l\, \mathsf{P}_{0,a}^0)^{\mathrm{aff}} \to V_l\, \mathsf{P}_{0,a}^0 \to (V_l\, \mathsf{P}_{0,a}^0)^{\mathrm{ab}} \to 0, \tag{14.2.2}$$

built from the Chevalley decomposition. Let us briefly re-emphasize that we are working with Deligne–Mumford stacks, which do not have canonical Chevalley decompositions, so this warrants some explanation.

Let $a \in \widetilde{\mathscr{A}}$ and choose any rigidification of $\mathsf{P}_{0,a}^0$, that is, a presentation of as a cokernel of smooth fiberwise-connected commutative group schemes

$$1 \to Q_{-1,a} \to Q_{0,a} \to \mathsf{P}_{0,a}^0 \to 1,$$

with Q_{-1} affine, and take the corresponding sequence of rational Tate modules

$$0 \to V_l\, Q_{-1,a} \to V_l\, Q_{0,a} \to V_l\, \mathsf{P}_{0,a}^0 \to 0.$$

Consider the Chevalley decomposition of $Q_{0,\overline{a}}$ at a geometric point $\overline{a}$ over a:

$$1 \to Q_{0,\overline{a}}^{\mathrm{aff}} \to Q_{0,\overline{a}} \to Q_{0,\overline{a}}^{\mathrm{ab}} \to 1.$$

As in the discussion before Definition 13.10, $Q_{-1,a}$ has trivial image in $Q_{0,\overline{a}}^{\mathrm{ab}}$ and the induced map on $V_l\, Q_{0,\overline{a}} \to V_l\, Q_{0,\overline{a}}^{\mathrm{ab}}$ descends to a surjection

$$V_l\, \mathsf{P}_{\overline{a}}^0 \to V_l\, Q_{0,\overline{a}}^{\mathrm{ab}} \to 0.$$

By considering its kernel one obtains a filtration of Tate modules over the geometric point into affine and abelian parts

$$0 \to (V_l\, \mathsf{P}_{\overline{a}}^0)^{\mathrm{aff}} \to V_l\, \mathsf{P}_{\overline{a}}^0 \to (V_l\, \mathsf{P}_{\overline{a}}^0)^{\mathrm{ab}} \to 0. \tag{14.2.3}$$

The decomposition is independent of choice of rigidification, and for any such rigidification with Chevalley decomposition as above the affine part is the image of $V_l\, Q_{0,\overline{a}}^{\mathrm{aff}}$ and the abelian part is $V_l\, Q_{0,\overline{a}}^{\mathrm{ab}}$.

We can extend the decomposition (14.2.3) to any Zariski point of $\mathscr{A}^\heartsuit$, in fact. This is slightly subtle because the Chevalley decomposition need only exist after a finite geometrically maximal normal affine subgroup need not be defined over the residue field, as in [Tot13] or [SGAIII, Exp. XVII, App. III, Cor. to Prop. C.5.1]. However, since finite radiciel extensions are universal homeomorphisms, the Tate modules and decomposition descend along such an extension.

The weights for these components of the Tate module are easy to compute using a choice of rigidification (and working over whatever extension of a necessary for the objects to be defined): since $Q_{0,a}^\flat$ is an abelian variety, its Tate module is dual to $H^1(Q_{0,a}^\flat, \mathbb{Q}_l)$, so is pure of weight -1. The affine part $R_{0,a}$, on the other hand, has l-torsion only in its toric part T_a, whose rational Tate module is isomorphic to $X_\star(T_a) \otimes_{\mathbb{Z}} \mathbb{Q}_l(1)$ with geometric Frobenius acting by finite order (with eigenvalues roots of unity) on the cocharacter lattice and by q^{-1} on the $\mathbb{Q}_l(1)$ factor. Overall, $V_l\, R_{0,a}$ is pure of weight -2. But we know that the action of $V_l\, \mathsf{P}^0_{0,a}$ takes $\mathscr{L}^n_a$, which is pure of weight n, to $\mathscr{L}^{n-1}_a$, which is pure of weight $(n-1)$. So the affine part of the Tate module acts *trivially* on the factors $\mathscr{L}^n_a$. Extending multiplicatively to $\Lambda^\bullet_a$, we have shown the following.

Proposition 14.2. *For each a in the socle of the ξ-truncated Hitchin fibration, the corresponding factor $\mathscr{L}^\bullet_a$ in the decomposition of (14.2.1) has an action of $\bigwedge^\bullet(V_l\, \mathsf{P}^0_{0,a})^{\mathrm{ab}}$, the cohomology of an abelian variety.*

14.3. Freeness of the cohomology over the Tate module.

The most technical part of Ngô's work on the support theorem is his analysis of this action. In the remainder of this section, we will show that this action makes $\mathscr{L}^\bullet_a$ into a free $\bigwedge^\bullet(V_l\, \mathsf{P}^0_{0,a})^{\mathrm{ab}}$-module. The crucial step is to to show a freeness result for the full cohomology of geometric closed fibers.

Proposition 14.3 ([CL12, Lemme 10.3.3, Théorème 10.5.1]). *For every geometric point $\overline{a} \in \mathsf{A}_0$ living over a closed point, the action of $\bigwedge^\bullet(V_l\, \mathsf{P}^0_{\overline{a}})^{\mathrm{ab}}$ on $H^\bullet(\mathsf{M}^\xi_{\overline{a}}, \overline{\mathbb{Q}}_l)$ makes it into a free module.*

This claim, of course, is checkable over algebraically closed k. We will deduce it from the following fundamental freeness result for étale-locally trivial families over an abelian variety.

Lemma 14.4 ([CL12, Lemme 10.3.3]). *Let Y a scheme or Deligne–Mumford stack over algebraically closed k, equipped with a proper k-morphism $f : Y \to B$ to an abelian variety. Assume that there exists an isogeny $\lambda : B' \to B$ which trivializes the family in the sense that after pullback along λ, the map $Y \times_B B' \to B'$ admits a translation-invariant section. Then the action of $H_\bullet(B', \overline{\mathbb{Q}}_l) = \bigwedge^\bullet V_l\, B' \cong \bigwedge^\bullet V_l\, B$ on $H^\bullet(X, \overline{\mathbb{Q}}_l)$ makes it into a free module.*

Proof. Factoring λ into purely inseparable and étale part, the purely inseparable part induces the identity on cohomology and so preserves our desired conclusion. So it suffices to show the result in the étale case. The local splitting condition is equivalent to an isomorphism $f^{-1}(0) \times_k B' \cong X \times_B B'$, i.e. a Cartesian square

$$\begin{tikzcd} f^{-1}(0) \times_k B' \arrow[r, "\lambda'"] \arrow[d, "\pi_2"] & X \arrow[d, "f"] \\ B' \arrow[r, "\lambda"] & B \end{tikzcd}$$

so that λ', the multiplication map, is an étale isogeny. That is, the fibration $f : X \to B$ is constructed by étale descent along the quotient by $\ker(\lambda)$. Étale descent on cohomology thereby identifies the pullback map on $\overline{\mathbb{Q}}_l$-cohomology with the inclusion of the $\ker(\lambda)$-invariants; we can then apply the Kunneth isomorphism:

$$\begin{aligned} H^\bullet(X) &\cong H^\bullet(f^{-1}(0) \times_k B')^{\ker(\lambda)} \\ &\cong (H^\bullet(f^{-1}(0)) \otimes H^\bullet(B'))^{\ker(\lambda)} \\ &\cong H^\bullet(f^{-1}(0))^{\ker(\lambda)} \otimes H^\bullet(B'). \end{aligned}$$

For the last isomorphism we observe that $\ker(\lambda)$ preserves $f^{-1}(0)$, and translations act trivially on $H^\bullet(B')$ by the usual homotopy argument. Clearly $H_\bullet(B')$ acts freely on this tensor product. □

Following Chaudouard–Laumon, we apply this lemma not to the truncated Hitchin fiber, but to a contracted product homeomorphic to it. We can use a slightly different construction (using G-affine Springer fibers rather than those for M) because the product formula of Ngô for anisotropic Hitchin fibers ([Ngô10, 4.15.1]) has been extended in [BČ22] to all of $\mathscr{A}^\heartsuit$. We recall the theorem. Given $v \in X$ and $a \in \mathscr{A}$, we denote $\mathscr{M}_{a,v}$ for the *affine Springer fiber* of a at the place v, an ind-scheme defined as in [Ngô10, §3.2]. We denote also $\mathscr{P}_{a,v}$ for the local Picard stack, parameterizing J_a-torsors on $\widehat{\mathscr{O}}_v$ equipped with trivializations on the punctured disk, as in [Ngô10, §3.3]. This acts on the affine Springer fiber with a dense orbit, just as in the global setting.

Theorem 14.5 ([BČ22, Theorem 4.3.8]). *For $a \in \mathscr{A}^{\heartsuit}(k)$, we write $j : U_a := a^{-1}(\mathfrak{c}_D^{\mathrm{reg}}) \hookrightarrow X$ for the open on which a is regular semisimple. Then Beauville–Laszlo gluing defines a universal homeomorphism of locally finite type algebraic k-stacks*

$$\mathscr{M}_a^{\mathrm{prod}} := \prod_{v \in X \setminus U_a} \mathscr{M}_{a,v}^{\mathrm{red}} \times^{\prod_{v \in X \setminus U_a} \mathscr{P}_{a,v}^{\mathrm{red}}} \mathscr{P}_a \longrightarrow \mathscr{M}_a.$$

It is evident from the construction that this map is equivariant for the action of $\mathscr{P}_a$ on the right factor and on $\mathscr{M}_a$. Pulling the map back to a homeomorphism $\mathsf{M}_a^{\mathrm{prod}} \to \mathsf{M}_a$ for (a,t) in the compatible locus, we can denote $\mathscr{M}_a^{\mathrm{prod},\xi} \to \mathscr{M}_a^{\xi}$ for its restriction to the inverse image of the ξ-stable locus.

Proof of Proposition 14.3. Consider the modified regular centralizer

$$J_a^{(S)} := j_\star j^\star J_a.$$

We remark that adelic uniformization/Beauville–Laszlo patching applies to this group scheme since it is generically reductive by the same reasoning as in Theorem 6.3 (though only over algebraically closed constants k), i.e.

$$\mathrm{Bun}_{J_a^{(S)}}(k) \cong [J_a(F) \backslash J_a(\mathbb{A}) / \prod_{v \notin S} J_a(\mathscr{O}_v) \prod_{v \in S} J_a(F_v)].$$

One advantage of the product formula for Hitchin fibers is that there is a natural Beauville–Lazlo patching map $\mathscr{M}_a^{\mathrm{prod}} \to \mathrm{Bun}_{J_a^{(S)}}$, which in general may not descend to $\mathscr{M}_a$.

One moreover has a map $J_a^\flat \to J_a^{(S)}$ adjoint to their identification over U_a; pushforward along it gives a surjective map of abelian stacks

$$\mathscr{P}_a^\flat = \mathrm{Bun}_{J_a^\flat} \to \mathrm{Bun}_{J_a^{(S)}}$$

which on k-points corresponds to the further quotient from the maximal bounded subgroup $J_a(F_v)^\flat$ to $J_a(F_v)$:

$$[J_a(F) \backslash J_a(\mathbb{A}) / \prod_{v \notin S} J_a(\mathscr{O}_v) \prod_{v \in S} J_a(F_v)^\flat] \to [J_a(F) \backslash J_a(\mathbb{A}) / \prod_{v \notin S} J_a(\mathscr{O}_v) \prod_{v \in S} J_a(F_v)].$$

This map is not an isogeny—valuation directions from split tori give a lattice of components in the fibers, but after pullback to A and restriction to the degree 0 part (the image of P^1) it is.

So we have a commutative diagram (with the horizontal maps actions)

$$\begin{tikzcd} \mathsf{P}^1 \arrow[r, "\sim"] \arrow[d] & \mathsf{M}_a^{\mathrm{prod}} \arrow[d] \\ \mathsf{P}^{\xi,\flat} \arrow[r, "\sim"] & \mathrm{Bun}_{J_a^{(S)}} \end{tikzcd}$$

By Poincaré reducibility as in *loc. cit.*, there exists a section of the left vertical map up to isogeny.[1] Such isogenies are invisible at the level of Tate module, and so the hypothesis of Lemma 14.4 is satisfied.

□

We next transfer this freeness result to the action of the Tate module on $\mathscr{L}_a^\bullet$. This relies on the crucial fact that on the Hitchin fibration, the Tate module $V_l\, \mathsf{P}_0^0$ is *polarizable*, in the sense that it admits an alternating form

$$V_l\, \mathsf{P}_0^0 \times V_l\, \mathsf{P}_0^0 \to \overline{\mathbb{Q}}_l(-1)$$

which is on each geometric fiber identically zero on the affine part and perfect on the abelian part. This is [Ngô10, Proposition 4.12.1]. The argument is an elegant reduction to the case of $\mathbb{G}_m$ using a map

$$J_a \to \pi'_{a,\star}(\mathbb{T} \times X'_a)$$

defined as in §12.3, where $\mathbb{T}$ is a split torus and we have chosen a splitting Γ-cover $X' \to X$ for the group and written $\pi'_a : X'_a := X' \times_X X_a \to X$ for the composite $\mathbb{W} \rtimes \Gamma$. The map induces an extension of scalars

[1] It is essential here that we are working over $k = \overline{\mathbb{F}}_q$: the extension class of Picard stacks associated with the map $\mathsf{P}^1 \to \mathsf{P}^{\xi,\mathrm{ab}}$ is torsion.

from J_a-torsors on X to $\mathbb{T}$-torsors on X'_a. This extension is a group homomorphism with respect to the group structure, so sends l^n-torsion points of $\mathscr{P}^\heartsuit$ to l^n-torsion points of $\mathrm{Bun}_T \cong \mathrm{Pic}_{X'_a} \otimes X_\star(\mathbb{T})$. Tensoring a symmetric invariant form on the lattice part with the classical Weil pairing on the Jacobian gives a form with the desired properties.

Ngô proves that polarizability, as well as the freeness on relative cohomology of Proposition 14.3, imply freeness of the Tate module action on each local system summand $\mathscr{L}_a^\bullet$. Its proof constitutes [Ngô10, §7.5-7.7], and would take us too far into the weeds to expound.

Proposition 14.6 ([CL10, Lemme 10.3.1], [Ngô10, 7.4.10]). *For any point a in the socle of the ξ-truncated Hitchin fibration, the fiber $\mathscr{L}_a^\bullet$ is a free module for $\bigwedge^\bullet(V_l\, \mathsf{P}_0^0)^{\mathrm{ab}}$.*

Corollary 14.7. *Given a point a in the socle of the truncated Hitchin fibration, one has the inequalities*

$$\mathrm{Amp}(\mathscr{L}_a^\bullet) \geq 2(d_f - \delta_a^{\mathrm{aff}})$$

and

$$\delta_a^{\mathrm{aff}} \geq \mathrm{codim}_{\mathsf{A}}(a).$$

Proof. The rank of the abelian part of the Tate module is 2 times the dimension of the abelian variety (after some rigidification), which is $d_f - \delta_a^{\mathrm{aff}}$ by Definition 13.10. Since the fiber is nonzero, choosing any nonzero element and taking its span under $\bigwedge^\bullet(V_l\, \mathsf{P}_0^0)^{\mathrm{ab}}$ gives nonzero elements spread across at least $2(d_f - \delta_a^{\mathrm{aff}})$ degrees.

Composing this lower bound on the amplitude with the upper bound of Proposition 14.1 gives

$$2(d_{f^\xi} - \delta_a^{\mathrm{aff}}) \leq 2(d_{f^\xi} - d_{\mathsf{A}} + d_a),$$

from which we deduce the codimension bound. □

In summary, Ngô's support theorem implies that the supports are large relative to their affine part.

Theorem 14.8 (cf. [CL12, Théorème 10.5.1]). *According to our running hypothesis, let G be reductive with anisotropic center. The socle (the set of generic points of support components) of the restriction of $Rf_\star \overline{Q}_{l,M}[d_{\widetilde{\mathscr{M}}^\xi}]$ to A^{good} is entirely contained within the elliptic locus $\mathsf{A}_{G\text{-}ell} \cap \mathsf{A}^{good}$.*

Proof. Let $a \in \mathsf{A}^{good}$ be a point in the socle. Since we are on the good locus of Definition 13.6, we have

$$\delta_a \leq \mathrm{codim}_{\mathsf{A}}(a).$$

The computation of 13.11 implies that if a is not G-elliptic

$$\delta_a^{\mathrm{aff}} < \delta_a \leq \mathrm{codim}_{\mathsf{A}}(a).$$

But the freeness of the action of the Tate module implies the bound of Corollary 14.7,

$$\delta_a^{\mathrm{aff}} \geq \mathrm{codim}_{\mathsf{A}}(a),$$

a contradiction. □

This implies that the perverse cohomology of the truncated Hitchin fibration is determined by its restriction to the elliptic locus, as below.

Corollary 14.9. *For each i, we have on A^{good} a canonical isomorphism*

$${}^p\mathscr{H}^i(Rf^\xi_\star \overline{\mathbb{Q}}_l[d_{\mathsf{M}^\xi}]) \cong j_{!\star} j^{\star p}\mathscr{H}^i(Rf^\xi_\star \overline{\mathbb{Q}}_l[d_{\mathsf{M}^\xi}]),$$

where $j : \mathsf{A}^{G\text{-}ell} \cap \mathsf{A}^{good} \hookrightarrow \mathsf{A}^{good}$ is the open immersion.

Proof. By the decomposition theorem and stability of perversity under intermediate extension and restriction to an open, both are semisimple perverse sheaves which agree on those summands whose supports intersect the elliptic locus. But the theorem implies all summands do, so the canonical map induces isomorphisms on all summands. □

14.4. Geometric endoscopy

Recall that associated to $\tilde{a} = (a, t) \in \widetilde{\mathscr{A}}_G$ (and its associated pointed cameral cover $(X_a, \widetilde{\infty}) \to (X, \infty)$) one has the monodromic invariant subgroup $W_{\tilde{a}} \subset W_\infty \rtimes \pi_1(U_a, \infty)$ of Definition 3.17, the normalizer of the irreducible component $C_{\tilde{a}} \subset X_a$ containing $\widetilde{\infty}$. Define the *inertia subgroup* $I_{\tilde{a}} \subset W_{\tilde{a}} \cap W_\infty$ to be the subgroup generated by elements which fix at least one point in $C_{\tilde{a}}$. Equivalently, it is the kernel of the composition $W_\infty \rtimes \pi_1(U_a, \infty) \to \pi_1(U_a, \infty) \to \pi_1(X, \infty)$.

Recall also that given $s \in \hat{T}_{tors}$, one has the coendoscopic connected-étale sequence

$$1 \longrightarrow \widehat{G}'_s \longrightarrow Z_{\widehat{G} \rtimes \mathrm{Out}(\widehat{G})}(s) \longrightarrow \pi_0^G(s) \longrightarrow 1 \ .$$

There is a corresponding short exact sequence of Weyl groups, noting that $\mathbb{W} \cong \widehat{\mathbb{W}}$ and $\mathrm{Out}(\mathbb{G}) \cong \mathrm{Out}(\widehat{G})$:

$$1 \to \mathbb{W}_{\widehat{G}_s} \to Z_{\mathbb{W} \rtimes \mathrm{Out}(\mathbb{G})}(s) \to \pi_0(s) \to 1.$$

Denote W_s for the middle term in this sequence and I_s the left term. By [Ngô06, Lemme 10.1], this short exact sequence is canonically split, so we have a canonical isomorphism $W_s \cong I_s \rtimes \pi_0(s)$.

Ngô showed ([Ngô10, 6.3.3], cf. Proposition 3.22) that for any $s \in \hat{T}_{tors}$ the (Frobenius-stable) locus

$$\widetilde{\mathscr{A}}_s := \{\tilde{a} : W_{\tilde{a}} \subseteq W_s, I_{\tilde{a}} \subseteq I_s\} \subset \widetilde{\mathscr{A}}$$

is exactly the (disjoint!) union of the images of the endoscopic closed embeddings

$$i_{(s,\rho_s)} : \widetilde{\mathscr{A}}_{G'_s} \to \widetilde{\mathscr{A}}_G$$

of Proposition 3.26 over all pointed endoscopic data $\rho_s : \pi_1(X, \infty) \to \pi_0(s)$ for that particular s. In particular, a point $\tilde{a} = (a, t) \in \widetilde{\mathscr{A}}_s$ gives rise to a natural map

$$\pi_1(U, \infty) \to W_{\tilde{a}}/I_{\tilde{a}} \to W_s/I_s = \pi_0(s)$$

which descends to $\rho_s : \pi_1(X, \infty) \to \pi_0(s)$, i.e. a pointed endoscopic datum for s. One can then show that $\tilde{a}$ is the image of a point in the Hitchin base $\widetilde{\mathscr{A}}_{H_s}$.

The above notions are all equally valid arithmetically (over k_0 rather than k), if one works with arithmetic endoscopic data in the sense of §2.10. For a pointed arithmetic endoscopic datum $\rho_s : \pi_1(X, \infty) \rtimes \mathrm{Gal}(k/k_0) \to \pi_1(s)$, the endoscopic group $G_s \to X$ descends to X_0 and the endoscopic locus $\widetilde{\mathscr{A}}_{\rho_s} \subset \widetilde{\mathscr{A}}_G$ likewise, i.e. is Frobenius-stable.

Remark 14.10. When $G = \mathbb{G} \times_k X$ is split (constant) $\rho_G = 1$ is the trivial cocycle, and so there is for each $s \in \hat{T}$ a canonical choice $\rho_s = 1$ (and so $G'_s = \mathbb{G}'_s$) of split endoscopic group. In [CL12], for each s the authors restrict to the open locus which contains only the endoscopic data associated to that split endoscopic group by manually removing the (finitely many by 3.29) closed loci $\widetilde{\mathscr{A}}_{\rho_s}$ associated to endoscopic data with $\rho_s \neq 1$. For a quasi-split group there is no such canonical choice, and indeed applications require a WFL for which neither G nor the endoscopic groups which appear need be split.

14.5. Geometric stabilization

With the requisite machinery built, and in particular Theorem 14.8 indicating that on the good locus all cohomological supports intersect the G-elliptic locus, one can now, transfer Chaudouard and Laumon's cohomological weighted fundamental lemma to our setting. Write $f_G^\xi : \mathsf{M}_G^\xi \to \mathsf{A}_G$ for the compatible ξ-stable Hitchin fibration associated to G. With the aid of ξ-truncation, we can extend Ngô's cohomological result, which we recall now, from the elliptic locus to A_G^{good}.

For each elliptic endoscopic datum $(s, \rho_s, \rho_s \to \rho_G)$, write $i_{\rho_s} : \widetilde{\mathscr{A}}_{G_s}^{G_s\text{-ell}} \hookrightarrow \mathsf{A}_G^{G\text{-ell}}$ for the closed embedding of Proposition 3.26, restricted to the elliptic loci.[2]

[2]For the reader who has been carefully tracking questions of compatibility in the sense of the discrepancy between A and $\widetilde{\mathscr{A}}$, note that compatibility is a vacuous condition on the elliptic locus, so $\widetilde{\mathscr{A}}_G^{G\text{-ell}} = \mathsf{A}_G^{G\text{-ell}} \subset \mathsf{A}_G$.

Theorem 14.11 (The cohomological Fundamental Lemma, [Ngô10, 6.4.1, 6.4.2]). *Let $s \in \hat{T}_{tors}$. On $\mathsf{A}_{G\text{-}ell}$ one has a canonical isomorphism of fine isotypic parts in the sense of Theorem 12.33:*

$$ {}^p\mathscr{H}^\bullet(Rf_\star^{G\text{-}ell}\overline{\mathbb{Q}}_l)_s \cong \bigoplus_{(s,\rho_s)} (i_{\rho_s})_\star {}^p\mathscr{H}^\bullet(Rf_{G'_{\rho_s},\star}^{G'_{\rho_s}\text{-}ell}\overline{\mathbb{Q}}_l)_1[-2r_s](-r_s), $$

where $r_s = \operatorname{codim}_{\widetilde{\mathscr{A}}_G}(\widetilde{\mathscr{A}}_{G'_{\rho_s}})$ and the sum ranges over (geometric) elliptic endoscopic data (s,ρ_s) with fixed component s, for G'_{ρ_s} the corresponding elliptic endoscopic group. Equivalently, for each (s,ρ_s) elliptic,

$$ (i_{\rho_s})^{\star p}\mathscr{H}^\bullet(Rf_\star^{G\text{-}ell}\overline{\mathbb{Q}}_l)_s \cong {}^p\mathscr{H}^\bullet(Rf_{G'_{\rho_s},\star}^{G'_{\rho_s}\text{-}ell}\overline{\mathbb{Q}}_l)_1[-2r_s](-r_s). $$

In fact, if a given geometric endoscopic datum extends to an arithmetic one over finite coefficient field k_0, then the second isomorphism above is defined over k_0, up to taking semisimplifications of both sides.

Remark 14.12. By the elementary bound of [Ngô10, Proposition 6.2.3], the support of the s-isotypic part on the elliptic locus lies within $\widetilde{\mathscr{A}}_s = \bigcup_{\rho_s} i_{\rho_s}(\widetilde{\mathscr{A}}_{G_{\rho_s}})$, the disjoint union of all the endoscopic loci for geometric endoscopic data extending s. Pulling back or pushing forward between one of these components justifies the equivalence between the two geometric statements.

Remark 14.13. We remark that under our running hypothesis that $A_G^{\mathrm{spl}} = 1$, $Z_{\widehat{G}}^{\Gamma,0} = 1$ and Ngô's decomposition is exactly the fine decomposition of 12.33. For groups with anisotropic centers, G-ellipticity in our sense coincides with anisotropicity. For groups with nontrivial split central tori, Ngô's anisotropic locus is empty, so we have lost no generality.

Now pick once and for all a semistandard Levi M. Recall that the open $U_M \subset \mathsf{A}_G$ (inside the compatible locus of Definition 3.9) is defined to be the union of the L-elliptic loci for $L \supseteq M$. It is Frobenius-stable by Corollary 2.21 and its consequence that the constructions of §3 are compatible with extending scalars. Write $f^{\xi,M\text{-par}}$ for the restriction of f^ξ to this open. Since the δ-invariant is likewise compatible with extending scalars, the good locus $\widetilde{\mathscr{A}}^{good}$ of Definition 13.6 is Frobenius-stable. So we can define over k_0 the open embedding

$$ j_{\mathrm{ell}} : \mathsf{A}_{G\text{-ell}}^{good} \hookrightarrow U_M^{good}. $$

We can now give a description of each coarse summand in terms of fine pieces on the G-elliptic locus.

Theorem 14.14. *Let G quasisplit reductive with $A_G^{\mathrm{spl}} = 1$ and $\xi \in \mathfrak{a}_T$ in general position. Let $\overline{s}_M \in (\hat{T}/Z_{\widehat{M}}^0)_{tors}$. Then as perverse sheaves on U_M^{good}, the corresponding coarse $\overline{s}_M$-isotypic component of Theorem 12.33 satisfies*

$$ {}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-}par}\overline{\mathbb{Q}}_l)_{\overline{s}_M} \cong \bigoplus_{\overline{s}} \bigoplus_{(s,\rho_s)} (j_{\mathrm{ell}})_{!\star}(i_{\rho_s})_\star {}^p\mathscr{H}^i(Rf_{G'_{\rho_s},\star}^{G'_{\rho_s}\text{-}ell}\overline{\mathbb{Q}}_l)_1[-2r_s](-r_s), $$

where

- *the sum s ranges over the fiber of $\overline{s}_M$ under $\hat{T}_{tors} \to (\hat{T}/Z_{\widehat{M}}^0)_{tors}$,*
- *(s,ρ_s) ranges over elliptic geometric pointed endoscopic data for G with fixed s.*
- *${}^p\mathscr{H}^i(Rf_{G'_{\rho_s},\star}^{G'_{\rho_s}\text{-}ell}\overline{\mathbb{Q}}_l)_1$ is the fine 1-isotypic part in the decomposition of Theorem 12.33 applied to the (quasisplit reductive) X-endoscopic G'_{ρ_s}, and*
- *$r_s = \operatorname{codim}_{\widetilde{\mathscr{A}}_G}(\widetilde{\mathscr{A}}_{G'_{\rho_s}})$.*

Proof. By Corollary 14.9, we know that on the good locus we have

$$ {}^p\mathscr{H}^i(Rf_\star^\xi\overline{\mathbb{Q}}_l) \cong (j_{ell})_{!\star}(j_{ell})^{\star p}\mathscr{H}^i(Rf_\star^\xi\overline{\mathbb{Q}}_l). $$

Once restricted to U_M^{good}, this splits across the coarse s-decomposition of Theorem 12.33, so writing $K_{\overline{s}_M} := {}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-par}}\overline{\mathbb{Q}}_l)_{\overline{s}_M}|_{U_M^{good}}$, it decomposes in terms of the *fine* summands on the G-elliptic locus:

$$ K_{\overline{s}_M} \cong (j_{\mathrm{ell}})_{!\star}(j_{\mathrm{ell}})^\star K_{\overline{s}_M} \cong (j_{\mathrm{ell}})_{!\star} \bigoplus_{\substack{s\in\widehat{T}_{tors}\\ s\mapsto\overline{s}_M}} {}^p\mathscr{H}^i(Rf_{G,\star}^{G\text{-ell}}\overline{\mathbb{Q}}_l)_s. $$

By the geometric Fundamental Lemma 14.11 of Ngô, the fine s-isotypic factor is a sum of stable factors over elliptic endoscopic data (s, ρ_s) for that fixed s

$$K_{\overline{s}_M} \cong (j_{\mathrm{ell}})_{!\star} \bigoplus_{\substack{s \in \widehat{T}_{tors} \\ s \mapsto \overline{s}_M}} \bigoplus_{(s,\rho_s)} (i_{\rho_s})_\star {}^p\mathscr{H}^i(Rf^{G_{\rho_s}\text{-ell}}_{G_{\rho_s},\star} \overline{\mathbb{Q}}_l)_1[-2r_s](-r_s)$$

where the index sets are as desired.

□

Remark 14.15. This theorem is stated geometrically, over k rather than k_0, and indeed not all the elliptic endoscopic data (s, ρ_s) which arise are arithmetic for a given k_0. Correspondingly, this decomposition is not fully defined over k_0. However, each term is defined over an appropriate finite extension, and there are finitely many such terms by Proposition 3.29, so the entire decomposition is defined over an appropriate finite extension of k_0. Our application will ultimately need an arithmetic version, making use of [Ngô10, 6.4.2] rather than [Ngô10, 6.4.1].

Remark 14.16. Arthur's statement of the WFL differs from this theorem in a few ways:

(1) The WFL contains only a single sum over s, with a single endoscopic datum for G induced from an endoscopic datum for M and the choice of s, whereas the theorem above is a double sum.
(2) The WFL allows only $s \in s_M Z^\Gamma_{\widehat{M}}$, whereas the theorem has no Galois-fixed requirement.
(3) The WFL allows translates by the full (Galois-fixed) center, whereas the theorem ranges only over $s \in s_M Z^0_{\widehat{M}}$.

All of the discrepancies in these remarks, except the bookkeeping between $Z^\Gamma_{\widehat{M}}$ and $(Z^0_{\widehat{M}})^\Gamma$, will naturally resolve when we specify a full endoscopic datum for M, rather than just s_M. We explain this in the next section.

14.6. The cohomological Weighted Fundamental Lemma

The Weighted Fundamental Lemma depends on an endoscopic datum $\mathfrak{M} = (M', ...)$ for a Levi M, and sums over a set of (representatives of) endoscopic data G'_s for G which are compatible with it (and in particular contain M' as a Levi) as defined in [Art98, §4]. One version of this construction was recalled in the Introduction. In order to give a statement of our cohomological theorem that more closely matches the WFL, we reinterpret this construction in terms of Ngô's formalism for endoscopic data. In particular, this version will imply a global version of the WFL in the stalk over a point which is M'-elliptic.

For the sake of the general construction and definitions, we briefly relax our hypothesis of anisotropic center and let G arbitrary quasisplit reductive. We fix a semistandard Levi $M \subset G$. Let $\mathfrak{M}' = (s_M, \rho_{s_M})$ be a pointed elliptic endoscopic datum for M, defining quasisplit endoscopic group $M' \to X$. That is, $s_M \in \hat{T}_{tors}$, defining the connected-étale short exact sequence

$$1 \longrightarrow \widehat{M}'_{s_M} \longrightarrow Z_{\widehat{M} \rtimes \mathrm{Out}(\widehat{M})}(s_M) \longrightarrow \pi_0^M(s_M) \longrightarrow 1 \ ,$$

and $\rho_{s_M} : \pi_1(X, \infty) \to \pi_0^M(s_M)$ whose composition with $\pi_0^M(s_M) \to \mathrm{Out}(\widehat{M})$ defines $M' \to X$ as a form of the split Langlands dual $\mathbb{M}'$ of $\widehat{M}'_{s_M}$. Each $s \in s_M Z^\Gamma_{\widehat{M}}/Z^\Gamma_{\widehat{G}}$ induces a corresponding endoscopic datum $(s, \rho_s, \rho_s \to \rho_G)$ for G, defining endoscopic group $G'_s \to X$, as follows. We have a connected-étale sequence of the centralizer

$$1 \longrightarrow \widehat{G}'_s \longrightarrow Z_{\widehat{G} \rtimes \mathrm{Out}(\widehat{G})}(s) \longrightarrow \pi_0^G(s) \longrightarrow 1 \ .$$

Choosing a representative $s = s_M z$ with $z \in Z(\widehat{M})^\Gamma \subseteq Z(\widehat{M} \rtimes \mathrm{Out}(\widehat{M}))$, we can see that

$$Z_{\widehat{M} \rtimes \mathrm{Out}(\widehat{M})}(s_M) \subseteq Z_{\widehat{G} \rtimes \mathrm{Out}(\widehat{G})}(s),$$

so we get an induced map

$$q_s : \pi_0^M(s_M) \to \pi_0^G(s), \tag{14.6.1}$$

not necessarily injective.

Definition 14.17. Given (pointed) elliptic endoscopic datum $\mathfrak{M}' = (s_M, \rho_{s_M})$ for M and $s \in s_M Z^{\Gamma}_{\widehat{M}}/Z^{\Gamma}_{\widehat{G}}$, define the *induced (pointed) endoscopic datum for* G to be the pair $\mathfrak{G}'_s = (s, \rho_s)$, where $\rho_s = (q_s)_\star \rho_{s_M}$ is the pushforward along (14.6.1). We denote $G'_s \to X$ the corresponding endoscopic group for G.

Denote *Arthur's parameter set* $\mathscr{E}_{\mathfrak{M}'}(G)$ to be the set of such

$$s \in s_M Z^{\Gamma}_{\widehat{M}}/Z^{\Gamma}_{\widehat{G}}$$

such that the induced datum is elliptic for G, in the sense of Definition 3.28. In particular, by Proposition 3.29 all such endoscopic groups have anisotropic center if G does.

It is useful to define two refinements of this set. Define the *coarse parameter set* $\mathscr{E}^c_{\mathfrak{M}'}(G)$ the set of

$$s \in s_M (Z^0_{\widehat{M}})^{\Gamma}/(Z^0_{\widehat{G}})^{\Gamma},$$

respectively the *fine parameter set* $\mathscr{E}^f_{\mathfrak{M}'}(G)$ the set of

$$s \in s_M (Z^0_{\widehat{M}})^{\Gamma}/Z^{\Gamma,0}_{\widehat{G}},$$

in both cases ranging only over s such that the induced datum is elliptic. Note that under our running hypothesis of anisotropic center, $Z^{\Gamma,0}_{\widehat{G}} = 1$.

If $\mathfrak{M}' = 1_M = (1, \rho_M)$ is the trivial endoscopic datum for M defining M itself, then denote $\mathscr{E}_M(G) := \mathscr{E}_{\mathfrak{M}'}(G)$ and $\mathscr{E}^f_M(G), \mathscr{E}^c_M(G)$ likewise.

Remark 14.18. The constructions and definition above are equally valid whether one works geometrically over k, or arithmetically. In other words, if the endoscopic datum $\mathfrak{M}'$ was defined over some finite constant field k_0, then the induced endoscopic data for G will all be defined over the same constant field.

As will be useful in §15 below, we observe that if the datum $\mathfrak{M}'$ is split at the point ∞ (i.e. if the endoscopic group M' is), then the induced data $\mathfrak{G}'_s$ are likewise.

Remark 14.19. For arbitrary reductive G and elliptic endoscopic datum $\mathfrak{M}'$ for $M \subset G$, the set $\mathscr{E}_{\mathfrak{M}'}(G)$ is finite: by consideration of sub-root systems, there are at most finitely many possibilities for $\widehat{G'_s} = Z_{\widehat{G}}(s)^0$, and for each, the ellipticity condition implies finitude of $Z^{\Gamma}_{\widehat{G'_s}}/Z^{\Gamma}_{\widehat{G}}$, so at most finitely many classes of s generate each $\widehat{G'_s}$. (The torsor ρ_s is then uniquely determined as the pushforward.)

Remark 14.20. There are natural maps

$$\mathscr{E}^f_{\mathfrak{M}'}(G) \to \mathscr{E}^c_{\mathfrak{M}'}(G) \to \mathscr{E}_{\mathfrak{M}'}(G).$$

The first map has fibers which are torsors under the dual residual torsion group of Proposition 12.28

$$\frac{(Z^0_{\widehat{G}})^{\Gamma}}{Z^{\Gamma,0}_{\widehat{G}}} = \widehat{\Xi}_G.$$

The second map has fibers which are torsors under the kernel

$$\frac{(Z^0_{\widehat{M}})^{\Gamma} \cap Z^{\Gamma}_{\widehat{G}}}{(Z^0_{\widehat{G}})^{\Gamma}}.$$

These discrepancies will create the expected scalar factors between our cohomological Theorem 14.24 indexed by $\mathscr{E}^f_{\mathfrak{M}'}(G)$ and the global Weighted Fundamental Lemma indexed by $\mathscr{E}_{\mathfrak{M}'}(G)$.

All the induced endoscopic G'_s naturally contain M'.

Proposition 14.21. *We have a natural identification of root data*

$$\Phi(\widehat{M'}, \hat{T}) = \Phi(\widehat{G'_s}, \hat{T}) \cap \Phi(\widehat{M}, \hat{T})$$

inside $\Phi(\widehat{G}, \hat{T})$. *Moreover, a cocharacter* $\lambda : \mathbb{G}_m \to \hat{T}$ *whose centralizer is* $\widehat{M} \subset \widehat{G}$ *also cuts out* $\widehat{M}'$ *inside* $\widehat{G}'$*.In particular,* M' *is naturally a semistandard Levi in each induced* G'_s.

Proof. Write $s = s_M z$ for $z \in Z_{\widehat{M}}$. By definition $\widehat{G'_s} = Z_{\widehat{G}}(s)^0$, so

$$\hat{\alpha} \in \Phi(\widehat{G'_s}, \hat{T}) \Longleftrightarrow \hat{\alpha}(s) = 1.$$

But for any absolute roots $\hat{\alpha}$ of $\widehat{M}$, we have $\hat{\alpha}(z) = 1$, so this last condition is equivalent to $\hat{\alpha}(s_M) = 1$, which defines $\Phi(\widehat{M'}, \hat{T})$. This gives the statement. But then by root space considerations, the same cocharacter which cuts out M inside G cuts out M' inside G'_s. As in the discussion of §2.8, we conclude the last statement. $\square$

We now seek to isolate a summand of the relative cohomology $K^i_{\overline{s}_M} := {}^p\mathscr{H}^i(Rf_\star^{\xi, M\text{-par}}\overline{\mathbb{Q}}_l)_{s_M}|_{U_M^{good}}$ corresponding to a full pointed endoscopic datum (s_M, ρ_{s_M}). We would like to define it to be the sum of those pieces which have support containing $\iota_{\rho_{s_M}}(\widetilde{\mathscr{A}}_{M'})$. Geometrically, this is the right definition, but in order to obtain a definition which descends to k_0, we need to take the semisimplification first.

Definition 14.22. Given a pointed elliptic endoscopic datum $\mathfrak{M}' = (s_M, \rho_{s_M})$ for M defined over k_0, define the *coarse semisimple* $\mathfrak{M}'$*-component* to be the maximal summand of the semisimplified coarse component

$$(K^i_{\overline{s}_M})^{\mathrm{ss}}_{\mathfrak{M}'} \subseteq (K^i_{\overline{s}_M})^{\mathrm{ss}}$$

consisting of those simple components whose geometric support contains $\iota_{\rho_{s_M}}(\widetilde{\mathscr{A}}^{M'\text{-ell}}_{M'})$.

Note that there is no subtlety about compatible loci here, since the M'-elliptic locus is vacuously compatible and embeds into the compatible M-stratum of G.

The definition above defines a Frobenius-stable summand because $\iota_{\rho_{s_M}}(\widetilde{\mathscr{A}}_{M'})$ is defined over k_0. Moreover, the summands which appear (out of those parameterized by elliptic endoscopic data (s, ρ_s) for G after Theorem 14.14) are exactly those represented by $s \in \mathscr{E}^f_{\mathfrak{M}'}(G)$.

Lemma 14.23. *Let s be a fine G-character occurring the restriction to the G-elliptic locus of the coarse $\overline{s}_M$-components, and let $\mathfrak{G}' = (s, \rho_s)$ be a geometric elliptic endoscopic datum for G which appears in Ngô's decomposition of that fine summand. The support of the $\mathfrak{G}'$-component contains $\iota_{\rho_{s_M}}(\widetilde{\mathscr{A}}^{M'\text{-}ell}_{M'}) \cap U_M^{good}$ if and only if $\mathfrak{G}'$ is induced from $\mathfrak{M}'$ by $s \in \mathscr{E}^f_{\mathfrak{M}'}(G)$ in the sense of Definition 14.17.*

In particular, if $a \in U_M^{good}$ is the image of a point $a_{M'} \in \widetilde{\mathscr{A}}^{M'\text{-}ell}_{M'}$, then

$$(K^i_{\overline{s}_M})^{\mathrm{ss}}_{\mathfrak{M}', a} = (K^i_{\overline{s}_M})^{\mathrm{ss}}_a.$$

Proof. If $\mathfrak{G}' = \mathfrak{G}'_s$ is induced, then M' is a Levi in G'_s. The Hitchin base is functorial in endoscopic groups and in particular in Levi subgroups, so we have a diagram of closed embeddings

$$\begin{tikzcd} & \widetilde{\mathscr{A}}_M \arrow[dr, "\chi^M_G"] & \\ \widetilde{\mathscr{A}}_{M'} \arrow[ur, "\iota_{\rho_{s_M}}"] \arrow[dr, "\chi^{M'}_{G'}"'] & & \widetilde{\mathscr{A}}_G \\ & \widetilde{\mathscr{A}}_{G'_s} \arrow[ur, "\iota_{\rho_s}"'] & \end{tikzcd}$$

We can easily see that this square commutes, so the $\mathfrak{G}'$-component has support containing the image of $\widetilde{\mathscr{A}}_{M'}$.

Conversely, if $\widetilde{\mathscr{A}}_{G'_s} \cap \widetilde{\mathscr{A}}^{M'\text{-ell}}_{M'} \neq \emptyset$ for some $s = s_M z$, with $z \in Z^0_{\widehat{M}}$, then we must have z Galois-fixed and so $z \in (Z^0_{\widehat{M}})^\Gamma$. Indeed, at any point a in the intersection, we have $W_{\tilde{a}} \subseteq W_s$, so W_a stabilizes $s = s_M z$ and induces ρ_s, but also stabilizes s_M. For any $\gamma \in \pi_1(U_a, \infty)$, let $w_\gamma \rtimes \gamma \in W_{\tilde{a}}$ of the monodromic subgroup lying over it. Since $\widetilde{\mathscr{A}}_{M'} \subset \widetilde{\mathscr{A}}_M$ and so $w_\gamma \in W_M$, we combine the above facts to obtain

$$s_M z = (w_\gamma \rtimes \gamma)(s_M z) = s_M (w_\gamma \rtimes \gamma)(z) = s_M \gamma(z)$$

and so $z = \gamma(z)$ for all $\gamma \in \Gamma$, as claimed.

Second, recall that by [Ngô10, 6.3.3] the images loci of the embeddings $\iota_{\rho_s} : \widetilde{\mathscr{A}}_{G'_s} \to \widetilde{\mathscr{A}}_G$ are disjoint as ρ_s ranges over the finite collection of torsors for a fixed s, and so at most one ρ_s-summand survives restriction to $\widetilde{\mathscr{A}}_{M'}^{M'\text{-ell}}$. But the commutativity above implies that $\iota_{\rho_{s_M}}(\widetilde{\mathscr{A}}_{M'}^{M'\text{-ell}})$ lies inside the $\iota_{\rho_s}(\widetilde{\mathscr{A}}_{G'_s})$ for the induced ρ_s, so it cannot for any non-induced torsor for s. $\square$

We deduce our main cohomological theorem, under our running hypothesis of G quasisplit reductive with anisotropic center.

THEOREM 14.24. *Let $\mathfrak{M}' = (s_M, \rho_{s_M})$ an arithmetic (defined over k_0) elliptic endoscopic datum for M with endoscopic group M'. We have an isomorphism of semisimplified Weil sheaves on U_M^{good}*

$$({}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-}par}\overline{\mathbb{Q}}_l)_{\overline{s}_M})^{\mathrm{ss}}_{\mathfrak{M}'} \cong \left(\bigoplus_{s\in\mathscr{E}^f_{\mathfrak{M}'}(G)} (j_{ell})_{!\star}(i_{\rho_s})_\star {}^p\mathscr{H}^i(Rf^{G'_s\text{-}ell}_{G'_s,\star}\overline{\mathbb{Q}}_l)_1[-2r_s](-r_s)\right)^{\mathrm{ss}},$$

where

- ρ_s *and* G'_s *are the compatible endoscopic data of Definition 14.17 for* $s \in \mathscr{E}^f_{\mathfrak{M}'}(G)$,
- ss *denotes the semisimplification of perverse sheaves,*
- $({}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-}par}\overline{\mathbb{Q}}_l)_{\overline{s}_M})^{\mathrm{ss}}_{\mathfrak{M}'}$ *is the coarse semisimple* $\mathfrak{M}'$*-component of Definition 14.22,*
- ${}^p\mathscr{H}^i(Rf^{G'_{\rho_s}\text{-}ell}_{G'_{\rho_s},\star}\overline{\mathbb{Q}}_l)_1$ *is the* 1*-isotypic part in the fine decomposition of Theorem 12.33 applied to the (quasisplit reductive) X-endoscopic* G'_{ρ_s},
- $r_s = \mathrm{codim}_{\widetilde{\mathscr{A}}_G}(\widetilde{\mathscr{A}}_{G'_{\rho_s}})$.

PROOF. Again denote $K^i_{\overline{s}_M} = {}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-par}}\overline{\mathbb{Q}}_l)_{\overline{s}_M}|_{U_M^{good}}$, viewed as a Weil sheaf. After base change to $k = \overline{k}_0$, we have a geometric decomposition

$$j^\star_{ell}(K^i_{\overline{s}_M})^{\mathrm{ss}}|_k = \bigoplus_s \bigoplus_{(s,\rho_s)} K^i_{G,s,\rho_s},$$

which coarsens to an arithmetic decomposition when we group the endoscopic data into Frobenius orbits. But the lemma above and Theorem 14.14 compute that when we restrict to geometric components with support $\iota_{\rho_{s_M}}(\widetilde{\mathscr{A}}_{M'})$, this corresponds to restricting to the set of induced data for $s \in \mathscr{E}^f_{\mathfrak{M}'}(G)$, which are each Frobenius-fixed by Remark 14.18, so define a Frobenius-stable summand K_{G,s,ρ_s}. We obtain an arithmetic decomposition (up to semisimplification)

$$j^\star_{ell}(K^i_{\overline{s}_M})^{\mathrm{ss}}_{\mathfrak{M}'} \cong \bigoplus_{s\in\mathscr{E}^f_{\mathfrak{M}'}(G)} K^{\mathrm{ss}}_{s,\rho_s}.$$

The last statement of Theorem 14.11 implies that each $K^{\mathrm{ss}}_{s,\rho_s}$, the semisimplified fine summand with support $\iota_{\rho_s}(\widetilde{\mathscr{A}}_{G'_s})$, is isomorphic as a Weil sheaf to

$$\left((i_{\rho_s})_\star {}^p\mathscr{H}^i(Rf^{G'_s\text{-ell}}_{G'_s,\star}\overline{\mathbb{Q}}_l)_1[-2r_s](-r_s)\right)^{\mathrm{ss}}.$$

Theorem 14.8 implies that

$$K^i_{\overline{s}_M} \cong (j_{ell})_{!\star} j^\star_{ell} K^i_{\overline{s}_M}$$

over k_0, i.e., as Weil sheaves, and likewise for its semisimplified summand $(K^i_{\overline{s}_M})^{\mathrm{ss}}_{\mathfrak{M}'}$, so intermediate extension gives the theorem. $\square$

Part 4

Arithmetic applications

CHAPTER 15

Point counts, s-orbital integrals, and the global WFL

In this chapter, we define the global s-orbital integrals for our setting and carry over roughly unchanged the Frobenius point-counting argument of Chaudouard–Laumon [CL12, §11, 12] for ξ-weighted Hitchin fibers. In the first section, we define s-orbital integrals, prove their invariance up to $Z^{\Gamma}_{\widehat{M}}$. In the second section, we express the trace of Frobenius on a *coarse* component of the relative ξ-stable Hitchin cohomology as a global (adelic) weighted orbital integral. For ξ in general position, the result is in terms of a global version of Arthur's weighted orbital integrals, and in particular is independent of ξ. This provides the arithmetic interpretation of the left side of the cohomological WFL 14.24. In the third subsection, we define recursively the stabilized weighted orbital integrals, and inductively prove that the trace of Frobenius on a *fine* stable summand. Together, these give in Theorem 15.16 a global incarnation of the WFL that matches exactly Arthur's expected form.

The moral of this chapter is that the techniques applied in the split case by [CL10] will apply equally well here, as long as we are working over the compatible locus and all weight conditions are defined relative to the *rational* notion of ξ-stability from Definition 5.7. This is absolutely necessary in order to interface with Arthur's weight functions, which are defined purely in terms of relative/rational data.

We recall our running situation and fix notation: let $k_0 = \mathbb{F}_q$ and $k = \overline{\mathbb{F}}_q$. Let X_0 be a curve over $k_0 = \mathbb{F}_q$ with generic point η_0, function field F_0, and adèles $\mathbb{A}_0$. Let G_0 a quasisplit[1] reductive X_0-group. We will often assume G_0 to have anisotropic center. Fix Borel B_0 and maximal torus T_0, and assume the characteristic of k_0 is good for G_0. We fix a geometrically connected Galois Γ-cover $X_0' \to X_0$ which splits G_0, cf. Proposition 2.20. Denote without a 0 the corresponding objects over $X = X_0 \times_{k_0} k$. We denote by τ the arithmetic Frobenius automorphism $x \mapsto x^q$ of k as well as the induced automorphism $1 \times \tau$ of X, G, B, etc. Proposition 2.20 and its Corollary imply that the torus T is no more generically split than T_0, that the sets of semistandard Levis and parabolics in G_0 and in G are in bijection, and that the rational character and cocharacter lattices remain the same when base changing from k_0 to k, i.e., the rational lattices satisfy

$$X^\star(M) = X^\star(M_0) \qquad \text{and} \qquad X_\star(M) = X_\star(M_0)$$

for each semistandard Levi M. The same invariance applies to the dual space $\mathscr{X}_\star(M)$ and the vector spaces $\mathfrak{a}_M$ and $\mathfrak{a}_M^\star$.

For each closed point $v_0 \in |X_0|$, write k_{v_0} for its residue field, z_{v_0} for a uniformizer at v_0, $\mathscr{O}_{v_0}$ for its (complete) local ring of integers and F_{v_0} for its local fraction field. Let L^+_{G,v_0}, L_{G,v_0}, and Gr_{G,v_0} denote the arc group, loop group, and affine Grassmannian of $G_0|_{\mathscr{O}_v}$, respectively. Likewise, write $L^+_{\mathfrak{g},v}$, $L_{\mathfrak{g},v}$ for the arc and loop Lie algebras, with k_{v_0}-points given by $\mathfrak{g}(\mathscr{O}_{v_0})$ and $\mathfrak{g}(F_{v_0})$. Pick an effective divisor

$$D_0 = \sum_{v_0 \in |X_0|} d_{v_0} v_0$$

which is even (i.e. d_{v_0} is even for all v_0) and disjoint from ∞, and denote its base change to X by D. Write 1_D the indicator function of $(z_v^{-d_v})_v \mathfrak{g}(\mathscr{O})$. Denote

$$K_v := L^+_{G,v}(k_v)$$

and $K := \prod_v K_v$ the compact open in $G(\mathbb{A}_0)$ corresponding to the integral group scheme G_0.

[1] This means pinned quasisplit, as in §2.

15.1. Global weighted s-orbital integrals

In this section, G is not required to have anisotropic center. We recall Chaudouard–Laumon's construction of global weighted s-orbital integrals. The results of this section follow exactly as in [CL12, §11], with adapted definitions (in particular, the monodromic subgroup W_{a_M} of Definition 3.17 which lies in $W_M \rtimes \Gamma$) and some careful tracking of Γ-invariants and coinvariants. We begin with an application of Kottwitz's Lemma 12.19 above.

Proposition 15.1 ([CL12, 11.2.1]). *Let $U_0 \subset X_0$ an open containing ∞, and consider the functor $B : \mathsf{Tori}_{U_0} \to \mathsf{Ab}$ given by*

$$B(Z) \mapsto (Z(F)\backslash Z(\mathbb{A}))_\tau.$$

This satisfies the hypotheses (1), (2) of Kottwitz's lemma, and so there is a canonical isomorphism of functors $\underline{X}_\star(-)(\infty)_{\pi_1(U_0,\infty)} \cong B(-)$. Moreover, there is a canonical map $\underline{X}_\star(-)_\infty \to B$ such that the natural triangle

$$\begin{tikzcd} \underline{X}_\star(-)_\infty \arrow[d] \arrow[dr] & \\ \underline{X}_\star(-)(\infty)_{\pi_1(U_0,\infty)} \arrow[r, "\sim"'] & B(-) \end{tikzcd}$$

commutes.

Choose now a semistandard Levi M and a point $a_M = (h_{a_M}, t) \in \widetilde{\mathscr{A}}_M(k_0) = \widetilde{\mathscr{A}}_M(k)^\tau$. The M-regular centralizer

$$J_{a_M} := h_{a_M}^\star J_D^M$$

is a torus over $U_{0,a_M} := h_{a_M}^{-1}(\mathfrak{c}_{M,D}^{G\text{-reg}})$, so by the previous Proposition we have an isomorphism

$$(J_{a_M}(F)\backslash J_{a_M}(\mathbb{A}))_\tau \cong \underline{X}_\star(J_{a_M})(\infty)_{\pi_1(U_{0,a_M},\infty)}.$$

As argued in the proof of 12.18, the choice of $\widetilde{\infty}$ over ∞ (that is, the choice of t) determines an isomorphism $\mathbb{X}_\star := \underline{X}_\star(T)|_\infty \cong \underline{X}_\star(J_a)|_\infty$, which identifies the π_1-action on the left side with the monodromy homomorphism $\pi_1(U_{a_M},\infty) \to W_M \rtimes \Gamma$, as in Definition 3.17. So we deduce an isomorphism

$$(J_{a_M}(F)\backslash J_{a_M}(\mathbb{A}))_\tau \cong (\mathbb{X}_\star)_{W_{a_M}}.$$

We can now define the invariant map which appears in the definition of s-orbital integrals. Consider the short-exact sequence of topological groups with Frobenius-actions

$$1 \longrightarrow J_{a_M}(\mathbb{A}) \longrightarrow G(\mathbb{A}) \longrightarrow J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}) \longrightarrow 1\ .$$

From its long-exact sequence in nonabelian Galois cohomology, one obtains a coboundary map

$$\epsilon : (J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^\tau \to \ker(J_{a_M}(\mathbb{A})_\tau \to G(\mathbb{A})_\tau) \subset J_{a_M}(\mathbb{A})_\tau$$

with fibers torsors under $G(\mathbb{A}_0)$. Explicitly, the map identifies a class represented by $g \in G(\mathbb{A})$ with the τ-conjugacy class $g\tau(g)^{-1} \in J_{a_M}(\mathbb{A})_\tau$, and two classes have the same image if and only if they differ by right translation by $G(\mathbb{A}_0)$.

Definition 15.2. Given $a_M \in \widetilde{\mathscr{A}}_M(k_0) = \widetilde{\mathscr{A}}_M(k)^\tau$, define

$$\mathrm{inv}_{a_M} : (J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^\tau \to (\mathbb{X}_\star)_{W_{a_M}}$$

to be the composition

$$(J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^\tau \xrightarrow{\ \epsilon\ } J_{a_M}(\mathbb{A})_\tau \longrightarrow (J_{a_M}(F)\backslash J_{a_M}(\mathbb{A}))_\tau \xrightarrow{\ \sim\ } (\mathbb{X}_\star)_{W_{a_M}}\ .$$

Write $\langle -,- \rangle$ for the pairing $\widehat{T}^{W_{a_M}} \times (\mathbb{X}_\star)_{W_{a_M}} \to \mathbb{C}^\times$ induced by the contragredient action of the monodromic subgroup W_{a_M} on $\widehat{T} = \mathrm{Hom}(\mathbb{X}_\star, \mathbb{C}^\times)$.

PROPOSITION 15.3 ([CL12, Lemme 11.4.2]). *For any* $a_M \in \widetilde{\mathscr{A}}_M(k_0)$ *and* $g \in (J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^\tau \to (\mathbb{X}_\star)_{W_{a_M}}$, *the invariant map takes values*

$$\mathrm{inv}_{a_M}(g) \in \mathrm{im}\left(\underline{X}_\star(T_{M_{\mathrm{sc}}})(\infty)_{W_{a_M}} \to \underline{X}_\star(T)(\infty)_{W_{a_M}}\right)$$

PROOF. Exactly as in *loc. cit.* □

Equip $G(\mathbb{A}_0)$ with the Haar measure which assigns volume 1 to the compact open K corresponding to global integral model G_0, and equip $J_{a_M}(\mathbb{A}_0)$ with the Haar measure which assigns volume 1 to its unique maximal compact open (recall it is commutative!). As in [CL12, §11.5], we observe that $J_{a_M}(\mathbb{A})_\tau$ is countable, so along the couboundary map $(J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^\tau$ is the disjoint union of countably many orbits under $G(\mathbb{A}_0)$, each of which can be equipped with the quotient measure from $G(\mathbb{A}_0)$. So we obtain a measure on $(J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^\tau$ which is right $G(\mathbb{A}_0)$-invariant.

Now we can proceed to define weighted s-orbital integrals. Recall from §6.3, Definition 6.8 the adelic characterization of the weight function of Arthur: there is for each rational semistandard parabolic P with Levi M a well-defined map

$$H_P : G(\mathbb{A}) \to \mathfrak{a}_M = \underline{\mathfrak{a}}_M^\Gamma$$

defined by the requirement that for each rational character $\lambda \in X^\star(P) = \underline{X}^\star(P)^\Gamma$,

$$\lambda(H_P(g)) = -\sum_v \mathrm{val}_v(\lambda(p_v))$$

where $g \in pG(\mathscr{O})$ in a (geometric) Iwasawa decomposition. Recall that we have fixed a choice of Weyl-invariant inner product on the rational character space $\mathfrak{a}_T$ and so on all its induced summands.

DEFINITION 15.4. For $g \in G(\mathbb{A})$, define *Arthur's weight* $v_M^G(g)$ to be the volume in $\mathfrak{a}_M^G$ (the *rational* cocharacter space) of the convex envelope spanned by the projections of the $-H_P(g)$ for P with Levi M, with respect to the measure on $\mathfrak{a}_M^G$ induced from the global choice in Definition 5.1.

In particular, the measure obtained in this way is invariant for the relative Weyl group, which is all that Arthur ever requires in his weight formalism.

Recall that $H_P|_{M(\mathbb{A})}$ is independent of P and satisfies $H_P(mg) = H_M(m) + H_P(g)$. It follows that left multiplication by $m \in M(\mathbb{A})$ translates each vertex $-H_P(g)$ by $-H_M(m)$, and so $v_M^G(mg) = v_M^G(g)$. The weight function is also clearly invariant under right multiplication by $G(\mathscr{O})$.

DEFINITION 15.5 ([CL12, Définition 11.7.1]). Let M be a semistandard Levi and $a_M \in \widetilde{\mathscr{A}}_M(k_0) = \widetilde{\mathscr{A}}_M(k)^\tau$. Let $s \in \hat{T}^{W_{a_M}}$. The *weighted s-orbital integral* associated to a_M is defined by

$$J_M^{G,s}(a_M) = q^{-\deg(D)|\Phi^G|/2} \int_{(J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^\tau} \langle s, \mathrm{inv}_{a_M}(g)\rangle 1_D(\mathrm{Ad}(g)^{-1} X_{a_M}) v_M^G(g)\, dg,$$

where

- Φ^G is the set of absolute roots of G,
- $X_{a_M} \in \mathfrak{m}(F)$ is the image of $a_{M,\eta}$ under the Kostant section, and
- the integral is taken with respect to the measure on $(J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^\tau$ defined above.

It will be helpful (and psychologically useful) to rewrite this as a sum of integrals over the values of the invariant. Given $a_M \in \mathsf{A}_{M\text{-ell}}$, recall the (finite) endoscopic symmetry group of Definition 12.27:

$$\Sigma_a := \mathrm{im}(\underline{X}_\star(T \cap D(M))(\infty)_{W_{a_M}} \to \underline{X}_\star(T)(\infty)_{W_{a_M}}).$$

As a corollary of Proposition 15.3, $\mathrm{inv}_{a_M}(g) \in \Sigma_a$ for every $g \in (J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^\tau$. So writing for $\lambda \in \Sigma_a$

$$(J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^{\tau,\lambda} := \left\{g \in (J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^\tau : \mathrm{inv}_{a_M}(g) = \lambda\right\}, \tag{15.1.1}$$

and equipping it with the induced measure as a closed and open subset of $(J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^\tau$, we deduce the following decomposition of the s-orbital integral.

Proposition 15.6. *Under the same notations as Definition 15.5, we have*

$$J_M^{G,s}(a_M) = q^{-\deg(D)|\Phi^G|/2} \sum_{\lambda\in\Sigma_a} \langle s,\lambda\rangle \int_{(J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^{\tau,\lambda}} 1_D(\mathrm{Ad}(g)^{-1}X_{a_M}) v_M^G(g)\, dg.$$

Finally, in order to obtain statements of the Weighted Fundamental Lemma which match Arthur's (in particular, ranging over $s \in s_M Z_{\widehat{M}}^{\Gamma}$ rather than $s \in s_M(Z_{\widehat{M}}^0)^{\Gamma}$, cf. Remark 14.16), it will be important to know that the s-orbital integrals satisfy a corresponding invariance.

Proposition 15.7 (cf. [CL12, Proposition 11.7.3]). *The map $s \in \hat{T}^{W_{a_M}} \mapsto J_M^{G,s}(a_M)$ is invariant under translation by $Z_{\widehat{M}}^{\Gamma}$.*

Proof. Let M_{sc} denote the simply connected X-cover of the derived group of M and $T_{M_{\mathrm{sc}}}$ the inverse image of T inside M_{sc}. By Proposition 15.3, one has

$$\mathrm{inv}_{a_M}(g) \in \mathrm{im}\left(\underline{X}_\star(T_{M_{\mathrm{sc}}})(\infty)_{W_{a_M}} \to \underline{X}_\star(T)(\infty)_{W_{a_M}}\right).$$

Unlike in the split case, W_{a_M} need not fix $Z_{\widehat{M}}$ pointwise because of its Γ-component. But dualizing the map of coinvariants shows that the pairing $\langle s, -\rangle$ is invariant under the annihilator

$$\ker\left(\hat{T}^{W_{a_M}} \to \hat{T}_{\mathrm{sc}}^{W_{a_M}}\right),$$

which is canonically $Z_{\widehat{M}}^{W_{a_M}}$ by left-exactness of fixed points. The W_M part, of course, acts trivially on the center, so

$$Z_{\widehat{M}}^{W_{a_M}} = Z_{\widehat{M}}^{\Gamma},$$

as desired. $\square$

15.2. The trace of Frobenius on the coarse $\overline{s}$-component

Fix semistandard Levi M, and assume that G has anisotropic center. In this section, we compute the alternating sum of traces of Frobenius on the stalk of the coarse s-decomposition of Theorem 12.33 over $a \in \mathsf{A}_{M\text{-ell}}(k_0)$. By Proposition 12.34, we know that $\overline{s}_M \in \hat{T}/Z_{\widehat{M}}^0$ can only contribute components to the stalk over a if s_M admits a lift $s \in \hat{T}^{W_a}$. We will perform the computation first in terms of such a lift. In this section we adapt Chaudouard–Laumon's point-counting argument to give the following theorem.

Theorem 15.8 (cf. [CL12, Théorème 12.1.1]). *Let $a_M \in \mathsf{A}_{M\text{-}ell}(k_0)$ with image $a \in \mathsf{A}_G$. Let $s \in \hat{T}^{W_a}$ have image $\overline{s} \in \hat{T}^{W_a}/(Z_{\widehat{M}}^0)^{\Gamma} \subset (\hat{T}/Z_{\widehat{M}}^0)_{tors}$. Then we have an equality*

$$\sum_i (-1)^i \operatorname{Tr}(\tau^{-1}, {}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-}par}\overline{\mathbb{Q}}_l)_{a,\overline{s}}) = \frac{c(J_{a_M})}{\mathrm{vol}(\mathfrak{a}_M/\mathscr{X}_\star(M))|\hat{T}^{W_a}/(Z_{\widehat{M}}^0)^{\Gamma}|} q^{\deg(D)|\Phi^G|/2} J_M^{G,s}(a_M),$$

where

- *$\mathfrak{a}_M$ is the relative cocharacter space of Definition 2.16,*
- *$\mathscr{X}_\star(M)$ is the dual of the relative character lattice $X^\star(M)$, as in Proposition 2.11 and Definition 12.8,*
- *J_{a_M} is the fiber of the M-regular centralizer of Ngô,*
- *Φ^G is the set of absolute roots for G,*
- *$c(J_{a_M})$ is a positive constant defined below (15.2.4).*

By the Theorem and Proposition 15.7, it follows that the left side depends only on the image of s in $\hat{T}/Z_{\widehat{M}}^{\Gamma}$.

Before we begin the proof, a few remarks comparing this section with [CL12, §12]. We follow the same strategy, with the natural corrections in order to keep everything relative. Note the appearance of $\mathscr{X}_\star(M)$, the natural target lattice for rational degree computations, in the statement of the theorem, and that $\overline{s}$ naturally lives in $\hat{T}^{W_a}/(Z_{\widehat{M}}^0)^{\Gamma}$ in our generality, where W_a is the monodromic invariant of Definition 3.17.

We order the argument slightly differently and try to offer a complementary exposition. In particular, we streamline the computation by performing Fourier inversion directly on the endoscopic symmetry group of Definition 12.27:
$$\Sigma_a := \mathrm{im}(\underline{X}_\star(T\cap D(M))(\infty)_{W_a} \to \underline{X}_\star(T)(\infty)_{W_a}),$$
rather than on $\underline{X}_\star(T\cap D(M))(\infty)_{W_a}$. By Proposition 12.28, we know that
$$\widehat{\Sigma}_a \cong \hat{T}^{W_a}/(Z^0_{\widehat{M}})^\Gamma,$$
finite when $a_M \in \mathsf{A}_{M\text{-ell}}$. Note the second constant factor in the denominator is therefore exactly
$$|\Sigma_a| = |\widehat{\Sigma}_a|.$$

Proof. Recall that as in §12.8, we have actions on the cohomological stalks
$$K^i_a := {}^p\mathscr{H}^i(Rf^{\xi,M\text{-par}}_\star \overline{\mathbb{Q}}_l)_a$$
through the composition
$$\mathbb{X}_{\star,D(M)} \twoheadrightarrow \Sigma_a \to \pi_0(\mathsf{P}^1_a).$$
Splitting across the summands of the s-decomposition, we know that $\lambda \in \mathbb{X}_{\star,D(M)}$ acts on the $\overline{s}$-component by $\langle s,\lambda\rangle$, where $\overline{s} \in (\hat{T}/Z^0_{\widehat{M}})_{tors}$. So for any $\lambda \in \mathbb{X}_{\star,D(M)}$ we have an equality of alternating traces
$$\mathrm{Tr}((\lambda\tau)^{-1}, K^\bullet_a) = \sum_{\overline{s}\in(\hat{T}/Z^0_{\widehat{M}})_{tors}} \langle \overline{s},\lambda\rangle^{-1}\,\mathrm{Tr}(\tau^{-1}, K^\bullet_{a,\overline{s}}).$$
But by Proposition 12.34 we know only $\overline{s} \in \widehat{\Sigma}_a \subset (\hat{T}/Z^0_{\widehat{M}})_{tors}$ contribute, and on the other hand λ acts through its image in Σ_a, so we can coarsen our decomposition to
$$\mathrm{Tr}((\lambda\tau)^{-1}, K^\bullet_a) = \sum_{\overline{s}\in\widehat{\Sigma}_a} \langle \overline{s},\lambda\rangle^{-1}\,\mathrm{Tr}(\tau^{-1}, K^\bullet_{a,\overline{s}}),$$
where the pairing is now that between Σ_a and $\widehat{\Sigma}_a$. By Fourier inversion, we recover
$$\mathrm{Tr}(\tau^{-1}, K^\bullet_{a,\overline{s}}) = \frac{1}{|\Sigma_a|}\sum_{\lambda\in\Sigma_a} \langle \overline{s},\lambda\rangle\,\mathrm{Tr}((\lambda\tau)^{-1}, K^\bullet_a).$$

By abstract nonsense, the alternating trace of an endomorphism on stalks of the perverse cohomology is equal to the alternating trace on stalks of the classical cohomology: in the Grothendieck group of the derived category of pairs $(K \in D^b_c(U_M), \phi : K \to K)$,
$$[Rf^{\xi,M\text{-par}}_\star \overline{\mathbb{Q}}_l, (\lambda\tau)^{-1}] := \sum_i (-1)^i [{}^p\mathscr{H}^i(Rf^\xi_\star \overline{\mathbb{Q}}_l), (\lambda\tau)^{-1}],$$
which pulls back along $i^\star_a$ to the corresponding equality in the stalk. But on the other hand, proper base change gives $i^\star_a Rf^{\xi,M\text{-par}}_\star \overline{\mathbb{Q}}_l \cong R\Gamma(\mathsf{M}^\xi_a, \overline{\mathbb{Q}}_l)$. Taking alternating trace therefore gives
$$\mathrm{Tr}((\lambda\tau)^{-1}, K^\bullet_a) = \mathrm{Tr}((\lambda\tau)^{-1}, H^\bullet(\mathsf{M}^\xi_a, \overline{\mathbb{Q}}_l)).$$
Grothendieck–Lefschetz then identifies
$$\mathrm{Tr}((\lambda\tau)^{-1}, H^\bullet(\mathsf{M}^\xi_a, \overline{\mathbb{Q}}_l)) = \#\mathsf{M}^\xi_a(k)^{\lambda\tau},$$
where the fixed-points and the cardinality are taken in the groupoid sense, cf. [Ngô10, §8.1].

Geometrically, we have already given a description of the fiber groupoid $\mathsf{M}^\xi_a(k)$ for $a \in \mathsf{A}_{M\text{-ell}}(k)$ in Proposition 6.13: it is the groupoid of $g \in G(\mathbb{A})/G(\mathscr{O})$ such that
$$1_D(\mathrm{Ad}(g)^{-1}(X_{a_M})) = 1 \qquad \text{and} \qquad \xi_M \in \mathrm{Cvx}((-H_P(g))_P) \tag{15.2.1}$$
under the left multiplication action of $J_{a_M}(F)$. The fixed-point groupoid $\mathsf{M}^\xi_a(k)^{\lambda\tau}$, then, has objects
$$(g \in G(\mathbb{A})/G(\mathscr{O}), \delta \in J_{a_M}(F)) : \delta\lambda\tau(g) = g, \tag{15.2.2}$$
such that g satisfies (15.2.1), with action as before on g and $(\lambda\tau)$-conjugation on δ. We now perform a pair of reductions.

As in [CL12, Lemme 12.3.2], the data t determines a canonical isomorphism $J_{a_M}|_{\mathscr{O}_\infty} \cong T|_{\mathscr{O}_\infty}$. Write $j_{\tilde{\lambda}} \in J_{a_M}(F_\infty)$ as the element corresponding to $\tilde{\lambda}(z_\infty)$ for $\tilde{\lambda} \in \mathbb{X}_{\star,D(M)}$. Its action on the Hitchin fiber is through the action of the Picard stack and naturally factors through Σ_a. As described in §12.5, this action is by taking the associated product with J_a-torsors, and as described in Proposition 12.20, the canonical map $\mathbb{X}_\star \to \widetilde{\mathscr{P}}$ is given by constructing a J_a-torsor with gluing data $j_{\tilde{\lambda}}$ at ∞ and trivial data at all other points. Arithmetically, then, the action of λ on the Beauville–Laszlo family $\tau(g)$ is simply left multiplication by $j_{\tilde{\lambda}}$, viewed as an adelic element trivial away from ∞. The Kottwitz isomorphism (cf. the discussion after 15.1)

$$(J_{a_M}(F)\backslash J_{a_M}(\mathbb{A}))_\tau \cong (\mathbb{X}_\star)_{W_a}$$

descends this, so we have a canonical twisted conjugacy class $j_\lambda \in (J_{a_M}(F)\backslash J_{a_M}(\mathbb{A}))_\tau$ for $\lambda \in \Sigma_a \subset (\mathbb{X}_\star)_{W_a}$. The condition in (15.2.2) can be rewritten as

$$\delta j_\lambda \tau(g) = g, \tag{15.2.3}$$

Now, as in [CL12, Lemme 12.3.3], we reduce to the action groupoid, denoted $[J_{a_M}(F)\backslash \mathscr{W}^\xi_\lambda]$, of pairs $(g \in G(\mathbb{A})/K, \delta \in J_{a_M}(F))$ satisfying (15.2.1) and (15.2.3) under the same action of $J_{a_M}(F)$. The only difference between this groupoid and the one above is the quotient by $K = G(\mathscr{O}_0)$ rather than $K = G(\mathscr{O})$. There is clearly a fully faithful functor from this groupoid to the one above; to see it is essentially surjective is to show that every class $(gG(\mathscr{O}), \delta)$ such that

$$\delta j_\lambda \tau(gG(\mathscr{O})) = gG(\mathscr{O})$$

is isomorphic to one satisfying the same condition with K. For any given representative g, one has

$$\delta j_\lambda \tau(g) = gu$$

for some $u \in G(\mathscr{O})$. By Lang's theorem for $G(\mathscr{O})$ (Lemma 15.9 below), we can write $u = x\tau(x)^{-1}$ for some $x \in G(\mathscr{O})$. Then taking $g' = gx$, we find a new representative that satisfies (15.2.3) on the nose, in particular, as cosets for K.

Combining the above steps, we have shown

$$\operatorname{Tr}(\tau^{-1}, K^\bullet_{a,\bar{s}}) = \frac{1}{|\Sigma_a|} \sum_{\lambda \in \Sigma_a} \langle \bar{s}, \lambda \rangle \#[J_{a_M}(F)\backslash \mathscr{W}^\xi_\lambda].$$

It remains to relate the cardinality of the groupoid $[J_{a_M}(F)\backslash \mathscr{W}^\xi_\lambda]$ to orbital integrals. We do this in Lemma 15.11 below, which implies the formula

$$\operatorname{Tr}(\tau^{-1}, K^\bullet_{a,\bar{s}}) = \frac{c(J_{a_M})}{\operatorname{vol}(\mathfrak{a}_M/\mathscr{X}_\star(M))|\Sigma_a|} \sum_{\lambda \in \Sigma_a} \langle \bar{s}, \lambda \rangle \int_{(J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^{\tau,\lambda}} 1_D(\operatorname{Ad}(g)^{-1} X_{a_M}) v^G_M(g)\, dg.$$

Combining along the decomposition of Proposition 15.6 completes the proof of Theorem 15.8. □

Lemma 15.9. *The Lang morphism of pro-algebraic group schemes $L = 1 - \tau : G(\mathscr{O}) \to G(\mathscr{O})$ is surjective.*

Proof. It suffices, for each closed point v_0 of degree d, to show that

$$1 - \tau : \prod_{v|v_0} L^+_{G,v}(k) \to \prod_{v|v_0} L^+_{G,v}(k)$$

is surjective, where v ranges over the Frobenius orbit of geometric points over v_0. The endomorphism τ permutes the factors cyclically, so a coordinate calculation shows surjectivity reduces to surjectivity of $1 - \tau^d$ on any given factor $L^+_{G,v}(k)$. Replacing k_0 with its degree-d extension (and τ with τ^d, we have reduced to checking Lang's theorem for an arc group at a single Frobenius-fixed place.

Recall that these arc groups are representable by group schemes of infinite type:

$$L^+_{G,v} = \lim_{m \in \mathbb{Z}_{\geq 0}} L^+_{m,G,v}$$

where $L^+_{m,G,v} : R \mapsto G(R[\![z_v]\!]/(z_v^{m+1}))$ are twisted jet schemes, representable by irreducible group varieties over k (cf. [Hei10, Prop. 2] for the representability). We construct a preimage orbit-by-orbit. For each m we have by Lang's theorem a nonhomomorphic morphism

$$L^+_{m,G,v}(k_0) \hookrightarrow L^+_{m,G,v}(k) \twoheadrightarrow L^+_{m,G,v}(k)$$

with finite fibers which are right torsors under $L^+_{m,G,v}(k_0)$. We wish to know that the right map remains surjective upon passage to the limit. This follows if the family $(L^+_{m,G,v}(k_0))_m$ satisfies the Mittag–Loeffler condition (cf. [Sta22, Tag 0594]). The reduction maps $L^+_{m',G,v}(k_0) \to L^+_{m,G,v}(k_0)$ are surjective by the formal smoothness criterion:

$$\begin{tikzcd} \operatorname{Spec} k_0[\![z_v]\!]/(t_v^{m-1}) \arrow[r] \arrow[d] & G \arrow[d] \\ \operatorname{Spec} k_0[\![z_v]\!]/(t_v^{m}) \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} k_0[\![z_v]\!] \end{tikzcd}.$$

Thus we can proceed along the proof of [Sta22, Tag 0598], which works without assuming commutativity. Explicitly, let $x \in L^+_{G,v}(k)$ arbitrary, and for a given m let $h_m \in L^+_{m,G,v}(k)$ satisfy $h_m\tau(h_m^{-1}) \equiv x \mod t^m$. We wish to show we can lift h_m to arbitrary level. For any $m' \geq m$ and any $h_{m'}$ mapping to x mod $t^{m'}$, the element $e_m := h_{m'}h_m^{-1} \in L^+_{m,G,v}(k)$ is in the kernel of the Lang isogeny $L^+_{m,G,v}(k_0)$ at level m. Choosing a lift $e_{m'}$ of e_m, the element $x_{m'}e_{m'}$ is a lift of x_m which maps to x. It follows that surjectivity of the Lang isogeny is preserved upon passing to the limit $L^+_{G,v}(k) \to L^+_{G,v}(k)$. □

Now we perform the fundamental computation of $\#[J_{a_M}(F)\backslash\mathscr{W}^\xi_\lambda]$ in terms of Arthur's orbital integrals. It is easier to first pass through an orbital integral more directly related to the definition of our stability condition.

Definition 15.10. Let $\xi_M \in \mathfrak{a}_M$ and define *Chaudouard–Laumon's weight function* on $G(\mathbb{A})$ by

$$w^\xi_M(g) = \#\{\mu \in \mathscr{X}_\star(M) : 1_{M,g}(\xi_M + \mu) = 1\};$$

this is the count of points in the shifted lattice $\xi_M + \mathscr{X}_\star(M)$ which land inside the compact convex $C_g \in \mathfrak{a}_M$.[2]

Since the left action of $M(\mathbb{A})$ translates the rational cones

$$-H_P(g) + {}^+\mathfrak{a}_P + \mathfrak{a}^P_T$$

by $-H_P(m) = -H_M(m)$, which lands inside $\mathscr{X}_\star(M)$, the weight function is left $M(\mathbb{A})$-invariant.

Lemma 15.11 (cf. [CL12, Proposition 12.2.2], [CL12, Corollaire 12.2.4]). *Let $a_M \in \mathsf{A}_{M\text{-}ell}(k_0)$, $\lambda \in \Sigma_a$, and consider $[J_{a_M}(F)\backslash\mathscr{W}^\xi_\lambda]$, the action groupoid of $J_{a_M}(F)$ on the set $\mathscr{W}^\xi_\lambda$ of pairs $(g,\delta) \in G(\mathbb{A})/K \times J_{a_M}(F))$ satisfying (15.2.1) and (15.2.3) above. The cardinality of this groupoid satisfies the following two formulae:*

(1) For any $\xi \in \mathfrak{a}_T$,

$$\#[J_{a_M}(F)\backslash\mathscr{W}^\xi_\lambda] = c(J_{a_M}) \int_{(J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^{\tau,\lambda}} 1_D(\operatorname{Ad}(g)^{-1}X_{a_M}) w^\xi_M(g)\, dg,$$

(2) For any $\xi \in \mathfrak{a}_T$ in general position,

$$\#[J_{a_M}(F)\backslash\mathscr{W}^\xi_\lambda] = \frac{c(J_{a_M})}{\operatorname{vol}(\mathfrak{a}_M/\mathscr{X}_\star(M))} \int_{(J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^{\tau,\lambda}} 1_D(\operatorname{Ad}(g)^{-1}X_{a_M}) v^G_M(g)\, dg$$

where $(J_{a_M}(\mathbb{A})\backslash G(\mathbb{A}))^{\tau,j}$ is the subset, considered above, of those cosets represented by g such that $\operatorname{inv}_{a_M}(g) = \lambda$ and

(15.2.4) $$c(J_{a_M}) := \operatorname{vol}(J(F_0)\backslash J(\mathbb{A}_0)^1) \cdot (\#\ker(J(F)_\tau \to J(\mathbb{A})_\tau)).$$

[2]Recall that this is indeed compact because G has anisotropic center and we are working in the rational space $\mathfrak{a}_M$.

Proof. The proof of (1) proceeds almost unchanged from [CL12, Proposition 12.2.2]. The essential new point is that our relative degree map $H_M : J(\mathbb{A}_0) \to \mathscr{X}_\star(M)$ is *surjective*. By the split rigidification $J|_{F_{0,\infty}} \cong T|_{F_{0,\infty}}$ at ∞, we can build a datum with specified geometric local degree: for any $\mu \in X_\star(T_\infty)$ we obtain an element z^μ_∞, whose degree when paired with any $\chi \in X^\star(T_\infty)$ is $\langle \mu, \chi \rangle$. Viewing this as an adelic element trivial except at ∞, it suffices to check surjectivity of

$$X_\star(T_\infty) = \mathrm{Hom}(X^\star(T_\infty), \mathbb{Z}) \to \mathrm{Hom}(X^\star(M), \mathbb{Z}).$$

This follows because $X^\star(M) \subset X^\star(T_\infty)$ is saturated; after all, it is characterized as the set of integral points satisfying linear equations.

For the sake of completeness, we recall the rest of the argument. Our $J = J_{a_M}$ is M-elliptic, so in particular anisotropic mod center, and $J(F_0)\backslash J(\mathbb{A}_0)^1$ is compact, so finite volume. Lemme 12.2.5 applies directly in our setting to give finiteness of $\ker(J(F)_\tau \to J(\mathbb{A})_\tau)$. In particular, $c(J)$ is finite.

Choose any representative $j \in J(\mathbb{A})$ for $j_\lambda \in (J(F)\backslash J(\mathbb{A}))_\tau$ and write $\psi_j : G(\mathbb{A}) \to G(\mathbb{A})$ for the map $g \mapsto g\tau(g)^{-1}j^{-1}$. The equation

$$\delta j \tau(g) = g \qquad \iff \qquad \delta = g\tau(g)^{-1}j^{-1} = \psi(g)$$

allows us to replace the data δ with the condition that $g \in \psi^{-1}(J(F))$. The stabilizer of such a pair is $J(F_0) \cap gKg^{-1}$, so

$$\#[J(F)\backslash \mathscr{W}^\xi_\lambda] = \sum_{g \in J(F)\backslash \psi^{-1}(J(F))/K} |J(F_0) \cap gKg^{-1}|^{-1} 1_D(\mathrm{Ad}(g^{-1})X_{a_M}) 1_{M,g}(\xi_M).$$

Equipping $J(F)\backslash\psi^{-1}(J(F))$ with the appropriate $G(\mathbb{A}_0)$-invariant quotient measure as in *loc. cit.*, this becomes an integral

$$\#[J(F)\backslash \mathscr{W}^\xi_\lambda] = \int_{J(F)\backslash\psi^{-1}(J(F))} 1_D(\mathrm{Ad}(g^{-1})X_{a_M}) 1_{M,g}(\xi_M)\, dg. \tag{15.2.5}$$

One decomposes this integral: $(J(F)\backslash J(\mathbb{A}))^\tau$ acts without fixed points on $J(F)\backslash\psi^{-1}(J(F))$, and the orbits are exactly the fibers of the natural map

$$J(F)\backslash\psi^{-1}(J(F)) \twoheadrightarrow (J(\mathbb{A})\backslash G(\mathbb{A}))^{\tau,j}.$$

The associated quotient is compatible with choices of measures, so Fubini's theorem allows the integral of (15.2.5) to be computed as a double integral. The inner integral is

$$\int_{(J(F)\backslash J(\mathbb{A}))^\tau} 1_D(\mathrm{Ad}(g^{-1})X_{a_M}) 1_{M,g}(\xi_M)\, dg,$$

but since all such g lie in $J(\mathbb{A})$ and centralize X_{a_M}, the left factor in the integrand is contant and can be pulled out. The domain $(J(F)\backslash J(\mathbb{A}))^\tau$ admits a filtration with subquotients (1) $J(F_0)\backslash J(\mathbb{A}_0)^1$, (2) $J(\mathbb{A}_0)^1\backslash J(\mathbb{A}_0)$, and (3) $\ker(J(F)_\tau \to J(\mathbb{A})_\tau)$, all compatible with the choices of measures.

(1) The first integral contributes a constant factor $\mathrm{vol}(J(F_0)\backslash J(\mathbb{A}_0)^1)$
(2) The second integral contributes, by the surjectivity of the first paragraph, the sum over integral translates of ξ_M which defines w^ξ_M.
(3) The third integral contributes, by $M(\mathbb{A})$-invariance of w^ξ_M, a constant factor of $\#\ker(J(F)_\tau \to J(\mathbb{A})_\tau)$.

In summary, then, (15.2.5) becomes

$$\#[J(F)\backslash \mathscr{W}^\xi_\lambda] = \mathrm{vol}(J(F_0)\backslash J(\mathbb{A}_0)^1) \cdot (\#\ker(J(F)_\tau \to J(\mathbb{A})_\tau)) \int_{(J(\mathbb{A})\backslash G(\mathbb{A}))^{\tau,j}} 1_D(\mathrm{Ad}(g^{-1})X_{a_M}) w^\xi_M(g)\, dg,$$

which is (1).

To deduce (2), it remains to see that for ξ in general position, the two kinds of weighted orbital integrals differ by the constant factor $\mathrm{vol}(\mathfrak{a}_M/\mathscr{X}_\star(M))$. This follows *mutatis mutandis* as in [CL10, §11.10-13]; indeed they follow closely arguments and constructions of Arthur which require no splitness hypotheses. We emphasize the few modifications necessary and provide citations:

- The calculations of [CL10, §11.10] make use of the sublattice $X_\star(M_{\mathrm{sc}}) \subset X_\star(M)$, which in the split case has the convenient property that $\Delta_P^\vee$ is a $\mathbb{Z}$-basis for each parabolic P with Levi M. In the relative setting, this may fail to be the case, but the role of $X_\star(M_{\mathrm{sc}})$ is auxiliary: in [Art91, §6], he performs the same computation with an arbitrary sufficiently divisible lattice $R_M = N\mathbb{Z}(\Delta_P^\vee) \subset \mathscr{X}_\star(M)$ (recall G is semisimple). One then gets by the same method a corresponding analogue of *loc. cit.*, Proposition 11.10.1:

$$w_M^\xi(g) = \lim_{\Lambda\to 0} \sum_{\mu_0 \in \mathscr{X}_\star(M)/R_M} \sum_{P\supset M} c_{P,N}(\Lambda)^{-1} e^{-\Lambda(H_P(g)+[\mu_0+\xi_M]_{P,N})},$$

where $c_{P,N}(\Lambda) := \prod_{\alpha\in\Delta_P}\left(e^{\Lambda(N\alpha^\vee)}-1\right)$ and for any $\lambda \in \mathfrak{a}_M$ the expression $[\lambda]_{P,N} \in R_M$ is defined with regards to the basis $\{N\alpha^\vee : \alpha \in \Delta_P\}$ for R_M, i.e.,

$$\lambda - [\lambda]_{P,N}$$

is a real linear combination of $N\alpha^\vee$ with coefficients in $(0,1]$.
- The formula for Arthur's weight functions as in §11.11 transfers directly to the relative setting. One should normalize d_P by the covolume of the appropriate auxiliary lattice R_M, and the formula [CL10, Proposition 11.11.1] adapts directly.
- The general formalism for (G,M)-families as in §11.12 is developed for general reductive groups in [Art81] relatively and without splitness hypotheses. Replace Chaudouard–Laumon's definition of w_P with

$$w_{P,N}(\mu,\Lambda) = \frac{d_P(\Lambda)}{c_{P,N}(\Lambda)} e^{-\Lambda([\mu]_{P,N})},$$

and obtain by the same proof the analog of Lemme 11.12.1 with conclusion

$$\Lambda([\mu]_{P,N}) = \Lambda([\mu]_{P',N}),$$

so that indeed $(w_{P,N})_P$ is a (G,M)-family.
- In §11.13, the same finite-index cancellation which produces a volume factor

$$\frac{\mathrm{vol}(\mathfrak{a}_L/X_\star(L))}{\mathrm{vol}(\mathfrak{a}_M/X_\star(M))}$$

cancels any terms involving the auxiliary lattices $R_M \subset \mathscr{X}_\star(M)$ and $R_L \subset \mathscr{X}_\star(L)$, and we deduce a version of Lemme 11.13.2 of the form

$$\sum_{\mu\in\mathscr{X}_\star(M)/R_M} w_L(\mu+\xi_M) = \frac{\mathrm{vol}(\mathfrak{a}_L/\mathscr{X}_\star(L))}{\mathrm{vol}(\mathfrak{a}_M/\mathscr{X}_\star(M))} w_L^\xi(1).$$

One requires the fact that $\mathscr{X}_\star(M) \to \mathscr{X}_\star(L)$ is surjective, which follows because $X^\star(L) \hookrightarrow X_\star(M)$ is saturated, just as argued at the start of this proof.
- With the previous modifications made, §11.14 goes through directly, and one deduces, for G semisimple, the desired constant factor $\mathrm{vol}(\mathfrak{a}_M/\mathscr{X}_\star(M))^{-1}$ as in Corollaire 11.14.3.

Details are left to the reader. □

15.3. Stabilization and the endoscopic point count

Now we wish to compare the stable ($s = 1$) components of the cohomology with stable orbital integrals. Our formalism for stabilization can be more simply inductive than [CL12, §13]. Recall the recursive definition, as in [Art02].

Definition 15.12. For any quasisplit reductive H/X satisfying the hypotheses at the top of the chapter and semistandard Levi $M \subset H$, define the *stable orbital integral* S_M^H for $a_M \in \mathsf{A}_{M\text{-ell}}$ (implicitly, recursively) by

$$J_M^{H,1}(a_M) = \sum_{s\in\mathscr{E}_M(H)} \left|Z_{\widehat{H}'_s}^\Gamma / Z_{\widehat{H}}^\Gamma\right|^{-1} S_M^{H'_s}(a_M),$$

where $\mathscr{E}_M(G) = \mathscr{E}_{1_M}(G)$ is the set of induced endoscopic groups for the trivial datum for M. That is, since the identity datum $1_H = (1, \rho_H)$ is included in the sum, the above identity is equivalent to

$$S_M^H(a_M) = J_M^{H,1}(a_M) - \sum_{\substack{s \in \mathscr{E}_M(H), \\ s \neq 1}} \left| Z^\Gamma_{\widehat{H}'_s} / Z^\Gamma_{\widehat{H}} \right|^{-1} S_M^{H'_s}(a_M). \tag{15.3.1}$$

A few remarks about the well-founded nature of this recursion are in order. By Remark 14.19, the sum appearing in the definition is finite. If a class $[s] \in \mathscr{E}_M(H)$ satisfied $H'_s = H$, then s would be central in $\widehat{H}$ and the class would be trivial, so every term on the right side of (15.3.1) is a proper endoscopic group. By Proposition 2.26, any proper endoscopic group has smaller dimension than H, so the recursion has only finite depth.

More subtly, the definition of $J_M^{H,1}$ required our range of standing assumptions on H, and we need to check those transfer down to endoscopic groups.

- We work always with quasisplit reductive X-groups, which is preserved by passage to endoscopic groups.
- Since our original H is assumed to be arithmetic, i.e. defined over a fixed X_0/k_0, then by the Remark after Definition 14.17, the various H'_s are likewise.
- Let $\Gamma_T := \operatorname{im}(\pi_1(X, \infty) \to \operatorname{Aut}_{\mathbb{Z}}(\underline{X}_\star(T)(\infty)))$, and $\Gamma_{T_{H'_s}}$ likewise for the corresponding maximal torus in H'_s. By the construction of the endoscopic group, one can embed $\mathbb{W}_{H'_s} \rtimes \Gamma_{T_{H'_s}} \subset \mathbb{W}_H \rtimes \Gamma$, so our standing assumption of good characteristic transfers from H to its endoscopic groups.
- Since H is assumed split at ∞, then $M \subset H$ is the same, so all H'_s (which contain M as a Levi by 14.21) contain the same split maximal torus at ∞.
- Many of our geometric results require H to have anisotropic center. This is inherited by the elliptic endoscopic groups, by Proposition 3.29.

Combining the above, we can see that the class of admissible groups for all the results proven so far is closed under passage to elliptic endoscopic groups.

Remark 15.13. The auxiliary weight factor in general position ξ is not naturally inherited or in general position for endoscopic groups. This is not necessary: in our cohomological Theorem 14.24 one considers only the elliptic part of the Hitchin fibration for the endoscopic groups, and this restriction is independent of a choice of ξ. When applying the inductive hypothesis, one can and must choose a different auxiliary ξ in general position in order to apply our cohomological theorems to each endoscopic group. But such a ξ certainly exists for each endoscopic group, and our results are all ultimately independent of the choice. In particular, the Definition 15.5 of s-orbital integrals above is stated in terms of Arthur's weight function and independent of ξ.

As we induct down to endoscopic groups, we will need to ensure we remain within the good locus.

Lemma 15.14 ([CL12, Lemme 14.3.1]). *Let (s, ρ_s) an endoscopic datum for H. Then under the closed immersion $i_{\rho_s} : \widetilde{\mathscr{A}}_{H'_s} \hookrightarrow \widetilde{\mathscr{A}}_H$, one has*

$$\widetilde{\mathscr{A}}_H^{good} \cap \widetilde{\mathscr{A}}_{H'_s} \subset \widetilde{\mathscr{A}}_{H'_s}^{good}.$$

Proof. Proof exactly as in *loc. cit.* Follows from the elementary computation of the dimension of the Hitchin base in Proposition 3.2 and the Ngô–Bezrukavnikov formula for the invariant δ_a. □

With these preliminaries resolved, we can now inductively compute the Frobenius trace on the endoscopic side of Theorem 14.24.

Theorem 15.15 (cf. [CL12, Théorème 14.1.1]). *Let G a quasisplit reductive X-group with anisotropic center, M a semistandard Levi, and $a_M \in \mathsf{A}_{M\text{-}ell}(k_0)$ with image $a \in \mathsf{A}_G$. Suppose that $a \in \mathsf{A}^{good}$. Let $j_{ell} : \mathsf{A}_{G\text{-}ell} \to \mathsf{A}_G$, an open immersion. We have an equality*

$$\operatorname{Tr}(\tau^{-1}, ((j_{ell})_{!\star}({}^p\mathscr{H}^\bullet(Rf_{G,\star}^{G\text{-}ell}\overline{\mathbb{Q}}_l)_1))_a) = \frac{c(J_{a_M}) \cdot q^{\deg(D)|\Phi^G|/2} |Z^\Gamma_{\widehat{M}} / (Z^0_{\widehat{M}})^\Gamma|}{\operatorname{vol}(\mathfrak{a}_M / \mathscr{X}_\star(M)) \cdot |\hat{T}^{W_a} / (Z^0_{\widehat{M}})^\Gamma| \cdot |Z^\Gamma_{\widehat{G}} / Z^{\Gamma,0}_{\widehat{G}}|} S_M^G(a_M), \tag{15.3.2}$$

where ${}^p\mathscr{H}^\bullet(Rf^{G\text{-}ell}_{G,\star}\overline{\mathbb{Q}}_l)_1$ *is the* fine *stable summand of Theorem 12.33.*

PROOF. We proceed by a streamlined version of the inductive strategy from *loc. cit.* For each $H \in \mathscr{E}_M(G)$ (elliptic endoscopic groups induced from the trivial endoscopic datum for M), define $\tilde{S}^H_M(a_M)$ so that equation (15.3.2) holds with $\tilde{S}^H_M(a_M)$ in place of $S^H_M(a_M)$. Note that by the considerations above, each such H is an admissible group and admits a truncated Hitchin fibration with an s-decomposition, well-defined s-orbital integrals, etc.

By induction on dimension, we may assume

$$\tilde{S}^H_M(a_M) = S^H_M(a_M)$$

for all proper endoscopic groups $H \neq G$.

We apply now the cohomological WFL to a collection of trivial endoscopic data for M. The pointed endoscopic data for M whose underlying endoscopic group is M itself are parameterized, modulo the coarse character equivalence, by $Z^\Gamma_{\widehat{M}}/(Z^0_{\widehat{M}})^\Gamma$. For each such class, choose a representative $z \in Z^\Gamma_{\widehat{M}}$. Then $\mathfrak{M}_z = (z, \rho_M)$ is a pointed endoscopic datum for M whose group is M, but ranging over different coarse characters $\overline{z} \in \hat{T}/Z^0_{\widehat{M}}$. Applying Theorem 14.24 to each $\mathfrak{M}_z$ and restricting to the stalk over a gives a decomposition in terms of fine parameters over the coarse parameter (note by the support Lemma 14.23 that only the summands induced from the various $\mathfrak{M}_z$ appear):

$$({}^p\mathscr{H}^i(Rf^{\xi,M\text{-par}}_\star\overline{\mathbb{Q}})_{\overline{z}})_a \cong \bigoplus_{\substack{s\in\mathscr{E}^f_{\mathfrak{M}_z}(G)\\ s\mapsto\overline{z}}} (j_{ell})_{!\star}(i_{\rho_s})_\star{}^p\mathscr{H}^i(Rf^{G'_s\text{-ell}}_{G'_s,\star}\overline{\mathbb{Q}}_l)_1[-2r_s](-r_s)_a. \tag{15.3.3}$$

We apply alternating trace of Frobenius to both sides (which is insensitive to semisimplification). One obtains by Theorem 15.8 that the left-hand side corresponding to z is

$$C\cdot q^{\deg(D)|\Phi^G|/2}J^{G,z}_M(a_M), \qquad \text{where} \qquad C = \frac{c(J_{a_M})}{\mathrm{vol}(\mathfrak{a}_M/\mathscr{X}_\star(M))\cdot|\hat{T}^{W_a}/(Z^0_{\widehat{M}})^\Gamma|}$$

By Proposition 15.7 we have $J^{G,z}_M(a_M) = J^{G,1}_M(a_M)$ and these are all equal. On the other side, the shift $-2r_s$ is even, so does not affect the alternating trace, and the Tate twist contributes q^{r_s}.

Consider the extended fine index set

$$\widetilde{\mathscr{E}}^f_M(G) := \bigsqcup_{z\in Z^\Gamma_{\widehat{M}}/(Z^0_{\widehat{M}})^\Gamma} \mathscr{E}^f_{\mathfrak{M}_z}(G) = \{s \in Z^\Gamma_{\widehat{M}}/Z^{\Gamma,0}_{\widehat{G}} : G'_s \text{ elliptic}\}.$$

Averaging the equations (15.3.3) over $z \in Z^\Gamma_{\widehat{M}}/(Z^0_{\widehat{M}})^\Gamma$ gives the combined identity

$$C\cdot q^{\deg(D)|\Phi^G|/2}J^{G,1}_M(a_M) = \frac{1}{|Z^\Gamma_{\widehat{M}}/(Z^0_{\widehat{M}})^\Gamma|}\sum_{\tilde{s}\in\widetilde{\mathscr{E}}^f_M(G)} q^{r_s}\,\mathrm{Tr}(\tau^{-1},(j_{ell})_{!\star}{}^p\mathscr{H}^\bullet(Rf^{G'_s\text{-ell}}_{G'_s,\star}\overline{\mathbb{Q}}_l)_{a,1}). \tag{15.3.4}$$

Now we collect the right hand side down to a sum indexed by Arthur parameters. There is a natural map to Arthur's index set from Definition 14.17

$$\mu : \widetilde{\mathscr{E}}^f_M(G) \to \mathscr{E}_M(G)$$

with the induced endoscopic datum depending only on the image. Indeed, shifting by $Z^\Gamma_{\widehat{G}}$ preserves not only the connected dual centralizer $\widehat{G'_s}$, but the full L-group centralizer. So the terms on the right side of (15.3.4) and the ellipticity of a given $\tilde{s}$ depend only on $s = \mu(\tilde{s})$. The fiber size can be calculated in two steps: passing from fine to coarse parameters gives a factor of

$$|\widehat{\Xi}_G| = \left|\frac{(Z^0_{\widehat{G}})^\Gamma}{Z^{\Gamma,0}_{\widehat{G}}}\right|,$$

and passing from coarse parameters to Arthur parameters gives a factor of

$$\left|\frac{Z^{\Gamma}_{\widehat{G}}}{(Z^0_{\widehat{G}})^{\Gamma}}\right|.$$

We can then write (15.3.4) as

$$C\cdot q^{\deg(D)|\Phi^G|/2} J_M^{G,1}(a_M) = \frac{|Z^{\Gamma}_{\widehat{G}}/Z^{\Gamma,0}_{\widehat{G}}|}{|Z^{\Gamma}_{\widehat{M}}/(Z^0_{\widehat{M}})^{\Gamma}|}\sum_{s\in\mathscr{E}_M(G)} q^{r_s}\operatorname{Tr}(\tau^{-1},(j_{ell})_{!\star}{}^p\mathscr{H}^{\bullet}(Rf^{G'_s\text{-ell}}_{G'_s,\star}\overline{\mathbb{Q}}_l)_{a,1}).$$

Substitute the definition of $\tilde{S}_M^{G'_s}(a_M)$ for each endoscopic group G'_s and cancel C and $|Z^{\Gamma}_{\widehat{M}}/(Z^0_{\widehat{M}})^{\Gamma}|$ to obtain

$$q^{\deg(D)|\Phi^G|/2} J_M^{G,1}(a_M) = \sum_{s\in\mathscr{E}_M(G)}\frac{|Z^{\Gamma}_{\widehat{G}}/Z^{\Gamma,0}_{\widehat{G}}|}{|Z^{\Gamma}_{\widehat{G'_s}}/Z^{\Gamma,0}_{\widehat{G'_s}}|} q^{r_s} q^{\deg(D)|\Phi^{G'_s}|/2}\tilde{S}_M^{G'_s}(a_M).$$

We have $r_s = \frac{\deg D}{2}(|\Phi^G| - |\Phi^{G'_s}|)$, so the powers of q agree and can be cancelled. Ellipticity implies $Z^{\Gamma,0}_{\widehat{G'_s}} = Z^{\Gamma,0}_{\widehat{G}}$ (indeed, both are zero under our anisotropic center hypothesis), and moreover $Z^{\Gamma}_{\widehat{G}}\subset Z^{\Gamma}_{\widehat{G'_s}}$, so the remaining factor can be rewritten

$$\frac{|Z^{\Gamma}_{\widehat{G}}/Z^{\Gamma,0}_{\widehat{G}}|}{|Z^{\Gamma}_{\widehat{G'_s}}/Z^{\Gamma,0}_{\widehat{G'_s}}|} = |Z^{\Gamma}_{\widehat{G'_s}}/Z^{\Gamma}_{\widehat{G}}|^{-1}.$$

By the inductive hypothesis, all terms satisfy $\tilde{S}_M^{G'_s}(a_M) = S_M^{G'_s}(a_M)$ except possibly $G'_s = G$, so rearranging gives

$$\tilde{S}_M^G(a_M) = J_M^{G,1}(a_M) - \sum_{\substack{s\in\mathscr{E}_M(G)\\ s\neq 1}} |Z^{\Gamma}_{\widehat{G'_s}}/Z^{\Gamma}_{\widehat{G}}|^{-1} S_M^{G'_s}(a_M)$$

and comparing with (15.3.1) gives $\tilde{S}_M^G(a_M) = S_M^G(a_M)$ by induction. □

15.4. The global WFL

With the results of the previous two sections, we can immediately deduce the global analog of the WFL for all quasisplit reductive G.

THEOREM 15.16 (The global Weighted Fundamental Lemma, cf. [CL12, Théorème 15.1.1]). *Let G_0/X_0 be a quasisplit reductive group such that G_0 is split at chosen point ∞ and the characteristic is good. Let $M_0 \subseteq G_0$ be a semistandard Levi with base change $M \subseteq G$. Let*

$$\mathfrak{M}' = (s_M, \rho_{s_M})$$

be an arithmetic elliptic endoscopic datum for M with endoscopic group M'. Assume moreover that M' is split at ∞. Let

$$a_{M'} \in \mathsf{A}_{M'}^{M'\text{-}ell}(k_0)$$

and denote $a_M = i_{\rho_{s_M}}(a_{M'}) \in \mathsf{A}_M(k_0)$ and $a = \chi_G^M(a_M) \in \mathsf{A}_G(k_0)$ its images. Assume

$$a \in \mathsf{A}_G^{good}.$$

For every $s \in \mathscr{E}_{\mathfrak{M}'}(G) \subset s_M Z^{\Gamma}_{\widehat{M}}/Z^{\Gamma}_{\widehat{G}}$, let $\mathfrak{G}'_s = (s,\rho_s)$ be the induced datum with endoscopic group G'_s. Then M' is naturally a semistandard Levi of G'_s and

$$J_M^{G,s_M}(a_M) = |Z^{\Gamma}_{\widehat{M'}}/Z^{\Gamma}_{\widehat{M}}| \sum_{s\in\mathscr{E}_{\mathfrak{M}'}(G)} |Z^{\Gamma}_{\widehat{G'_s}}/Z^{\Gamma}_{\widehat{G}}|^{-1} S_{M'}^{G'_s}(a_{M'}).$$

Remark 15.17. Note the hypothesis that M' should be split at infinity. This is necessary to apply our previous point-counting arguments to the pair $M' \subset G'_s$. Of course, no such hypothesis exists in the local WFL. We will later absorb this into our globalization argument, showing that we can engineer an arbitrary local endoscopic setup M'_v, M_v, G_v at one place on a curve while mandating that M' should be split at ∞.

Proof. First, assume that G has anisotropic center, so that our various geometric results apply. The claim that $M' \subset G'_s$ as a Levi is Proposition 14.21. The scalar quotients appearing are finite by ellipticity of the endoscopic groups. We consider the stalk of the coarse summand ${}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-par}}\overline{\mathbb{Q}}_l)_{\overline{s}_M}$ at a. By the last claim of Lemma 14.23, we have

$$\left({}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-par}}\overline{\mathbb{Q}}_l)^{\mathrm{ss}}_{\overline{s}_M}\right)_a = \left({}^p\mathscr{H}^i(Rf_\star^{\xi,M\text{-par}}\overline{\mathbb{Q}}_l)^{\mathrm{ss}}_{\overline{s}_M}\right)_{\mathfrak{M}',a}.$$

We proceed by the same yoga of fine, coarse, and Arthur parameters as in the proof of Theorem 15.15. To be precise, we will average the results of the cohomological WFL 14.24 over twisted data

$$\mathfrak{M}'_z = (s_M z, \rho_{s_M}), \qquad z \in Z^\Gamma_{\widehat{M}}/(Z^0_{\widehat{M}})^\Gamma$$

defining the same endoscopic group M'. The argument of Lemma 14.23 moreover implies that W_a fixes each $s_M z$, and each has the same point-count, so our first fixed-point computation Theorem 15.8 applies and gives on the average

$$C_M \cdot q^{\deg(D)|\Phi^G|/2} J_M^{G,s_M}(a_M) = \frac{1}{|Z^\Gamma_{\widehat{M}}/(Z^0_{\widehat{M}})^\Gamma|} \sum_{\tilde{s} \in \widetilde{\mathscr{E}}^f_{\mathfrak{M}'}(G)} q^{r_s} \operatorname{Tr}(\tau^{-1}, (j_{ell})_{!\star}{}^p\mathscr{H}^\bullet(Rf^{G'_s\text{-ell}}_{G'_s,\star}\overline{\mathbb{Q}}_l)_{1,a}). \tag{15.4.1}$$

where $\widetilde{\mathscr{E}}^f_{\mathfrak{M}'}(G) = \{s \in s_M Z^\Gamma_{\widehat{M}}/Z^{\Gamma,0}_{\widehat{G}} : G'_s \text{ elliptic}\}$ and

$$C_M = \frac{c(J_{a_M})}{\operatorname{vol}(\mathfrak{a}_M/\mathscr{X}_\star(M)) \cdot |\hat{T}^{W_a}/(Z^0_{\widehat{M}})^\Gamma|}.$$

By the good locus Lemma 15.14, the stable point-count Theorem 15.15 applies to the pair $M' \subset G'_s$. The right side of (15.4.1) then becomes

$$\frac{1}{|Z^\Gamma_{\widehat{M}}/(Z^0_{\widehat{M}})^\Gamma|} \sum_{\tilde{s} \in \widetilde{\mathscr{E}}^f_{\mathfrak{M}'}(G)} q^{r_s} \frac{c(J_{a_{M'}}) \cdot q^{\deg(D)|\Phi^{G'_s}|/2} |Z^\Gamma_{\widehat{M'}}/(Z^0_{\widehat{M'}})^\Gamma|}{\operatorname{vol}(\mathfrak{a}_{M'}/\mathscr{X}_\star(M')) \cdot |\hat{T}^{W_a}/(Z^0_{\widehat{M'}})^\Gamma| \cdot |Z^\Gamma_{\widehat{G'_s}}/Z^{\Gamma,0}_{\widehat{G'_s}}|} S^{G'_s}_{M'}(a_{M'}).$$

Several factors cancel:

- the factors of $|\hat{T}^{W_a}/(Z^0_{\widehat{M}})^\Gamma|$ and $|\hat{T}^{W_a}/(Z^0_{\widehat{M'}})^\Gamma|$ cancel to $|(Z^0_{\widehat{M'}})^\Gamma/(Z^0_{\widehat{M}})^\Gamma|$
- the powers of q on the two sides are equal by the codimension computation as before,
- $c(J_{a_M}) = c(J_{a_{M'}})$ because the regular centralizers are generically isomorphic, see [Ngô10, Proposition 2.5.1], and
- $\mathfrak{a}_M = \mathfrak{a}_{M'}$ and $\mathscr{X}_\star(M) = \mathscr{X}_\star(M')$, compatibly with the measures (this is essentially by definition of ellipticity; note that $\mathscr{X}_\star(M)$ is dual to $(Z^\Gamma_{\widehat{M}})^0$, and likewise for M' – these duals are equal by ellipticity).

So (15.4.1) becomes, collecting the $|Z^\Gamma_{\widehat{G}}/Z^{\Gamma,0}_{\widehat{G}}|$ fine terms for each Arthur parameter,

$$J_M^{G,s_M}(a_M) = \frac{|Z^\Gamma_{\widehat{M'}}/(Z^0_{\widehat{M'}})^\Gamma| \cdot |(Z^0_{\widehat{M'}})^\Gamma/(Z^0_{\widehat{M}})^\Gamma|}{|Z^\Gamma_{\widehat{M}}/(Z^0_{\widehat{M}})^\Gamma|} \sum_{s \in \widetilde{\mathscr{E}}_{\mathfrak{M}'}(G)} \frac{|Z^\Gamma_{\widehat{G}}/Z^{\Gamma,0}_{\widehat{G}}|}{|Z^\Gamma_{\widehat{G'_s}}/Z^{\Gamma,0}_{\widehat{G'_s}}|} S^{G'_s}_{M'}(a_{M'}).$$

Both remaining constants simplify to those in the statement of the Theorem.

Now assume that G_0 is a general quasisplit reductive group (but otherwise satisfying our running hypotheses). By the same reduction as [Ngô10, §8.6.1], the quotient $\overline{G} = G/A^{\mathrm{spl}}_G$ has anisotropic center, and likewise $\overline{M} = M/A_G^{\mathrm{spl}}$ and $\overline{M}'$ and each $\overline{G}'_s$ descend, and the same for all recursively defined endoscopic groups. At the level of dual tori we have

$$1 \longrightarrow \hat{\overline{T}} \longrightarrow \hat{T} \longrightarrow \widehat{A_G^{\mathrm{spl}}} \longrightarrow 1 \ ,$$

so a given s_M may not lie within $\hat{\overline{T}}$. However, the s-orbital integrals are invariant under translation by $Z^{\Gamma}_{\widehat{M}}$ after Proposition 15.3, and the natural map $Z^{\Gamma}_{\widehat{M}} \to \widehat{A_G^{\mathrm{spl}}}$ is surjective because the dual map $X^\star(\widehat{A_G^{\mathrm{spl}}}) \to X_\star(A_G^{\mathrm{spl}}) \to \pi_1(M)_\Gamma = X^\star(Z_{\widehat{M}})$ can be checked to be injective. So assume $s_M \in \hat{\overline{T}}$

The weighted s-orbital integrals $J_M^{G,s}(a_M)$ and $J_{\overline{M}}^{\overline{G},s}(\overline{a_M})$ are equal, by the following observations:

- the root systems and therefore the relevant cocharacter spaces $\mathfrak{a}_M^G$ are equal,
- the value of the invariant lies in the image of the cocharacter lattice of $T_{M_{\mathrm{sc}}}$ by Proposition 15.3, hence is insensitive to our central quotient,
- the parabolics and Levis are in bijection,
- the H_P are equal modulo the common central direction, so $v_M^G(g) = v_{\overline{M}}^{\overline{G}}(g)$,
- since $A_G^{\mathrm{spl}} \subset J_{a_M}$, the adelic quotient in the domain of integration is canonically isomorphic in a way compatible with the hyperspecial normalization of measures, and
- X necessarily splits as $\overline{X} + Z$ for central component Z which is fixed under conjugation. Since a is a global section of $\mathfrak{c}_{G,D}$, one can check that Z satisfies D-integrality, so X is D-integral if and only if $\overline{X}$ is.

The constants appearing in the recursive definition of the stable orbital integrals are likewise invariant under passage to the quotient, so the stable orbital integrals are equal. We conclude that the global WFL for G, M, M', ... follows from that for $\overline{G}$, $\overline{M}$, $\overline{M}'$, ... $\square$

CHAPTER 16

Local-to-global and the Weighted Fundamental Lemma

16.1. Local orbital integrals and the statement of the WFL

In this section, let $F_{0,v} = k_v(\!(z_v)\!)$ and $\mathscr{O}_{0,v} = k_v[\![z_v]\!]$ for finite field $k_v = \mathbb{F}_{q^d}$, and denote F_v, $\mathscr{O}_v$ the completed base changes to $k = \overline{\mathbb{F}}_q$. Assume always that p does not divide the (split) cardinality of the Weyl group $|W_G|$. Let $G/\mathscr{O}_{0,v}$ a quasisplit reductive group (unramified), split by a finite quotient Γ_v of $\mathrm{Gal}(F_v/F_{0,v})$. Fix $M \subset G$ a Levi. Fix a Kostant section over $\mathscr{O}_{0,v}$

$$\epsilon_M : \mathfrak{c}_M \to \mathfrak{m}.$$

We now imitate the weighted s-orbital formalism, as in §15.1, locally. Let $b \in \mathfrak{c}_M(\mathscr{O}_{0,v}) \cap \mathfrak{c}_M^{G\text{-reg}}(F_{0,v})$ be a generically G-regular semisimple parameter with $Y_b = \epsilon_M(b)$ its Kostant image. Consider the local regular centralizer $J_b = Z_M(Y_b) = Z_G(Y_b)$ over $F_{0,v}$. Choosing a geometric point $\tilde{b}$ of the local cameral cover

$$\mathfrak{t} \to \mathfrak{c}_M$$

over b_η (and in particular, a geometric point over v), we obtain an identification

$$X_\star(J_b)_{F_{0,v}^{\mathrm{s}}} \cong \mathbb{X}_\star := X_\star(\mathbb{T}).$$

The action of $\Gamma_v = \mathrm{Gal}_{F_{0,v}}$ can permute the fibers of the cameral cover and act by the splitting Galois group; as in §3.5, these package together into a homomorphism

$$\rho_{b,v} : \Gamma_{0,v} \to W_{\mathbb{M}} \rtimes \Gamma_v.$$

we define the *local monodromic subgroup* $W_{b,v}$ to be the image of this homomorphism. Then

$$X_\star(J_{b,F_v^{\mathrm{s}}})_{\Gamma_{0,v}} \cong (\mathbb{X}_\star)_{W_{b,v}},$$

so dually

$$\hat{J}_b^{\Gamma_{0,v}} \cong \hat{T}^{W_{b,v}}.$$

These constructions are compatible with the global ones upon restriction to a geometric point lying over v.

We can give a local version of the Kottwitz invariant construction: writing τ_v for the arithmetic Frobenius, we have an exact sequence of pointed sets

$$1 \longrightarrow J_b(F_{0,v}) \longrightarrow G(F_{0,v}) \longrightarrow (J_b(F_v)\backslash G(F_v))^{\tau_v} \longrightarrow \ker(J_b(F_v)_{\tau_v} \to G(F_v)_{\tau_v}) \longrightarrow 1 \ .$$

By [Kot97, §7.6], or by an argument parallel to Proposition 15.1, we have an isomorphism

$$J_b(F_v)_{\tau_v} \cong X_\star(J_{b,F_v^{\mathrm{s}}})_{\Gamma_{0,v}} \cong (\mathbb{X}_\star)_{W_{b,v}}.$$

As in the global setting, equipping $J(F_v)_{\tau_v}$ with the counting measure and $J(F_{0,v})\backslash G(F_{0,v})$ with the quotient of Haar measures normalized to give the maximal compact in $J(F_{0,v})$ and to give $G(\mathscr{O}_{0,v})$ measure 1, the exact sequence above induces a right-$G(F_{0,v})$-invariant measure on $(J_b(F_v)\backslash G(F_v))^{\tau_v}$.

Definition 16.1. Define the invariant map to be the composition

$$\mathrm{inv}_{b,v} : (J_b(F_v)\backslash G(F_v))^{\tau_v} \to \ker(J_b(F_v)_{\tau_v} \to G(F_v)_{\tau_v}) \subseteq J_b(F_v)_{\tau_v} \cong (\mathbb{X}_\star)_{W_{b,v}}.$$

For Y_b and M as above and $s \in \hat{J}_b^{\Gamma_{0,v}} \cong \hat{T}^{W_{b,v}}$, the *local weighted s-integral* is defined to be

$$J_{M,v}^{G,s}(b) = J_{M,v}^{G,s}(Y_b) = |D^G(Y_b)|_{F_{0,v}}^{1/2} \int_{(J_b(F_v)\backslash G(F_v))^{\tau_v}} \langle s, \mathrm{inv}_{b,v}(g)\rangle 1_{\mathfrak{g}(\mathscr{O}_v)}(\mathrm{Ad}(g)^{-1}Y_b) v_M^G(g)\, dg,$$

where v_M^G is Arthur's weight function from Definition 15.4 and the pairing is that between $\hat{T}^{W_{b,v}}$ and $(\mathbb{X}_\star)_{W_{b,v}}$ (or more invariantly, between $\hat{J}_b^{\Gamma_{0,v}}$ and $X_\star(J_{b,F_v^{\mathrm{s}}})_{\Gamma_{0,v}}$).

Proposition 16.2. *Choose an unramified local endoscopic datum $\mathfrak{M}'_v = (s_M, \rho_{v,s_M})$ with endoscopic group $M'/\mathscr{O}_{0,v}$. Let $Y' \in \mathfrak{m}'(F_{0,v})$ semisimple and G-regular. Then, writing $b' = \chi_{M'}(Y') \in \mathfrak{c}_{M'}(F_{0,v})$ with image $b \in \mathfrak{c}_M(F_{0,v})$, denote*

$$Y'_0 := \epsilon_{M'}(b'), \qquad Y_0 := \epsilon_M(b).$$

we have

$$J_{M,v}^{G,s_M}(Y_0) = \sum_{[Y]} \Delta_{M',M}(Y',Y)|D^G(Y)|_{F_{0,v}}^{1/2} \int_{J_Y(F_{0,v})\backslash G(F_{0,v})} 1_{\mathfrak{g}(\mathscr{O}_{0,v})}(\mathrm{Ad}(g)^{-1}Y) v_M^G(g)\, dg,$$

the sum taken over $M(F_{0,v})$-conjugacy classes of G-regular semisimple elements in $\mathfrak{m}(F_{0,v})$. Here $\Delta_{M',M}$ is Waldspurger's variant of the Langlands–Shelstad transfer factor for Lie algebras with the factor Δ_{IV} omitted, as in [Wal97].

Proof. This is standard and follows as at the start of the proof of [CL12, Theorem 19.1.1]. For completeness, we give more details. To compare the two formulae, we need to translate the invariant formalism above into the more traditional language of stable conjugacy. The invariant defined above is valued in $\ker(J_b(F_v)_{\tau_v} \to G(F_v)_{\tau_v})$, where the subscripts denote τ_v-conjugacy classes. A standard argument in Galois cohomology shows that

$$\ker(J_b(F_v)_{\tau_v} \to G(F_v)_{\tau_v}) \cong \ker(H^1(F_{0,v}, J_b) \to H^1(F_{0,v}, M))$$

We give an argument in Lemma 16.3 for exposition's sake.

Choosing the basepoint Y_0, we can identify $\ker(H^1(F_{0,v}, J_b) \to H^1(F_{0,v}, M))$ with rational classes in the M-stable class of Y_0. We denote $\mathrm{inv}(Y_0, Y)$ for the image of $\mathrm{inv}_s(Y)$ along this identification. The value of the invariant decomposes the domain of the integral across rational classes $[Y]$ in the M-stable class of Y_0:

$$(J_{Y_0}(F_v)\backslash G(F_v))^{\tau_v} = \bigsqcup_{[Y]} (J_Y(F_{0,v})\backslash G(F_{0,v})).$$

So, using invariance of D^G under stable conjugacy,

$$J_{M,v}^{G,s_M}(Y_0) = \sum_{[Y]} \langle s_M, \mathrm{inv}(Y_0,Y)\rangle |D^G(Y)|_{F_{0,v}}^{1/2} \int_{J_Y(F_{0,v})\backslash G(F_{0,v})} 1_{\mathfrak{g}(\mathscr{O}_{0,v})}(\mathrm{Ad}(g)^{-1}Y) v_M^G(g)\, dg.$$

The comparison between our pairing and the transfer factor of Langlands–Shelstad was done in [Ngô10, §1.5-1.11]. We outline the steps. For $s_M \in \hat{T}^{W_{b,v}}$, the character $\langle s_M, -\rangle : J_b(F_v)_{\tau_v} \to \mathbb{C}^\times$ is given by passing through the Kottwitz identification

$$J_b(F_v)_{\tau_v} \cong X_\star(J_b)_\Gamma \cong X_\star(\mathbb{T})_{W_{b,v}}.$$

Under the isomorphism above, the corresponding character of $\ker(H^1(F_{0,v}, J_b) \to H^1(F_{0,v}, M))$ is simply the restriction to the kernel of the Newton map, and can be checked to be equal to the Tate–Nakayama dual with the same normalization as in [Ngô10, §1.6, 1.9].

By Kottwitz's normalization of the transfer factor [Kot99], $\Delta_{M',M}(Y'_0, Y_0) = 1$, and of course $\langle s_M, \mathrm{inv}(Y_0, Y_0)\rangle = 1$. The Langlands–Shelstad variation formula [LS87, cf. §3.4, §3.7] shows that the transfer factor varies by a multiplicative factor of the Tate–Nakayama pairing

$$\frac{\Delta_{M',M}(Y'_0, Y)}{\Delta_{M',M}(Y'_0, Y_0)} = \langle s_M, \mathrm{inv}(Y_0, Y)\rangle$$

and so

$$\Delta_{M',M}(Y'_0, Y) = \langle s_M, \mathrm{inv}(Y_0, Y)\rangle$$

over all classes. Moreover, the factor is invariant under stable conjugacy in the endoscopic variable, so we may replace Y'_0 with Y'. The result is the desired formula. $\square$

Lemma 16.3. *For J a torus inside Levi $M \subseteq G$ as above, we have a canonical bijection*

$$\ker(J(F_v)_{\tau_v} \to G(F_v)_{\tau_v}) \cong \ker(H^1(F_{0,v}, J) \to H^1(F_{0,v}, M)).$$

Proof. First, we claim that $M(F_v)_{\tau_v} \to G(F_v)_{\tau_v}$ has trivial kernel, so the kernel in question reduces to

$$\ker(J(F_v)_{\tau_v} \to M(F_v)_{\tau_v}).$$

Choose a parabolic P for M. The existence of the projection $P \to M$ and section $M \to P$ ensures the map $M(F_v)_{\tau_v} \to P(F_v)_{\tau_v}$ is injective, so it suffices to check injectivity with M replaced by P. Assume $p \in P(F_v)$ and for some $g \in G(F_v)$ we have

$$gp\tau_v(g^{-1}) = 1.$$

Then $gp = \tau_v(g)$, so $gP \in (G/P)(F_{0,v})$. Since $G \to G/P$ is surjective on points over any field (indeed, it has Zariski-local sections, see for example [CF, Prop. 7.2.1]) we can find a $g_0 \in G(F_{0,v})$ and a $p_0 \in P(F_v)$ such that $g_0 p_0 = g$. We have

$$p = g^{-1}\tau_v(g) = p_0^{-1}\tau_v(p_0),$$

as desired.

Second, we claim natural isomorphisms

$$\ker(J(F_v)_{\tau_v} \to M(F_v)_{\tau_v}) \cong \ker(H^1_{un}(F_{0,v}, J) \to H^1_{un}(F_{0,v}, M)) \cong \ker(H^1(F_{0,v}, J) \to H^1(F_{0,v}, M))$$

so that the invariant above is constant on rational conjugacy classes within a given stable conjugacy class. The right isomorphism follows from Steinberg's theorem and the inflation-restriction sequence. The first is an exercise in Kottwitz's Newton map formalism ([Kot85], [Kot97]): there is a natural map for any reductive H

$$\nu_H : H(F_v)_{\tau_v} \to (\mathrm{Hom}_{F_v}(\mathbb{D}, H)/H(F_v))^{\tau_v},$$

where $\mathbb{D}$ is the pro-torus with character group $\mathbb{Q}$. Here naturality means that given $H \to G$, the natural map on τ_v-conjugacy classes corresponds to post-composition of homomorphisms. The kernel of the Newton map is $H^1_{un}(F_{0,v}, H)$ by, for example, [HK21, Lemma 4.7]. If $[j] \in \ker(J(F_v)_{\tau_v} \to M(F_v)_{\tau_v})$, then $\nu_M([j]) = 0$, which means that the postcomposed homomorphism $i \circ \nu_J : \mathbb{D} \to J \to M$ is trivial. But $J \to M$ is injective, so $\nu_J([j]) = 0$. □

As in the global case, we have our recursive definition of stable orbital integrals. Recall that $\Gamma = \Gamma(F_v/F_{0,v})$ captures the monodromy of our unramified group G. Locally, we will only consider *unramified* elliptic endoscopic data $\mathfrak{M}'$ for semistandard Levi $M \subset G$. As defined in Definition 14.17, it is clear that the induced endoscopic data $\mathscr{E}_{\mathfrak{M}'}(\mathfrak{G})$ are also split by the Γ-extension. So taking Γ to be either $\mathrm{Gal}(F_v/F_{0,v})$ or a finite quotient corresponding to an extension large enough to split both G and M', all actions considered will factor through Γ, and we can write

$$(-)^{\Gamma_{0,v}} = (-)^{\Gamma}$$

unambiguously.

Definition 16.4. For any quasisplit reductive H/X satisfying the hypotheses at the start of the chapter and semistandard $M \subset H$ with Y_b as above, define the *stable orbital integral* $S^H_{M,v}$ for Y_b implicitly by

$$J^{H,1}_{M,v}(Y_b) = \sum_{s \in \mathscr{E}_M(H)} |Z^{\Gamma}_{\widehat{H}'_s}/Z^{\Gamma}_{\widehat{H}}|^{-1} S^{H'_s}_{M,v}(Y_b),$$

where $\mathscr{E}_M(G) = \mathscr{E}_{1_M}(G)$ is the set of induced endoscopic groups for the trivial datum for M. Equivalently,

$$S^H_{M,v}(Y_b) = J^{H,1}_{M,v}(Y_b) - \sum_{\substack{s \in \mathscr{E}_M(H) \\ s \neq 1}} |Z^{\Gamma}_{\widehat{H}'_s}/Z^{\Gamma}_{\widehat{H}}|^{-1} S^{H'_s}_{M,v}(Y_b).$$

This recursive definition is well-founded as in the discussion after Definition 15.12.

We now state the version of the Weighted Fundamental Lemma which is the primary goal of this work.

THEOREM 16.5 (The Weighted Fundamental Lemma for Lie algebras). *Assume that p does not divide the (split) cardinality of the Weyl group $|W_G|$. Choose an unramified local endoscopic datum $\mathfrak{M}'_v = (s_M, \rho_{v,s_M})$ with endoscopic group $M'/\mathscr{O}_{0,v}$, and let Γ_v be an unramified quotient prime to p large enough to split G and M'. Let $b' \in \mathfrak{c}_{M'}(\mathscr{O}_{0,v}) \cap \mathfrak{c}_{M'}(F_{0,v})^{G\text{-}reg}$ be a generically G-regular parameter, and write $b \in \mathfrak{c}_M(\mathscr{O}_{0,v})$ for its image under the natural map $\mathfrak{c}_{M'} \to \mathfrak{c}_M$. Denote*

$$Y_{b'} = \epsilon_{M'}(b') \in \mathfrak{m}'(\mathscr{O}_{0,v}), \qquad Y_b = \epsilon_M(b) \in \mathfrak{m}(\mathscr{O}_{0,v})$$

for the Kostant lifts. Then

$$J^{G,s_M}_{M,v}(Y_b) = |Z^{\Gamma_v}_{\widehat{M'}}/Z^{\Gamma_v}_{\widehat{M}}| \sum_{s \in \mathscr{E}_{\mathfrak{M}'}(G)} |Z^{\Gamma_v}_{\widehat{G'_s}}/Z^{\Gamma_v}_{\widehat{G}}|^{-1} S^{G'_s}_{M',v}(Y_{b'}). \tag{16.1.1}$$

In the case where G and the endoscopic datum $\mathfrak{M}'$ are split, this is [CL12, Théorème 19.1.1].

16.2. Splitting lemmas

In [CL12, §16], the authors develop a full formalism for scindage and descent over finite sets of places on the curve X_0. Some part of the same formalism (in our relative, nonsplit setting) is unavoidable, but for the sake of brevity, and we hope, transparency of the ideas, we try for a minimalist path to the WFL.

In this section and the next, we keep the running global geometry $(X_0, D_0, ...)$ described at the top of §15, in particular G split at infinity, (but not necessarily having anisotropic center). In particular, $M \subset G$ will be fixed, and we fix also $\mathfrak{M}' = (s_M, \rho_{s_M})$ a global elliptic endoscopic datum for M also split at ∞. Until the end of this section, we write Γ for a finite quotient of the global Galois group $\pi_1(X, \infty)$ through which $G, \mathfrak{M}'$, and consequently every induced datum from $\mathfrak{M}'$ factor. Assume now that we have a parameter $a_{M'} \in \mathsf{A}^{M'\text{-ell}}_{M'}(k_0)$ with images $a_M \in \mathsf{A}_M$, $a \in \mathsf{A}_G$.

We will need Arthur's coefficient d defined in [Art88, §7-9].

DEFINITION 16.6. For semistandard Levis

$$R \subset L_1, L_2 \subset H,$$

define $d^H_R(L_1, L_2) = 0$ unless the natural map

$$\mathfrak{a}^{L_1}_R \oplus \mathfrak{a}^{L_2}_R \to \mathfrak{a}^H_R$$

is an isomorphism, in which case $d^H_M(L_1, L_2)$ is the absolute value of the (real) determinant of this natural map with respect to the chosen Euclidean structures on each (cf. Definition 5.1). In particular, $d^H_R(H, R) = d^H_R(R, H) = 1$.

REMARK 16.7. Recall that if G' is any elliptic endoscopic group for G, we have a natural isomorphism $\mathfrak{a}_{G'} \cong \mathfrak{a}_G$, and that we equipped $\mathfrak{a}_{G'}$ with the measure which makes this map an isometry. If we likewise have elliptic endoscopic groups R' for R, L'_1 for L_1, and L'_2 for L_2, then by construction we have

$$d^H_M(L_1, L_2) = d^{H'}_{M'}(L'_1, L'_2).$$

DEFINITION 16.8. For any nonempty set of closed points $V \subset |X_0|$ and $s \in \hat{T}^{W_{a_M}}$, define the *weighted s-orbital integral on V* by

$$J^{G,s}_{M,V}(a_M) := q^{-\deg(D|_V)|\Phi^G|/2} |D^G(X_{a_M})|^{1/2}_V \int_{(J_{a_M}(\mathbb{A}_V)\backslash G(\mathbb{A}_V))^\tau} \langle s, \mathrm{inv}_{a_M}(g)\rangle 1_D(\mathrm{Ad}(g^{-1})X_{a_M}) v^G_M(g)\, dg,$$

where $|-|_V$ denotes the product of the local absolute values and $X_{a_M} \in \mathfrak{m}(\mathbb{A}_V)$ is the image of $a_M|_V$ under the Kostant section. Define $S^G_{M,V}$ by the same recursion of Definition 15.12.

REMARK 16.9. The above definition interpolates between the global s-orbital integral of Definition 15.5 when $V = |X_0|$ and the local s-orbital integral of Definition when $V = \{v\}$ with residue field k_v, except for a mild renormalization on the support of the divisor D. To be precise about this last case, if $i_v : \mathbb{D}_v \to X$ is a complete disk centered at v and $X_{a_M} \in \mathfrak{m}(F_v)$ is the local Kostant image, then writing $Y := z_v^{d_v} X_{a_M}$ for d_v the multiplicity of D at v, we have

$$1_D(\mathrm{Ad}(g^{-1})X_{a_M}) = 1_{\mathfrak{g}(\mathscr{O}_v)}(\mathrm{Ad}(g^{-1})Y).$$

Moreover, the scalar multiplication does not change the centralizer, the invariant map, or Arthur's weight. Our choice of normalizing factor exactly cancels the change of the discriminant valuation on replacing X_{a_M} with Y, so we have

$$J^{L,s}_{M,\{v\}}(a_M) = J^{L,s}_{M,v}(Y),$$

where the left side is the partial adelic s-orbital integral defined immediately prior and the right side is the local definition from the previous subsection. For $V = \{v\}$ where $v \notin \mathrm{supp}(V)$, we have directly

$$J^{L,s}_{M,\{v\}}(a_M) = J^{L,s}_{M,v}(a_M)$$

The inductive structure we will use is the following.

DEFINITION 16.10. For arbitrary $V \subseteq |X_0|$, we say that our *the $\mathfrak{M}'$-WFL holds on V for $M \subset G$* to mean that for every semistandard Levi $M \subseteq L \subseteq G$, the WFL holds for the V-orbital integrals, with L in place of G and Γ the *global* monodromy action, i.e.,

$$J^{L,s_M}_{M,V}(a_M) = |Z^{\Gamma}_{\widehat{M}'}/Z^{\Gamma}_{\widehat{M}}| \sum_{s\in\mathscr{E}_{\mathfrak{M}'}(L)} |Z^{\Gamma}_{\widehat{L'_s}}/Z^{\Gamma}_{\widehat{L}}|^{-1} S^{L'_s}_{M',V}(a_{M'}). \tag{16.2.1}$$

Say that V is *peelable* if in addition $J^{M,s_M}_{M,V}(a_M) \neq 0$. See Lemma 16.13 below for the meaning of the name.

REMARK 16.11. We warn that despite appearances, when V is a single place the above definition does *not* automatically capture the local WFL of Theorem 16.5. This is because the theorem is stated in terms of *local* orbital integrals, it is stated using *global* endoscopic parameters. The Γ appearing in the above definition is the global monodromy of G and $\mathfrak{M}'$, while the relevant Galois group to recover the actual WFL at v is $D_v \subseteq \Gamma$, the decomposition group. Not only would this give us the wrong constants, in general, but it also gives us only a subset of the endoscopic terms: a parameter

$$s = s_M z, \qquad z \in Z^{D_v}_{\widehat{M}}$$

need not be fixed by Γ, so defines a perfectly valid induced local endoscopic datum which does not extend to the global setting. Moreover, even the orbital integrals appearing are not exactly local orbital integrals: the Arthur weight function in question is the global one, defined by $H_{P,v}$ which pair only with characters which are invariant under the global monodromy Γ.

We will avoid these concerns in our globalization step below by engineering our global monodromy to satisfy $D_v = \Gamma$. Under this hypothesis, the constants appearing in (16.2.1) are correct, the sum ranges correctly over all local endoscopic parameters, and the weight function is the correct one which pairs with all local characters. When we then prove that the WFL holds on $V = \{v\}$ for $M \subset G$ in the sense of Definition 16.10 with respect to a global elliptic endoscopic parameter $\mathfrak{M}'$ which restricts to our given unramified elliptic endoscopic local parameter $\mathfrak{M}'_v$, the result will be Theorem 16.5.

LEMMA 16.12 (Splitting Lemma, [Art88], [Art99], [CL12, Proposition 16.4.1, 17.4.1]). *If $V = V_1 \bigsqcup V_2$ is a disjoint decomposition of $V \subset |X_0|$, then one has for any a_M and s as above*

$$J^{G,s}_{M,V}(a_M) = \sum_{L_1,L_2\supseteq M} d^G_M(L_1,L_2) J^{L_1,s}_{M,V_1}(a_M) J^{L_2,s}_{M,V_2}(a_M). \tag{16.2.2}$$

Writing for any intermediate Levi

$$E^L_{M,V}(a_{M'}) = |Z^{\Gamma}_{\widehat{M}'}/Z^{\Gamma}_{\widehat{M}}| \sum_{s\in\mathscr{E}_{\mathfrak{M}'}(L)} |Z^{\Gamma}_{\widehat{L'_s}}/Z^{\Gamma}_{\widehat{L}}|^{-1} S^{L'_s}_{M',V}(a_{M'})$$

for the endoscopic side of the WFL, these moreover satisfy the same splitting formula

$$E^G_{M,V}(a_{M'}) = \sum_{L_1,L_2\supseteq M} d^G_M(L_1,L_2) E^{L_1}_{M,V_1}(a_{M'}) E^{L_2}_{M,V_2}(a_{M'}) \tag{16.2.3}$$

In particular, when $G = M$ we have the product formulas

$$J^{M,s}_{M,V}(a_M) = J^{M,s}_{M,V_1}(a_M) J^{M,s}_{M,V_2}(a_M), \qquad E^M_{M,V}(a_{M'}) = E^M_{M,V_1}(a_{M'}) E^M_{M,V_2}(a_{M'})$$

Proof. That this holds for the weighted orbital integrals on V is asserted in the proof of [Art02, Proposition]. To be precise, the original splitting formula is proven for (G, M)-families in [Art88, Corollary 7.4], then applied for weighted orbital integrals in [Art99, Proposition 4.3]. That the stable orbital integrals satisfy a corresponding splitting identity which combines to give (16.2.3) is the content of [Art99, Theorem 6.1].

In the split case, the first claim is [CL12, 17.4.1], and their proof carries over verbatim (though interpreted in the relative spaces, of course). That is the only actual analysis required – the citations of Arthur above, and in particular the proof of [Art99, Theorem 6.1], demonstrate that the combinatorics which deduce the corresponding splitting formula for stable orbital integrals, and then recombine to obtain the original splitting formula for the endoscopic side, do not require any splitness hypotheses. Carrying over the argument into our notation is left as an exercise to the reader. □

In order to inductively remove places, one uses the following criterion.

Lemma 16.13 (Peeling lemma). *Fix all notations from the beginning of this section, and let $V = V_1 \bigsqcup V_2$ be a nontrivial decomposition of sets of places. Suppose*

(1) that the $\mathfrak{M}'$-WFL holds on V for $M \subset G$,
(2) that the $\mathfrak{M}'$-WFL holds on V_2 for $M \subset G$,
(3) that

$$J^{M,s_M}_{M,V_2}(a_M) \neq 0.$$

Then the $\mathfrak{M}'$-WFL holds on V_1 for $M \subset G$.

In other words, if the WFL holds on $V = V_1 \sqcup V_2$ for $M \subset G$ and V_2 is peelable, then the WFL holds on V_1 for $M \subset G$.

Proof. Fix an intermediate Levi $M \subset L \subset G$. First we consider the base case $L = M$. Then by the product case of the splitting formula we have

$$J^{M,s_M}_{M,V}(a_M) = J^{M,s_M}_{M,V_1}(a_M) J^{M,s_M}_{M,V_2}(a_M).$$

Since the identity holds for V, we can compute and apply the splitting formula again

$$J^{M,s_M}_{M,V}(a_M) = S^{M'}_{M',V}(a_{M'}) = J^{M',1}_{M',V}(a_{M'}) = J^{M',1}_{M',V_1}(a_{M'}) J^{M',1}_{M',V_2}(a_{M'}),$$

but the identity also holds on V_2 so $J^{M',1}_{M',V_2}(a_{M'}) = J^{M,s_M}_{M,V_2}(a_M)$. Combining gives

$$J^{M,s_M}_{M,V_1}(a_M) J^{M,s_M}_{M,V_2}(a_M) = J^{M',1}_{M',V_1}(a_{M'}) J^{M,s_M}_{M,V_2}(a_M).$$

The nonvanishing gives the desired cancellation, so the WFL holds on V_1 when $L = M$.[1]

By induction on dimension, we can now assume (16.2.1) holds on V_1 for all proper $K \subsetneq L$ as the ambient group, and wish to show it holds for L. By the splitting formulas, we know the equality

$$\sum_{L_1, L_2 \supseteq M} d^L_M(L_1, L_2) J^{L_1,s_M}_{M,V_1}(a_M) J^{L_2,s_M}_{M,V_2}(a_M) = \sum_{L_1, L_2 \supseteq M} d^L_M(L_1, L_2) E^{L_1}_{M,V_1}(a_{M'}) E^{L_2}_{M,V_2}(a_{M'}).$$

We cancel term-by-term. By the inductive hypothesis, we have equality of the V_1-terms whenever $L_1 \subsetneq L$, and we have equality of the V_2-terms always. The only remaining term with nonzero coefficient occurs when

$$(L_1, L_2) = (L, M).$$

There, the equality becomes (using the WFL on V_2 in the base case)

$$J^{L,s_M}_{M,V_1}(a_M) J^{M,s_M}_{M,V_2}(a_M) = E^L_{M,V_1}(a_{M'}) E^M_{M,V_2}(a_{M'}) = E^L_{M,V_1}(a_{M'}) J^{M,s_M}_{M,V_2}(a_M)$$

so again the nonvanishing gives the result. □

[1] Note that this base case did not use the classical fundamental lemma! We will, of course, use it later to show the hypotheses of the peeling lemma hold.

REMARK 16.14. It is easy to check that if V_1, V_2 are disjoint and peelable, then $V_1 \sqcup V_2$ is peelable, but we won't use this, so we don't prove it.

16.3. Peeling the regular places

In this section, we collect the local computations which we will use to show various places are peelable. In light of the Peeling Lemma, this will allow us to inductively reduce the global WFL of Theorem 15.16 down to one place and deduce Theorem 16.5. Of course, the claim that a place is peelable includes the fact that a version of the WFL holds at that place. Thus, our proof should still be considered as an application of Ngô's "Perverse Continuation Principle" – by packaging the WFL into families and understanding the support summands in the relative cohomology, we can by hand prove the identity for sufficiently simple characteristic polynomials and then transfer its validity to arbitrarily complicated characteristic polynomials. Thus, the results of this section and the next should be interpreted as essentially affirming the validity of the WFL at good places (except for the discrepancy discussed in Remark 16.11), from which its global validity spreads along the uniformity imposed by our cohomological theorems.

Recall that associated to an endoscopic group H for G we have the discriminant divisor $\mathfrak{D}_H$ of Definition 2.29 and the resultant divisor $\mathfrak{R}_H^G$ of Definition 2.31, both on $\mathfrak{c}_H$. In particular, these specialize to take $H = M$ a standard Levi, or $H = M'$ an endoscopic group associated to Levi M. In any case, these provide a measure of complexity for

$$b \in \mathfrak{c}_H(\mathscr{O}_{0,v}) \cap \mathfrak{c}_H^{G\text{-reg}}(F_{0,v})$$

a generically G-regular parameter. We write

$$v(\mathfrak{D}_H(b))$$

for the multiplicity of the closed point in the Cartier divisor $(b)^\star\mathfrak{D}_H$ on $\operatorname{Spec}\mathscr{O}_{0,v}$, and

$$v(\mathfrak{R}_H^G(b))$$

similarly. Of course, if f is any local function cutting out the divisor, then the above multiplicity is computed by

$$v(f(b')),$$

so the notation is only mildly abusive.

The basic local computations are encoded in the following Lemma.

LEMMA 16.15. *Let $b \in \mathfrak{c}_M(\mathscr{O}_{0,v}) \cap \mathfrak{c}_M^{G\text{-}reg}(F_{0,v})$, and write $Y := \epsilon_M(b)$.*

(1) (Unit-discriminant normalization) Suppose

$$v(\mathfrak{D}_M(b)) = 0.$$

Then for every $s \in \hat{T}^{W_{b,v}}$ we have

$$J_{M,v}^{M,s}(Y) = 1.$$

More precisely, every $[m] \in (J_b(F_v)\backslash M(F_v))^\tau$ contributing to the integral has a representative in $M(\mathscr{O}_{0,v})$ and in particular has invariant 0.

(2) (Unit-resultant vanishing) Let $M \subsetneq L$ an intermediate Levi, so b is generically L-regular semisimple. If

$$v(\mathfrak{R}_M^L(b)) = 0$$

then for every $s \in \hat{T}^{W_{b,v}}$ we have

$$J_{M,v}^{L,s}(Y) = 0.$$

More precisely, every $g \in L(F_v)$ contributing to the integral lies in $M(F_v)L(\mathscr{O}_v)$.

PROOF. For the first claim, let $[m] \in (J_b(F_v)\backslash M(F_v))^\tau$ be a coset contributing to the integral, and choose representative $m \in M(F_v)$. Put $Y_1 = \operatorname{Ad}(m^{-1})Y \in \mathfrak{m}(\mathscr{O}_v)$. Since the coset is τ-fixed, there exists $j \in J_b(F_v)$ such that $\tau(m) = jm$. Since j centralizes Y,

$$\tau(Y_1) = Y_1 \qquad \Longrightarrow \qquad Y_1 \in \mathfrak{m}(\mathscr{O}_{0,v}).$$

So Y and $Y_1 = \mathrm{Ad}(m^{-1})Y$ are rational elements which have the same (integral) characteristic polynomial. Reducing mod z_v, we have regular semisimple elements over finite residue field with the same characteristic polynomial. These are rationally conjugate by Lang's theorem, and the transporter is smooth because it is a torsor for the centralizer, a torus, so Henselian lifting implies conjugacy by $m_1 \in M(\mathscr{O}_{0,v})$. Then $m_1 m^{-1} \in Z_M(Y)(F_{0,v}) = J_Y(F_{0,v})$, so

$$m \in J_Y(F_v)M(\mathscr{O}_{0,v}).$$

So every such coset $[m]$ has a rational representative m_1, and the only elements contributing range over a single J_Y-orbit. The local invariant map is then identically 0 on the domain by definition, so $\langle s, \mathrm{inv}_{b,v}(g)\rangle = 1$ identically for each s. The discriminant factor of course contributes 1, and we compute simply that with our normalization of measure on the invariant kernel,

$$J^{M,s}_{M,v}(b) = 1$$

for every s.

For the second claim, we will show that the resultant condition implies $J^{L,s}_M(Y) = 0$ unless $M = L$. Let $g \in L(F_{0,v})$ be an element such that

$$\mathrm{Ad}(g^{-1})Y \in \mathfrak{l}(\mathscr{O}_{0,v}).$$

Choose a parabolic $P = M \rtimes N$ with Levi M and consider an Iwasawa decomposition $g = mnk$. The decomposition

$$\mathrm{Ad}(g^{-1})Y = (\mathrm{Ad}(n^{-1})Z - Z) + Z$$

where $Z = \mathrm{Ad}(m^{-1})Y \in \mathfrak{m}(F_{0,v})$ and $(\mathrm{Ad}(n^{-1})Z - Z) \in \mathfrak{n}(F_{0,v})$ implies that both components are integral.

The resultant hypothesis implies, after the formula of Proposition 2.33

$$\det(\mathrm{ad}(Y)|_{\mathfrak{n}}) = \pm R^L_M(\chi_M(Y)),$$

that $\mathrm{ad}(Y)$ and $\mathrm{ad}(Z)$, maps $\mathfrak{n}(\mathscr{O}_{0,v}) \to \mathfrak{n}(\mathscr{O}_{0,v})$, are integral isomorphisms.

We claim that $n \in N(\mathscr{O}_v)$. Over the maximal unramified extension, filter N by absolute-root height, $N_1 \supset N_2 \supset \cdots$. The map

$$\phi : n \mapsto \mathrm{Ad}(n^{-1})Z - Z$$

preserves the filtration, and on the associated graded it becomes a homomorphism

$$\phi(n_1 n_2) = \mathrm{Ad}(n_2^{-1})\phi(n_1) + \phi(n_2) = \phi(n_1) + \phi(n_2),$$

indeed a linear $\mathscr{O}_v$-isomorphism $N_i/N_{i+1} \to \mathfrak{n}_i/\mathfrak{n}_{i+1}$ which is equal to its differential $\mathrm{ad}(Z)$ under the identification $N_i/N_{i+1} \cong \mathfrak{n}_i/\mathfrak{n}_{i+1}$. We can then prove by induction that $n \in N(\mathscr{O}_v)$: we know that $\phi(n)$ is integral, and on the first subquotient it induces an $\mathscr{O}_v$-isomorphism, so the class of n is integral. The multiplicative difference, then, lies in $N_2(F_v)$ and has integral image under ϕ. Inductively, we can show all the differences are integral.

The original element g therefore lies in $M(F_v)L(\mathscr{O}_v)$. But this implies that all $H_P(g)$ are equal, and Arthur's weight $v^L_M(g)$ vanishes unless $M = L$, so we have $J^{L,s}_M(Y) = 0$ for any $L \supsetneq M$. ☐

These local computations quickly imply the peelability of all regular places and a reduction of our global WFL down to finitely many places. Fix global $\mathfrak{M}'$, $M \subset G$ and $a_{M'}$ satisfying the assumptions from the top of §16.2.

Corollary 16.16. *Let $V \subseteq |X_0|$ be any set (finite or infinite) of regular places, in the sense that a_M avoids $\mathfrak{D}_G$ on V. Then V is peelable.*

In particular, the $\mathfrak{M}'$-WFL holds on the finite set $B \subset |X_0|$ for $M \subset G$, where

$$B = \mathrm{supp}(a^\star_M \mathfrak{D}_M + a^\star_M \mathfrak{R}^G_M).$$

Proof. Up to the renormalization on $\mathrm{supp}(D)$ described in Remark 16.9, the restriction of a_M (resp. $a_{M'}$) to a formal disk around each $v \in V$ defines a valid local parameter b_v (resp. b'_v). These are valid local parameters in the sense of the first section. Pick an intermediate Levi L. By the decomposition $\mathfrak{D}_G|_{\mathfrak{c}_M} = \mathfrak{D}_M + 2\mathfrak{R}^G_M$ and likewise for M', the local parameters satisfy

$$v(\mathfrak{D}_{M'}(b'_v)) = v(\mathfrak{D}_M(b_v)) = 0, \qquad v(\mathfrak{R}^L_{M'}(b'_v)) = v(\mathfrak{R}^L_M(b_v)) = 0,$$

so the Lemma applies at each place.

Adelically on V, then, we can deduce from the statements beginning "more precisely" that the partial *adelic* s-orbital integral satisfies

$$J^{L,s}_{M,V}(a_M) = \delta_{L=M}.$$

For the case $L = M$, the support of the V-orbital integral modulo $J_{a_M}(\mathbb{A}_V)$ is the trivial class represented by 1 by the result at each place, so

$$J^{M,s}_{M,V}(a_M) = 1.$$

For the case $L \supsetneq M$, we have shown that every g_v which can occur in the support of the integral can be written as $g_v = m_v k_v$ for $m_v \in M(F_v), k_v \in L(\mathscr{O}_v)$. In particular,

$$H_{P,v}(g_v) = H_{M,v}(g_v)$$

is independent of parabolic P for M, so each adelic point $(g_v)_v$ in the support of the integral satisfies that

$$H_{P,V}(g) = \sum_v H_{P,v}(g_v)$$

is independent of P. The adelic convex hull defining $v^L_M(g)$ in $\mathfrak{a}^L_M$ is therefore a single point, with volume zero unless $L = M$. We have shown

$$J^{L,s}_{M,V}(a_M) = \delta_{L=M}.$$

Now we collate the computation

$$J^{L,s}_{M,V}(a_M) = \delta_{L=M}$$

into the form of the WFL. We have, by the same avoidance of discriminants and resultants for M' that the same Lemma applies, and repeating the argument above we have

$$J^{L',1}_{M',V}(a_{M'}) = \delta_{L'=M'}$$

for induced endoscopic L'.

The $\mathfrak{M}'$-WFL on V is then the following identity for each L:

$$J^{L,s}_{M,V}(a_M) = |Z^\Gamma_{\widehat{M}'}/Z^\Gamma_{\widehat{M}}| \sum_{s \in \mathscr{E}_{\mathfrak{M}'}(L)} |Z^\Gamma_{\widehat{L'_s}}/Z^\Gamma_{\widehat{L}}|^{-1} S^{L'_s}_{M',V}(a_{M'}),$$

where

$$S^H_{M',V}(a_{M'}) = J^{H,1}_{M',V}(a_{M'}) - \sum_{\substack{s \in \mathscr{E}_{M'}(H) \\ s \neq 1}} |Z^\Gamma_{\widehat{H}'_s}/Z^\Gamma_{\widehat{H}}|^{-1} S^{H'_s}_{M',V}(a_{M'})$$

Consider this last recursion. If $M' \subsetneq H$, then any induced elliptic group H'_s is not equal to M' (the split central torus changed!) so we will never get a nonzero term in the recursion defining $S^H_{M',V}(a_{M'})$, and it is identically 0. On the other hand, $S^{M'}_{M',V}(a_{M'}) = 1$. Considering now the WFL, both the left and right side vanish for $M \neq L$ and the WFL is $0 = 0$. When $M = L$, the identity reduces to

$$J^{M,s}_{M,V}(a_M) = J^{M,1}_{M,V}(a_M)$$

and both are 1. So we have checked the $\mathfrak{M}'$-WFL holds on the set of regular points. And of course, we have shown the nonvanishing condition for peelability, so V is peelable.

The last claim is now immediate from the Peeling Lemma 16.13 and the global WFL of Theorem 15.16: the $\mathfrak{M}'$-WFL on $|X_0|$ is exactly Theorem 15.16 applied to $\mathfrak{M}', M$, and each intermediate Levi L, each of which inherits the required hypotheses for G. Then we can peel off the set of regular places by the Peeling Lemma. $\square$

16.4. Transverse places

If we could arrange a globalization of our local parameter at v which was regular at all other points, then we would now be done, but this is too much to hope for. Instead, as in [Ngô10, §8.5], consider the divisor

$$\Delta = (\mathfrak{D}_G|_{\mathfrak{c}_{M',D}})_{\mathrm{red}} = (\mathfrak{D}_{M'} + \mathfrak{R}^G_{M'})_{\mathrm{red}} = (\mathfrak{D}_{M'} + \mathfrak{R}^M_{M'} + (\chi^{M'}_M)^\star \mathfrak{R}^G_M)_{\mathrm{red}}.$$

We will show below that a Bertini theorem allows us to choose a globalization of our local parameter which intersects this divisor *transversally* at smooth points of the divisor. If b' is a local parameter given by restriction of $a_{M'}$ to a formal disk around the place v and $v((\mathfrak{D}_G|_{\mathfrak{c}_{M',D}})_{\mathrm{red}}(b')) = 1$, then exactly one of

$$v(\mathfrak{D}_{M'}(b')), \qquad v(\mathfrak{R}^M_{M'}(b')), \qquad v(\mathfrak{R}^G_M(\chi^{M'}_M(b')))$$

is equal to 1 and the others vanish. We need to show that such places are peelable by hand. The following three lemmas do this.

Lemma 16.17 ([Ngô10, Lemme 8.5.7], cf. [CL12, Lemme 17.7.3]). *Let $v \in |X_0|$ be a place disjoint from* $\mathrm{supp}(D)$*, and write b (resp. b') for the restriction of the global parameter a_M (resp. $a_{M'}$) to the formal disk around v. Suppose*

$$v(\mathfrak{D}_{M'}(b')) = 1, \qquad v(\mathfrak{R}^M_{M'}(b')) = 0, \qquad v(\mathfrak{R}^G_M(b)) = 0. \tag{16.4.1}$$

Then v is peelable.

Proof. The hypothesis of the second claim in Lemma 16.15 applies both for M and M', so we know for $L \neq M$

$$J^{L,s_M}_{M,v}(Y_b) = 0$$

and likewise for M'-orbital integrals. On the endoscopic side, the recursive definition of $S^{L'_s}_{M',v}$ will therefore never contribute any nonzero terms, so the endoscopic side also vanishes.

On the other hand, when $L = M$ the WFL identity (16.2.1) reduces to

$$J^{M,s_M}_{M,v}(Y_b) = J^{M',1}_{M',v}(Y_{b'}).$$

This is exactly the (unweighted) Langlands–Shelstad fundamental lemma in the simple case of transverse discriminant, case (2) in the proof of [Ngô10, Lemme 8.5.7], which is proven by hand in *loc. cit.*: trathe affine Springer fibers in question are zero-dimensional and therefore are equal to their regular locus, which transfers from M' to M. Note that our normalizations absorb the standard power of q. Note also that the integral is nonzero: the stable class is a sum (with coefficients 1) of rational classes, and an open compact neighborhood of the identity contributes nontrivially to the distinguished rational class:

$$J^{M',1}_{M',v}(Y_{b'}) \geq |D^{M'}(Y_{b'})|_v^{1/2} \int_{J_{b'}(F_{0,v})\backslash M'(F_{0,v})} 1_{\mathfrak{m}'(\mathscr{O}_{0,v})}(\mathrm{Ad}(g^{-1})Y_{b'})\, dg > 0.$$

So v is peelable. ☐

Lemma 16.18 (cf. [Ngô10, §8.3], [CL12, Lemme 17.7.2]). *Let $v \in |X_0|$ be a place disjoint from* $\mathrm{supp}(D)$*, and write b (resp. b') for the restriction of the global parameter a_M (resp. $a_{M'}$) to the formal disk around v. Suppose*

$$v(\mathfrak{D}_{M'}(b')) = 0, \qquad v(\mathfrak{R}^M_{M'}(b')) = 1, \qquad v(\mathfrak{R}^G_M(b)) = 0. \tag{16.4.2}$$

Then v is peelable.

Proof. If $v(\mathfrak{R}^M_{M'}(b')) = 1$, then

$$v(\mathfrak{D}_M(b)) = 2.$$

As above, the case $L = M$ becomes a special case of the Langlands–Shelstad Fundamental Lemma, case (3) in the proof of [Ngô10, Lemme 8.5.7], checked by hand using the one-dimensional affine Springer fiber computation of [Ngô10, §8.3], and the right side is 1 by unit-discriminant for M'. When $L \supsetneq M$, then we have

$$J^{L,s_M}_{M,v}(Y_b) = 0$$

by the unit-resultant part of Lemma 16.15. We need to check that we also have unit resultant on the endoscopic side, $v(\mathfrak{R}^{L'_s}_{M'}(b')) = 0$. This intersection multiplicity can be checked after passage to $\mathscr{O}_v$, at which point we can use the root-theoretic formula for the resultant as discussed after Proposition 2.32: we can choose $Y' \in \mathfrak{t}(\mathscr{O}_v)$ a *diagonal* lift of b' by étale lifting, and then

$$v(\mathfrak{R}^{L'_s}_{M'}(b')) = \sum_{\{\pm\alpha\}\subset\Phi(L'_s,T)\setminus\Phi(M',T)} v(d\alpha(Y')).$$

Since $v(\mathfrak{R}^L_M(b)) = 0$, the corresponding product over $\Phi(L,T)\backslash\Phi(M,T)$ is equal to 0, which implies that

$$v(d\beta(Y')) = 0 \quad \forall\beta \in \Phi(L,T) \setminus \Phi(M,T).$$

So it suffices to check the containment of geometric roots

$$\Phi(L'_s,T) \setminus \Phi(M',T) \subseteq \Phi(L,T) \setminus \Phi(M,T).$$

This can be checked in terms of coroots on the dual side. But for sub-root systems defined from connected centralizers of torus elements, a coroot being contained in the sub-coroots is equivalent to the corresponding root being contained in the sub-roots. So it suffices to check

$$\Phi(\widehat{L'_s},\hat{T}) \setminus \Phi(\widehat{M'},\hat{T}) \subseteq \Phi(\widehat{L},\hat{T}) \setminus \Phi(\widehat{M},\hat{T}),$$

which is exactly the root computation of Proposition 14.21. We have shown that $v(\mathfrak{R}^{L'_s}_{M'}(b')) = 0$, so the unit-resultant part of Lemma 16.15 also applies to all the integrals appearing on the endoscopic side, and the WFL $0 = 0$ holds for $L \supsetneq M$; we have shown v is peelable in this subcase. ☐

The final case is the only genuinely weighted special case, and is also the place where the most new work is required in order to handle the nonsplit case.

Lemma 16.19 (cf. [CL12, Lemme 17.6.2, Lemme 17.8.1]). *Let $v \in |X_0|$ be a place disjoint from* supp(D), *and write b (resp. b') for the restriction of the global parameter a_M (resp. $a_{M'}$) to the formal disk around v. Suppose*

$$v(\mathfrak{D}_{M'}(b')) = 0, \qquad v(\mathfrak{R}^M_{M'}(b')) = 0, \qquad v(\mathfrak{R}^G_M(b)) = 1. \tag{16.4.3}$$

Then v is peelable.

Proof. In the case $L = M$, we have done all the work:

$$v(\mathfrak{D}_M(b)) = v(\mathfrak{D}_{M'}(b')) + 2v(\mathfrak{R}^M_{M'}(b')) = 0$$

so both b and b' are regular and the unit-discriminant lemma gives

$$J^{M,s_M}_{M,v}(Y_b) = 1 = J^{M',1}_{M',v}(Y_{b'}).$$

It remains only to check the WFL identity (16.2.1) holds for $L \supsetneq M$. If $v(\mathfrak{R}^L_M(b)) = 0$, then the unit-resultant Lemma 16.15.(2) implies the WFL reduces to $0 = 0$. So assume $v(\mathfrak{R}^L_M(b)) = 1$.

The proof is rather involved, and will occupy us until the end of the section. For the reader's convenience, we outline the argument. It will broadly have two steps, along the same strategy as in [CL12, Lemma 17.6.2, 17.8.1]. In the first step, we identify a pair of bad roots $\{\pm\alpha\}$ contributing to $v(\mathfrak{R}^L_M)$, build a Levi containing only that pair of roots, and then relate our orbital integrals to integrals for a corresponding semisimple rank 1 group R. In order to do this compatibly with Arthur's splitting formalism, it is necessary to adapt this construction to the *relative root datum*, thereby reducing to the case of *split semisimple rank 1*, but possibly higher geometric rank. We are nevertheless able to show that our s-orbital integrals are equal to a constant times a certain integral for the smaller group, and corresponding identities for the endoscopic groups.

These smaller integrals are not quite weighted orbital integrals, but shockingly, we will use no intrinsic analytic property they satisfy. Instead, the second step is an application of the combinatorial descent formalism as in [CL12, Proposition 16.3.7]. Rather than prove the general formalism, we extract its consequences to prove the $\mathfrak{M}'$-WFL directly. The essential structure of the argument is to show that the family of formulae

$$J^{L,s_M}_{M,v}(Y_b) = dC_R, \qquad J^{L'_s,1}_{M',v}(X_{b'}) = dC_{R'_s}$$

indexed over $\mathscr{E}_{\mathfrak{M}'}(L) \cup \{L\}$ is actually expressing this family as an *induced* family from numbers $C_R, C_{R'_s}$, which are indexed over endoscopic groups $\mathscr{E}_{\mathfrak{T}'}(R) \cup \{R\}$ in the sense of the statement of [CL12, Proposition 16.3.7] (nearly all coefficients in the sums vanish). Arthur's combinatorial argument shows that if the family J is induced from a family on a Levi, with coefficients d, then the stabilized family S is induced from the corresponding stabilized family on the Levi, with closely related coefficients e. Then we will use that the $\mathfrak{T}'$-WFL on R for any family of numbers is vacuously true because the endoscopic parameter s_M is central – so the family of identities contains only the recursion defining the stabilized numbers (16.4.4), and substitute our induced formulae for J and S to deduce the $\mathfrak{M}'$-WFL from this definition. Thus, the proof proceeds along the same lines as Chaudouard–Laumon's, though the geometry is more subtle and we require substantial new arguments (and a nonsplit version of the descent formulas due to Arthur) in order to carry it out.

Before beginning the reduction to R, we reduce our s-orbital integral to the distinguished rational component in the stable class by the same argument as claim (1) of Lemma 16.15: given any coset $[g] \in (J_b(F_v)\backslash L(F_v))^\tau$ with Iwasawa decomposition of a representative

$$g = mnk \in M(F_v)N(F_v)L(\mathscr{O}_v)$$

relative to parabolic $P = MN$, the nonvanishing of $\mathfrak{D}_M$ on the special fiber implies by Henselian lifting that $m \in J_b(F_v)M(\mathscr{O}_v)$. The cocycle defining $\mathrm{inv}_{s_M}(g)$ can then be representated by an element of $J_b(\mathscr{O}_v)$. But this is a smooth unramified torus, so Lang's theorem kills the cocycle and $\mathrm{inv}(g) = 0$. In particular, the s_M-orbital integral reduces from $(J_b(F_v)\backslash L(F_v))^\tau$ to its distinguished fiber, $J_b(F_{0,v})\backslash L(F_{0,v})$.

16.4.1. Step 1: Construction of R and computation of $J^{L,s_M}_{M,v}$ in terms of R. Choose a conjugation of $Y_b = \epsilon_M(b) \in \mathfrak{m}(\mathscr{O}_{0,v}) \subset \mathfrak{m}(\mathscr{O}_v)$ by $m \in M(\mathscr{O}_v)$ into $\mathfrak{t}(\mathscr{O}_v)$, which exists by Hensel-lifting a transporting element from the (algebraically closed) special fiber. Let τ denote the Frobenius action on the geometric based root datum, which generates the unramified Galois action Γ. The original Y_b was fixed by Frobenius τ, so the conjugate $Y = \mathrm{Ad}(m)Y_b$ is fixed by

$$\theta = \widetilde{w} \circ \tau,$$

where $\widetilde{w} = m\tau(m)^{-1} \in N_M(T)(\mathscr{O}_v)$ with image $w \in W_M(\overline{v})$. We translate our orbital integral from Y_b to Y along conjugation by m. The discriminant factor clearly remains unchanged. Since $m \in M(\mathscr{O}_v)$, conjugation by it preserves $\mathfrak{l}(\mathscr{O}_v)$ and sends the maximal compact in J_b to $T(\mathscr{O}_v)^\theta$. Quotient measures normalized to give these volume 1 transfer without a scalar. Moreover, the Arthur weight is unchanged: it is right-invariant by $L(\mathscr{O}_v)$ and left-invariant by $M(F_v)$, so is conjugation-invariant by $m \in M(\mathscr{O}_v)$:

$$v^L_M(mgm^{-1}) = v^L_M(g).$$

In summary,

$$\begin{aligned} J^{L,s_M}_{M,v}(Y_b) &= |D^L(Y_b)|^{1/2} \int_{J_b(F_{0,v})\backslash L(F_{0,v})} 1_{\mathfrak{l}(\mathscr{O}_v)}(\mathrm{Ad}(g^{-1})Y_b) v^L_M(g)\, dg \\ &= |D^L(Y)|^{1/2} \int_{T(F_v)^\theta\backslash L(F_v)^\theta} 1_{\mathfrak{l}(\mathscr{O}_v)}(\mathrm{Ad}(g^{-1})Y) v^L_M(g)\, dg. \end{aligned}$$

The rank 1 construction now adapts to the nonsplit setting as follows. Recall our fixed maximal torus T, and write $S \subset T$ for the maximal split subtorus. By the differential formula for the resultant, there exists a *unique* pair of geometric roots $\{\pm\alpha\} \subset \Phi(G_{F_v}, T_{F_v}) \setminus \Phi(M_{F_v}, T_{F_v})$ such that

$$v(d\alpha(Y)) = 1$$

and the others are units. We claim $\theta(\alpha) = \alpha$. On the one hand, we must have

$$v(d(w\tau\alpha)(Y)) = v(d\alpha(Y)) = 1$$

so $w\tau\alpha = \pm\alpha$. Choose a rational parabolic $P = MN$ over $\mathscr{O}_{0,v}$ for M, and define $+\alpha$ to be the geometric root contained in N_{F_v}. Then N_{F_v} is stable under τ and $w \in W_M$, so $w\tau\alpha \in \Phi(N_{F_v})$. So $\theta(\alpha) = \alpha$. We define now *two* reductive groups: the *geometric* rank-one Levi is

$$R_\alpha^{\mathrm{geom}} := \langle T, U_\alpha, U_{-\alpha} \rangle \subset L_{F_v}.$$

This is a θ-stable (but not τ-stable!) semistandard Levi over F_v of absolute semisimple rank 1. The *relative* rank-one Levi is, for $\beta = \alpha|_S$ the relative root corresponding to α,

$$R_\beta^{\mathrm{rel}} = Z_L((\ker\beta)^0).$$

This is by definition $F_{0,v}$-rational, and has relative semisimple rank 1, but the containment

$$R_\alpha^{\mathrm{geom}} \subseteq (R_\beta^{\mathrm{rel}})_{F_v}$$

is in general proper. When T is split, the difference is invisible. Our analysis of the integral will be controlled by the geometric group, but Arthur's descent formalism is controlled by the relative group.

The support of the θ-orbital integral above is controlled by R_α^{geom}. Choose geometric parabolic $Q \subset L_{F_v}$ with Levi R_α^{geom} and unipotent radical N. Every root value of Y on $\mathfrak{n}$ is a unit. By Iwasawa and the same filtration argument as in the proof of Lemma 16.15.(2), one can show that all $g \in L(F_v)$ with $\mathrm{Ad}(g^{-1})Y \in \mathfrak{l}(\mathscr{O}_v)$ are of the form

$$g = lk, \qquad l \in R_\alpha^{\mathrm{geom}}(F_v), \quad k \in L(\mathscr{O}_v).$$

If moreover g is θ-fixed (as is necessary to contribute to the integral) then

$$\theta(lk) = lk \qquad \Longrightarrow \qquad l^{-1}\theta(l) \in R_\alpha^{\mathrm{geom}}(\mathscr{O}_v).$$

Lang's theorem applies to $R_\alpha^{\mathrm{geom}}(\mathscr{O}_v)$ with Frobenius θ, so we can arrange $l = \theta(l)$ (and $k = \theta(k)$). Our orbital integral therefore reduces to

$$J_{M,v}^{L,s_M}(Y_b) = |D^L(Y)|^{1/2} \int_{T(F_v)^\theta \backslash R_\alpha^{\mathrm{geom}}(F_v)^\theta} 1_{\mathrm{Lie}(R_\alpha^{\mathrm{geom}})(\mathscr{O}_v)}(\mathrm{Ad}(l^{-1})Y) v_M^L(l)\, dl.$$

It remains to write Arthur's weight function in terms of one for one of our smaller Levis. Because R_α^{geom} may not descend to an $F_{0,v}$-Levi, we cannot apply Arthur's splitting formula with it, but must instead use the relative group. We have [Art88, §7-8, Cor. 7.2]

$$v_M^L(l) = \sum_R d_T^L(M,R) v_T^{Q_R}(l),$$

where R ranges over $F_{0,v}$-Levis containing T and

$$v_T^{Q_R}(l) := v_T^R(r)$$

for a Q_R-Iwasawa decomposition $l = rk \in Q_R(F_v)L(\mathscr{O}_v)$. Consider the terms which can contribute for $l \in R_\alpha^{\mathrm{geom}}(\mathscr{O}_v)^\theta$. The intersection $Q_R \cap R_\alpha^{\mathrm{geom}}$ is over F_v a parabolic inside the semisimple-rank 1 group R_α^{geom}, so is either a Borel or the entire group. If the intersection is a Borel, then the Iwasawa decomposition above would put $r \in T$, and $v_T^{Q_R}(l)$ must vanish. So the only Levis R contributing nontrivially have

$$R_\alpha^{\mathrm{geom}} \subset R_{F_v}.$$

Since such R are also rational, they contain every τ-conjugate of $U_{\pm\alpha}$, and so

$$R_\beta^{\mathrm{rel}} \subset R.$$

Recall that $v_T^R(l)$ is defined to be the volume of the rational convex hull of $-H_B(l)$, or equivalently, of the projection of the geometric convex hull into the rational space. Geometrically, this convex is at most a line since the $H_B(l)$ differ only along the line corresponding to the single geometric root $\pm\alpha$. The rational convex, then, cannot have positive volume unless

$$\dim \mathfrak{a}_T^R = 1.$$

We conclude that $R = R^{\mathrm{rel}}_\beta$, and our formula becomes

$$v^L_M(l) = d^L_T(M, R^{\mathrm{rel}}_\beta) v_T^{R^{\mathrm{rel}}_\beta}(l).$$

Since $D_M(b)$ is a unit and the bad pair is unique, all roots of L other than $\{\pm\alpha\}$ contribute units to the discriminant, so

$$|D^L(Y)| = |D^{R^{\mathrm{geom}}_\alpha}(Y)| = |D^{R^{\mathrm{rel}}_\beta}(Y)|.$$

We have therefore reduced our orbital integral to a kind of integral for a relative semisimple rank 1 group:

$$J^{L,s_M}_{M,v}(Y_b) = d^L_T(M,R)|D^R(Y)|^{1/2} \int_{T(F_v)^\theta \backslash R^{\mathrm{geom}}_\alpha(F_v)^\theta} 1_{\mathrm{Lie}(R)(\mathscr{O}_v)}(\mathrm{Ad}(l^{-1})Y) v^R_T(l)\, dl = d^L_T(M,R) C_R,$$

where $R := R^{\mathrm{rel}}_\beta$. The indicator function in the integrand could be made relative because both relative and absolute rank 1 groups have hyperspecial integral models over $\mathscr{O}_v$ and so

$$\mathrm{Lie}_{R^{\mathrm{geom}}_\alpha}(\mathscr{O}_v) = \mathrm{Lie}_{R^{\mathrm{rel}}_\alpha}(\mathscr{O}_v) \cap \mathrm{Lie}_{R^{\mathrm{geom}}_\alpha}(F_v).$$

Note that these C_R are not quite standard orbital integrals for the relative Levi: their domain of integration is genuinely controlled by the θ-stable absolute-rank-one subgroup, smaller than the relative group (which need not even be θ-stable).

On the endoscopic side, we cannot identify the same rational torus, rank-one group, and reduced orbital integral inside each L'_s the way that is possible in the split case. We can nevertheless imitate the same argument above for each induced endoscopic group $L'_s \supset M'$. By transitivity of resultants, we have $v(\mathfrak{R}^{L'_s}_{M'}(b')) \in \{0,1\}$. In the zero-case, the unit-resultant Lemma implies

$$J^{L'_s,1}_{M',v}(X_{b'}) = 0.$$

Otherwise, the bad root pair lives in $\Phi(L'_s, T') \backslash \Phi(M', T')$. Performing the same constructions as we did above gives $S'_s \subset T'_s$ rational tori inside L'_s, a pair of relative roots $\pm\beta'_s$ by the image of $\pm\alpha$, and a relative rank 1 group $R'_s := Z_{L'_s}((\ker \pm\beta'_s)^0)$. Then the orbital integrals likewise reduce to

$$J^{L'_s,1}_{M',v}(X_{b'}) = d^{L'_s}_{T'}(M', R'_s) C_{R'_s}.$$

We claim that in fact each R'_s is an elliptic endoscopic group for R. Indeed, by the general machinery of Lemma 16.20, R'_s is necessarily elliptic endoscopic for *some* $Q_s \subset L$. By consideration of its geometric roots and ellipticity, we can see that we must have $Q_s = R$: indeed, dually

$$\Phi(\widehat{R'_s}, \hat{T}) = \Phi(\widehat{R}, \hat{T}) \cap \Phi(\widehat{L'_s}, \hat{T}).$$

The same Lemma also implies that T'_s is an elliptic endoscopic group for a Levi $T_0 \subset R$, corresponding to the restricted pointed datum $\mathfrak{T}' := (s_M, \rho^T_{s_M})$. For each term with $d^{L'_s}_{T'}(M', R'_s) \neq 0$, ellipticity and a dimension calculation implies that we must have $T_0 = T$. More precisely, they are all rationally isomorphic to the torus T – elliptic endoscopy for a torus is trivial because Proposition 2.27 implies the Galois actions on the identified split torus coincide. By Remark 16.7, it follows that the corresponding endoscopic spaces all correspond inside $\mathfrak{a}_T = \mathfrak{a}_{T'_s}$, and so

$$d^{L'_s}_{T'}(M', R'_s) = d^L_T(M, R) \neq 0$$

for all s. Denote d for this value. Again, these $C_{R'_s}$ are not quite the values of endoscopic orbital integrals, because their domain of integration is generally smaller. Nevertheless, we can form stable versions of these numbers by the usual recursion for the containment $T \subset R$:

$$C_R = \sum_{u \in \mathscr{E}_T(R)} |Z^\Gamma_{\widehat{R'_u}} / Z^\Gamma_{\widehat{R}}|^{-1} S_T(R'_u). \tag{16.4.4}$$

This definition is the content of the observation that J_T is vacuously stabilizing in the proof of [CL12, Proposition 17.8.1]. The remainder of the proof consists of extracting the essence of [CL12, Proposition 16.3.7], the descent property of the stabilization formalism, in our setting, and showing that an appropriate repackaging of this definition gives the WFL for the $dC_{R'_s}$.

16.4.2. Step 2: Deduction of the $\mathfrak{M}'$-WFL by Arthur's descent formulae. The combinatorics of repackaging (16.4.4) into the $\mathfrak{M}'$-WFL is governed by the natural map of Arthur's endoscopic parameters corresponding to the construction $s \mapsto L'_s \mapsto R'_s$ above.This gives a map

$$p : \left\{(s \in \mathscr{E}_{\mathfrak{M}'}(L), T' \subseteq Q' \subseteq L'_s) : d_{T'}^{L'_s}(M', Q') \neq 0\right\} \to \left\{(T \subseteq Q \subseteq L, u \in \mathscr{E}_{\mathfrak{T}'}(Q)) : d_T^L(M, Q) \neq 0\right\},$$

such that $R'_u = R'_s$. As argued above, the transverse-resultant computation shows that the only valid Q in the codomain is $Q = R$, and then we must have $Q' = R'_s$. So p reduces to

$$p : \{s \in \mathscr{E}_{\mathfrak{M}'}(L)\} \to \{u \in \mathscr{E}_{\mathfrak{T}'}(R)\}.$$

The fibers of p are torsors for the kernel

$$\ker(p) = \frac{Z^\Gamma_{\widehat{M}} \cap Z^\Gamma_{\widehat{R}}}{Z^\Gamma_{\widehat{L}}},$$

so they all have the corresponding cardinality.

More interestingly, the map p is surjective onto the codomain: since our map on parameters is the natural projection

$$s_M Z^\Gamma_{\widehat{M}}/Z^\Gamma_{\widehat{L}} \to s_M Z^\Gamma_{\widehat{T}}/Z^\Gamma_{\widehat{R}}$$

this is asking for an equality of Γ-fixed centers:

$$Z^\Gamma_{\widehat{T}} = Z^\Gamma_{\widehat{M}} Z^\Gamma_{\widehat{R}}. \tag{16.4.5}$$

In general, taking Γ-fixed points is not right exact; there can be a cohomological obstruction to this surjectivity. However, since we have restricted to parameters such that $d_T^L(M, R) \neq 0$, the equality

$$\mathfrak{a}_T = \mathfrak{a}_M + \mathfrak{a}_R$$

implies by dimension-counting that

$$(Z^\Gamma_{\widehat{T}})^0 = (Z^\Gamma_{\widehat{M}})^0 (Z^\Gamma_{\widehat{R}})^0.$$

We are then saved by [Art99, Lemma 1.1]:

$$Z^\Gamma_{\widehat{T}} = Z^\Gamma_{\widehat{R}} (Z^\Gamma_{\widehat{T}})^0$$

and (16.4.5) follows. This is enough for surjectivity of p: given any $u = s_M z_T$, writing $u = s_M z_M z_R$, the parameter $s = s_M z_M$ is a preimage of u. Moreover, since s defines elliptic parameters M' for M and R'_s for R it will also define an elliptic parameter for L:

$$\mathfrak{a}_{L'_s} \subset \mathfrak{a}_{M'} \cap \mathfrak{a}_{R'_s} = \mathfrak{a}_M \cap \mathfrak{a}_R = \mathfrak{a}_L,$$

the last equality following from $d_T^L(M, R) \neq 0$, and so $s \in \mathscr{E}_{\mathfrak{M}'}(L)$.

We now at last have all the ingredients that we need to deduce the $\mathfrak{M}'$-WFL holds for $L \supsetneq M$ on v. Take (16.4.4) and multiply by d. We will repackage the right side to give the endoscopic side of the WFL. As noted above, R'_s (and the corresponding stable distributions) depend only upon $u = p(s)$. So it suffices to show that

$$d|Z^\Gamma_{\widehat{R'_u}}/Z^\Gamma_{\widehat{R}}|^{-1} S_T(R'_u)$$

is equal to the part of the endoscopic side of the WFL ranging over $s \in p^{-1}(u)$. As in Arthur's proof of the Splitting Lemma, the stable distributions satisfy a corresponding descent formula (cf. [Art99, Theorem 7.1]) which in our case simplifies to a single nonzero term

$$S_{M'}(L'_s) = e_{T'}^{L'_s}(M', R'_u) S_T(R'_u), \qquad e_s = e_{T'}^{L'_s}(M', R'_u) S_T(R'_u) := d \left| \frac{Z^\Gamma_{\widehat{M'}} \cap Z^\Gamma_{\widehat{R'_u}}}{Z^\Gamma_{\widehat{L}}} \right|^{-1},$$

see in particular Equation (7.1) in *loc. cit.* This identity really does apply here: the left side are stabilized values of real weighted orbital integrals, so do satisfy their appropriate stabilization formula in terms of the coefficients e. And moreover, the real weighted orbital integrals do satisfy a real descent formula down to the family C in terms of the coefficients d; in our case it simplified to

$$J^{L',1}_{M',v} = dC_{R'_s}$$

because all other coefficients vanished. Arthur's purely combinatorial argument then interchanges the descent formula for the J's into an equivalent descent formula for the corresponding stabilized objects in terms of coefficients e, which is the above equality. Exactly as in [CL12, Lemme 16.7.1] we then have the completely formal identity

$$\begin{aligned}\frac{d}{e_s} = \left|\frac{Z^\Gamma_{\widehat{M'}} \cap Z^\Gamma_{\widehat{R'_u}}}{Z^\Gamma_{\widehat{L}}}\right| &= |Z^\Gamma_{\widehat{M'}}/Z^\Gamma_{\widehat{M}}||Z^\Gamma_{\widehat{R'_u}}/Z^\Gamma_{\widehat{R}}||Z^\Gamma_{\widehat{L'_s}}/Z^\Gamma_{\widehat{L}}|^{-1}\left|\frac{Z^\Gamma_{\widehat{M}} \cap Z^\Gamma_{\widehat{R}}}{Z^\Gamma_{\widehat{L}}}\right| \\ &= |Z^\Gamma_{\widehat{M'}}/Z^\Gamma_{\widehat{M}}||Z^\Gamma_{\widehat{R'_u}}/Z^\Gamma_{\widehat{R}}||Z^\Gamma_{\widehat{L'_s}}/Z^\Gamma_{\widehat{L}}|^{-1}|p^{-1}(u)|\end{aligned}$$

Substituting this into the right side of (16.4.4) and expanding the factor of $|p^{-1}(u)|$ into a sum over $s \in p^{-1}(u)$ gives the $\mathfrak{M}'$-WFL for $M \subsetneq L$ on v. We have shown v is peelable. ☐

We made use of the following standard result in the proof.

Lemma 16.20 (cf. [CL12, Lemme 16.3.3(3)]). *Let H' be an elliptic endoscopic group for quasisplit H from pointed datum $\mathfrak{H}' = (s, \rho_s)$, and let $Q' \subset H'$ be a semistandard Levi containing the distinguished endoscopic torus. Then there is a unique semistandard Levi $Q \subset H$ and a unique factorization*

$$\rho_{s,Q} : \Gamma \to \pi_0^Q(s) \to \pi_0^H(s)$$

such that

$$Z_{\widehat{Q}}(s)^0 = \widehat{Q}'.$$

The pointed datum $(s, \rho_{s,Q})$ is elliptic for Q and has endoscopic group Q'.

Proof. A semistandard Levi $Q' \subset H'$ is also an endoscopic group for H: on the dual side this is, for example, [CE12, Lemma 3.9]. But every endoscopic group for G is an elliptic endoscopic group for some semistandard Levi by an "envelope" construction, as in [Art99]. Details left to the reader. ☐

16.5. Local constancy of weighted orbital integrals

We prove a very weak version of local constancy of local s-orbital integrals over finite residue field which is sufficient for our purposes.

Proposition 16.21 (cf. [CL12, §18.7]). *Let $b \in \mathfrak{c}_M(\mathscr{O}_{0,v}) \cap \mathfrak{c}_M^{G\text{-}reg}(F_{0,v})$, and write $X = \epsilon_M(b) \in \mathfrak{m}(\mathscr{O}_{0,v})$. There exists an integer n such that for every $b' \in \mathfrak{c}_M(\mathscr{O}_{0,v})$ satisfying*

$$b' \equiv b \mod z_v^n,$$

and the corresponding lift $X' \in \epsilon_M(b')$, there exists

$$h \in M(\mathscr{O}_{0,v}), \qquad h \equiv 1 \mod z_v$$

such that

1. *conjugation by h identifies the regular centralizers $J_b \cong J_{b'}$,*
2. *for every $g \in G(F_{0,v})$,*

$$\mathrm{Ad}(g^{-1})X \in \mathfrak{g}(\mathscr{O}_{0,v}) \iff \mathrm{Ad}(h^{-1}g^{-1})X' \in \mathfrak{g}(\mathscr{O}_{0,v}),$$

3. *for every $F_{0,v}$-parabolic P with Levi M,*

$$H_P(hg) = H_P(g).$$

In particular, the local s-weighted orbital integrals $J^{G,s}_{M,v}$ associated to b and b' are equal.

Proof. Consider the action map over $F_{0,v}$

$$\Phi : M \times \mathfrak{j} \to \mathfrak{m}, \quad (m, Y) \mapsto \mathrm{Ad}(m)Y,$$

where $\mathfrak{j} = \mathrm{Lie}(J_b)$. A differential calculation and semisimplicity of X shows this is smooth at $(1, X)$. Fix arbitrarily small neighborhoods

$$U_h := \ker(M(\mathscr{O}_{0,v}) \to M(\mathscr{O}_{0,v}/z_v^N)) \subset M(F)$$

of 1 and

$$U_Z := X + z_v^{N_1} \,\mathrm{Lie}(J_b)(\mathscr{O}_{0,v}) \subset \mathfrak{j}(F)$$

of X. Hensel's lemma as a p-adic inverse function theorem says that $\Phi(U_h, U_Z)$ contains a sufficiently small ball around $\mathrm{Ad}(1)X = X$. In other words, for any $X' \equiv X \mod z_v^n$, there exist $h \approx 1 \in U_h, Z \approx X \in U_Z$ such that

$$X' = \mathrm{Ad}(h)Z.$$

Since ϵ_M is defined over $\mathscr{O}_{0,v}$, this applies to our nearby Hensel lift X'. By definition, $Z \in \mathfrak{j}(F)$, so that $[X, Z] = 0$, and $Z - X \in z_v^{N_1} \,\mathrm{Lie}(J_b)(\mathscr{O}_{0,v})$.

Since $Z \in \mathfrak{j}(F)$, we have at least

$$J_b \subseteq Z_M(Z).$$

But Z is sufficiently close to regular semisimple X, so it is also regular semisimple and its centralizer is a maximal torus. We conclude

$$J_b = Z_M(Z),$$

So $\mathrm{Ad}(h)$ sends J_b to $Z_M(X') = J_{b'}$ over $F_{0,v}$. But h is integral, so the conjugation extends to an $\mathscr{O}_{0,v}$-morphism

$$\mathrm{Ad}(h) : J_b \to M_{\mathscr{O}_{0,v}}.$$

Generically it lands in $J_{b'}$, and J_b is flat, so the morphism factors through $J_{b'}$. The same argument applied to the inverse shows $J_b \cong J_{b'}$.

The second claim is the heart of the proof, and proceeds as in [CL12, Proposition 18.7.1]. We will show only the forward claim; the converse is almost a perfectly symmetric argument. We want to show that $\mathrm{Ad}(g^{-1})(X) \in \mathfrak{g}(\mathscr{O}_{0,v})$ if and only if $\mathrm{Ad}(g^{-1})(Z) \in \mathfrak{g}(\mathscr{O}_{0,v})$. Fix an $F_{0,v}$-parabolic P with Levi M and consider the iwasawa decomposition $P = MN$ of g

$$g = mnk, \qquad k \in G(\mathscr{O}_{0,v}).$$

Suppose that $\mathrm{Ad}(g^{-1})(X) \in \mathfrak{g}(\mathscr{O}_{0,v})$. We can discard k. By assumption, writing $Y = \mathrm{Ad}(m^{-1})(X)$ we have

$$\mathrm{Ad}((mn)^{-1})(X) = (\mathrm{Ad}(n^{-1})Y - Y) + Y.$$

The difference lies in $N(F_{0,v})$, and Y lies in $M(F_{0,v})$. By the Levi decomposition there is no intersection, so since the sum of the two terms is integral each term must be likewise. The $\mathscr{O}_{0,v}$-morphism $n \mapsto \mathrm{Ad}(n^{-1})Y - Y$ is an $F_{0,v}$-isomorphism whose inverse has denominators of valuation bounded by a fixed power of $\det(\mathrm{Ad}(Y)|_{\mathfrak{n}})$. This boundedness can be checked after a finite unramified extension which splits the group, where we use [CL12, Lemme 18.7.2]. Note that Y and X have the same determinant valuation on $\mathfrak{n}$ because conjugation by m intertwines the endomorphisms. So we have shown that

$$\mathrm{Ad}(g^{-1})(X) \in \mathfrak{g}(\mathscr{O}_{0,v})$$

implies n lies in a bounded subset of $B \subset N(F_{0,v})$, independent of g.

Now we need to bound $\delta_m = \mathrm{Ad}(m^{-1})\delta$, where $\delta = Z - X$. The universal property of the regular centralizer and integrality of Y implies that the $F_{0,v}$-map

$$\mathrm{Ad}(m^{-1}) : J_b \to Z_M(Y)$$

extends over $\mathscr{O}_{0,v}$. It follows that

$$\mathrm{Ad}(m^{-1}) \,\mathrm{Lie}(J_b)(\mathscr{O}_{0,v}) \subset \mathfrak{m}(\mathscr{O}_{0,v}).$$

Since $\delta \in z_v^{N_1} \operatorname{Lie}(J_b)(\mathscr{O}_{0,v})$, we have $\delta_m \in z_v^{N_1}\mathfrak{m}(\mathscr{O}_{0,v})$. Choosing N_1 large enough relative to the bounded subset $B \subset N(F_{0,v})$ that n lives in, we have

$$\operatorname{Ad}(n^{-1})(Y + \delta_m) = Y + (\operatorname{Ad}(n^{-1}Y - Y) + \delta_m + (\operatorname{Ad}(n^{-1}\delta_m - \delta_m)$$

with all four pieces integral, i.e., in $\mathfrak{g}(\mathscr{O}_{0,v})$. But

$$\operatorname{Ad}(g^{-1})Z = \operatorname{Ad}(n^{-1})\operatorname{Ad}(m^{-1})(X + \delta) = \operatorname{Ad}(n^{-1})(Y + \delta_m),$$

so this gives the second claim.

For the third claim, since $h \equiv 1 \mod z_v$ is an integral element of M, it vanishes in every rational character and satisfies $H_M(h) = 0$. But

$$H_P(hg) = H_M(h) + H_P(g),$$

so both the weight function v_M^G of Arthur and the weight function w_M^ξ of Chaudouard–Laumon 15.10 are invariant under h.

It remains to show $J_{M,v}^{G,s}(Y_b) = J_{M,v}^{G,s}(Y_{b'})$. The isomorphism $\operatorname{Ad}(h) : J_b \to J_{b'}$ induces a bijection of the domains and identifies bounded subgroups, so preserves the normalizations of the measures. The discriminant valuation factor is continuous, i.e., locally constant. Finally, the exact sequence in τ-cohomology and the Kottwitz isomorphism are natural along $\operatorname{Ad}(h)$, so

$$\operatorname{inv}_{b'}(hg) = \operatorname{inv}_b(g).$$

If one prefers the presentation of the invariant pairing in terms of $\hat{T}^{W_{b,v}}$, then one should be cautious about the Weyl group ambiguity implicit in the choice of geometric point $\tilde{b} \mapsto b$ upstairs in the cameral cover. However, making a choice of lift for b induces a corresponding such choice for every b' sufficiently close to b since the Weyl group is finite—choose a neighborhood of b small enough that its preimage is a disjoint union of lifts, and then the component containing $\tilde{b}$ is the distinguished one. We have shown that all requisite parts of the s-orbital integral satisfy the desired local constancy. □

16.6. Globalization of the local parameter and the proof of the WFL

We are at last prepared to prove Theorem 16.5, the Lie algebra version of Arthur's WFL for Lie algebras. The final stage is that of globalizing our arbitrary local parameter in a way that satisfies all our global hypotheses, and is fully analogous to [CL12, Theorem 19.1.1], [Ngô10, §8.6]. We begin from our local setting as in §16.1. In particular, $\mathscr{O}_{0,v}$ is the ring of integers of equal-characteristic local field $F_{0,v}$ with residue field k_v. As in the statement of Theorem 16.5, let $\mathfrak{M}'_v = (s_M, \rho_{v,s_M})$ be an unramified local endoscopic datum with endoscopic group $M'/\mathscr{O}_{0,v}$. Let Γ_v be an unramified (hence cyclic) Galois group large enough to split G and M', assumed prime to p.

16.6.1. Globalization of the group and endoscopic datum. We recall the requisite globalization statement. It is well-known but we provide a proof for completeness.

Lemma 16.22 (cf. [Ngô10, §8.6]). *Let k_v finite of characteristic p, and $\Gamma_v = \langle\gamma\rangle \cong \mathbb{Z}/n\mathbb{Z}$ for $(n,p) = 1$. Then there exists X_0/k_v smooth, projective, geometrically connected, two distinct points $v, \infty \in X_0(k_v)$, and a geometrically connected finite étale Γ_v-torsor*

$$X'_0 \to X_0$$

such that $\tau_v \mapsto \gamma$, $\tau_\infty \mapsto 1$. Equivalently, $D_v = \Gamma_v$ and $D_\infty = 1$.

Proof. Choose any elliptic curve E/k_v. For some finite extension k_m/k_v we have $E[n](k_m) \cong (\mathbb{Z}/n)^2$. So $A = \operatorname{Res}_{k_m/k_v}(E_{k_m})$ is an abelian variety such that there exists $\psi : A(k_v) \twoheadrightarrow \Gamma_v$. Choose $a \in A(k_v)$ mapping to generator γ, and consider the Lang isogeny

$$L_A = \operatorname{Frob}_q - 1 : A \to A.$$

Quotienting the Lang cover by $H = \ker(\psi) \subset A(k_v)$ gives an étale Γ_v-cover

$$A/H \to A.$$

Frobenius acts on the fiber over $x \in A(k_v)$ by $\psi(x)$, so at the identity $x = e$ it acts trivially and at $x = a$ it acts by γ. By [Poo04, Corollary 3.4] there exists a smooth projective geometrically integral curve $X_0 \subset A$ containing both e and a. Writing $\infty := e$ and $v := a$ and pulling back gives our desired étale cover

$$X_0' = X_0 \times_A (A/H) \to X_0,$$

The arithmetic Frobenius has trivial image (forced by the split point over ∞) but the arithmetic monodromy is surjective $\pi_1(X_0, \infty) \twoheadrightarrow \Gamma_v$. The juxtaposition ensures a surjection from the fundamental group of $X = (X_0)_{\overline{k_v}}$:

$$\pi_1(X, \overline{\infty}) \twoheadrightarrow \Gamma_v$$

so the cover is geometrically connected. □

Our local datum prescribes an action of Γ_v on a split pinned model whose corresponding twist along the unramified cover of the disk $\operatorname{Spec} \mathscr{O}_{0,v}$ defines $(G_v, M_v, \mathfrak{M}_v')$. Instead twisting the constant objects by the same action over the cover $X_0' \to X_0$ constructs

(1) G/X_0 quasisplit reductive group with X_0-maximal torus T satisfying the global characteristic hypotheses of §2.1, split at ∞, and such that $G|_{\operatorname{Spec} \mathscr{O}_{0,v}} \cong G_v$;
(2) M/X_0 semistandard Levi such that $M|_{\operatorname{Spec} \mathscr{O}_{0,v}} \cong M_v$;
(3) $\mathfrak{M}'$ an elliptic endoscopic datum for M on X_0 with endoscopic group M'/X_0 such that $M'|_{\operatorname{Spec} \mathscr{O}_{0,v}} \cong \mathfrak{M}_v'$ and M' is split at ∞.

We fix a finite set $S \subset |X_0| \setminus \{v, \infty\}$ of points. For convenience, by Chebotarev we are free to choose them to split in X' and to have sufficiently high residue field such that they admit regular elements in the split torus. We will define D to be a large even effective divisor with $\operatorname{supp}(D) = S$, say by fixing such a D_0 to begin with and then defining D to be a sufficiently positive even multiple of D_0.

16.6.2. Specifying finitely many jets. Now fix a local parameter $b' \in \mathfrak{c}_{M'}(\mathscr{O}_{0,v}) \cap \mathfrak{c}_{M'}^{G\text{-reg}}(F_{0,v})$, write $b \in \mathfrak{c}_M(\mathscr{O}_{0,v})$ for its image, and write

$$Y_{b'} = \epsilon_{M'}(b') \in \mathfrak{m}'(\mathscr{O}_{0,v}), \qquad Y_b = \epsilon_M(b) \in \mathfrak{m}(\mathscr{O}_{0,v})$$

for the images under Kostant sections. Let n be large enough to satisfy the conclusion of Proposition 16.21 for not only $M \subset G$, but for every endoscopic pair which appears in the (finite) recursion defining the local stable orbital integrals of Definition 16.4 and in the statement of the WFL 16.1.1. In particular, this ensures that if a global parameter $a_{M'}$ agrees with the local parameter b' up to order n, then it and all its images under any Chevalley maps will satisfy a sufficiently high congruence to identify the corresponding orbital integral values.

For any finite effective divisor Z on X_0, the restriction map

$$\operatorname{res}_Z : \mathscr{A}_{M'}(k_v) = H^0(X_0, \mathfrak{c}_{M',D}) \to H^0(Z, \mathfrak{c}_{M',D}|_Z)$$

is surjective for $D >> 0$. In particular, we can prescribe finite-order jets for the restriction to finitely many points on the curve, and for $D >> 0$ we will be able to find a global section of $\mathscr{A}_{M'}(k_v)$ mapping to it. In particular, we prescribe the following conditions:

- At v, we prescribe
$$a_{M'}|_{nv} \equiv b',$$
where n is the local constancy parameter. In other words, we prescribe the n-th order jet $a_{M'}|_{\mathscr{O}_{0,v}} \equiv b' \mod z_v^n$.
- At ∞, we prescribe $a_{M'}(\infty) = \chi_{M'}(t_\infty)$ for a fixed $t_\infty \in \mathfrak{t}_{M',D,\infty}^{G\text{-reg}}(k_v)$.
- At each $x \in S = \operatorname{supp}(D)$, we prescribe $a_{M'}(x)$ to match a given arbitrary regular section of $\mathfrak{t}_D(x)$, and in particular can certainly enforce G-regularity at each such point with a 0-jet.
- At one other auxiliary x_{ell}, chosen to split completely in $X' \to X$ and have sufficiently large residue degree, we enforce ellipticity in the special fiber as follows: choose an elliptic Weyl element $w \in W_{M'}(\overline{x}_{\text{-ell}}) = W_{M'}(x_{\text{ell}})$, and consider the $k_{x_{\text{ell}}}$-rational torus $T_w \subset M'|_{x_{\text{ell}}}$ (such that

Frobenius acts on the character lattice by w). This torus is elliptic, and for the residue field chosen sufficiently large there exists a G-regular

$$Y \in \mathrm{Lie}(T_w)(k_{x_{\mathrm{ell}}}).$$

We prescribe the 0-jet $a_{M'}(x_{\mathrm{ell}}) = \chi_{M'}(Y)$.

That is, $Z := n[v] + \infty + S + x_{\mathrm{ell}}$. The last condition ensures M'-ellipticity of the global section $a_{M'}$: the decomposition group at x_{ell} contributes an elliptic element $(w, 1)$ to the global cameral monodromy. This shows that $a_{M'}$ does not have monodromy subgroup lying within $W_{L'}(\infty) \rtimes \Gamma$ for any proper Levi, since $w \notin W_{L'}(\infty)$, so by Proposition 3.22, the element is M'-elliptic with *any* choice of rigidifying datum t. In particular, it lies in the compatible locus of Definition 3.9 for M' vacuously, and its image is M-elliptic by Proposition 3.27.

16.6.3. Bertini–Poonen transversality. So far we have only enforced finitely many conditions, so for $D = mD_0$ sufficiently positive (supported on S) the map res_Z is surjective. Moreover, the fiber $V_m \subset H^0(X_0, \mathfrak{c}_{M',mD_0})$ is an affine space, a torsor for $H^0(X_0, \mathfrak{c}_{M',mD_0}(-Z))$. It remains to handle the transversality condition, which we choose to do directly on $\mathbb{F}_q$ using a Bertini–Poonen sieve as in [Poo04]. Consider the reduced divisor

$$\Delta = (\mathfrak{D}_G|_{\mathfrak{c}_{M',D}})_{\mathrm{red}} = \left(\mathfrak{D}_{M'} + \mathfrak{R}^M_{M'} + (\chi^{M'}_M)^\star \mathfrak{R}^G_M\right)_{\mathrm{red}}.$$

At every closed point $x \notin \mathrm{supp}(Z)$, we wish to impose that $a_{M'}$ meets Δ with multiplicity at most 1. This is a first-jet condition.

Lemma 16.23 ([HLN14, Proposition 5.1.1], cf. [Poo04, Theorem 1.2], [Poo03, Theorem 8.1]). *Write $D = mD_0$ for D_0 finite effective and supported on S. For m sufficiently positive, there exists*

$$a_{M'} \in H^0(X_0, \mathfrak{c}_{M',mD_0}) = \mathscr{A}_{M'}(k_v)$$

such that $a_{M'}$ has the prescribed jets above on Z and $a^\star_{M'}\Delta$ is reduced on $X_0 \setminus |Z|$.

Proof. This is essentially the first result cited. We outline how to adapt the arguments of Poonen to give the Lemma. For each $x \notin |Z|$, we wish to bound the size of

$$B_x(m) = \{a \in V_m : v_x(a^\star\Delta) \geq 2\}.$$

Let r_m denote the rank of $\mathfrak{c}_{M',D}$ as a vector bundle, and write q_x for the size of the residue field at x. The first-order jet space for a at x has $q_x^{2r_m}$ points. If $y = a_{M'}(x)$ lies in a smooth point of Δ_x, then there are $O(q_x^{r_m-1})$ points y can equal, and tangency imposes a single linear condition on the derivative, leaving $q_x^{r_m-1}$ values of the derivative. In total, there are

$$O(q_x^{2r_m-2})$$

bad jets at smooth points. On the other hand, since Δ is defined inside $\mathfrak{c}_{M',D}$ by twisting from the constant divisor, inside each fiber we have

$$\mathrm{codim}_{(\mathfrak{c}_{M',D})_x}(\Delta_x^{\mathrm{sing}}) \geq 2;$$

the divisor is reduced so its singular locus is of higher codimension. So the set of singular values y can take is of order $O(q_x^{r_m-2})$, and the derivative can be chosen freely. So we have again

$$O(q_x^{2r_m-2})$$

bad jets. In particular, if the space of jets is equidistributed, then the probability of a bad jet at the point x is

$$\mathbb{P}(B_x(m)) = q_x^{-2} = q^{-2\deg(x)}.$$

We divide the closed points of X into 3 ranges, as in *loc. cit.*

(1) (Low-degree points). Fix R, independent of m, and consider the finite collection of points satisfying $\deg(x) < R$. Just as done above, for $m >> 0$ the first-order jets at such points can be imposed directly, and for $m >> 0$ the restriction map

$$V_m \to \prod_{\substack{x \notin Z \\ \deg(x) < R}} H^0(2x, \mathfrak{c}_{M', mD_0})$$

is surjective. The map is affine-linear, so its fibers are equal in size and points $a_{M'} \in V_m$ have images which equidistribute over the codomain. The probability of avoiding bad jets at all low-degree points is then

$$\mathbb{P}(\text{good at all } x, \deg(x) < R) = \prod_{\deg(x) < R} (1 - q^{-2\deg(x)}) > 0.$$

(2) (Medium-degree points). There is some constant $c > 0$ such that for points with $\deg(x) \leq cm$, the restriction map $V_m \to H^0(2x, \mathfrak{c}_{M', mD_0})$ is still surjective: the surjectivity holds whenever

$$H^1(X_0, \mathfrak{c}_{M', mD_0}(-Z - 2x)) = 0,$$

and the Chevalley bundle has degree growing linearly in m. For each degree $R \leq e \leq cm$, the number of $\mathbb{F}_{q^e}$-points on the curve is bounded by $O(q^e)$ (cf. [Poo04, Lemma 2.4]), and each closed point of degree e contributes e such. Summing, we see by the union bound that the probability that there exists a point with a bad jet in the range $R \leq \deg(x) \leq cm$ is

$$\mathbb{P}(\exists x, R \leq \deg(x) \leq cm, B_x(m)) \leq \sum_{e=R}^{cm} O(q^e/e) q^{-2e} \leq \sum_{e \geq R} q^{-e} = O(q^{-R}).$$

In particular, we can choose R originally such that

$$\limsup_{m \to \infty} \mathbb{P}(\exists x, R \leq \deg(x) \leq cm, B_x(m)) < \epsilon,$$

so taking m sufficiently large the bad probability will be bounded arbitrarily close to this limit, say $< 2\epsilon$.

(3) (High-degree points). Consider now the set of points satisfying $\deg(x) > cm$, for which jet-surjectivity fails. Here we use a different estimate of Poonen for the density of squarefree values of a squarefree multivariable function. Deleting finitely many additional points, we can assume $\mathfrak{c}_{M', D}$ is trivial on $U := X_0 \setminus (S \cup |Z| \cup S_1)$. Since D was supported on S, the resulting family of sections is independent of m. Choosing a trivialization as a family of fiber coordinates $y_1, \dots, y_r$, the reduced discriminant divisor has on U an equation

$$\Delta|_U = V(F(y_1, \dots, y_r))$$

with $F \in H^0(U, \mathcal{O}_U)[y_1, \dots, y_r]$ squarefree. Then [Poo03, Theorem 8.1] implies that a positive proportion of sections over U are squarefree. Moreover, since the allowed degrees of poles on S go to infinity, the proof shows that the sections can be chosen to globalize to $H^0(X_0, \mathfrak{c}_{M', D})$. See the proof of [HLN14, Proposition 5.1.1] for more details.

Combining these three, we find that a positive proportion of sections in V_m are transverse to Δ on $X_0 \setminus |Z|$, and in particular such a globalization exists. □

16.6.4. Goodness. Finally, it remains to check that when $\deg(D)$ is chosen sufficiently large relative to the complexity of the parameter b', our globalizations $a_{M'}$ necessarily land inside the good locus $\mathscr{A}_G^{good}$ of Definition 13.6. This follows from Theorem 13.24, taken with $S = \{v\}$, as long as

$$\deg(D) > 2g - 2 + \frac{|W_G|}{2} v(\mathfrak{D}_G(a)),$$

which is determined by the local parameter b'.

16.6.5. Proof of the WFL.

Proof of Theorem 16.5. Any globalized parameter $a_{M'}$ as in the conclusion of Lemma 16.23 is M'-elliptic by the condition at x_{ell} and lies in the good locus by Theorem 13.24. Thus, choosing any t to give a parameter $(a_{M'}, t) \in \mathsf{A}_{M'}$, the image $(a_M, t) \in \mathsf{A}_M$ is M-elliptic by Proposition 3.27. The parameter (as well as the endoscopic datum $\mathfrak{M}'$) therefore satisfy all the hypotheses of Theorem 15.16, which implies that the $\mathfrak{M}'$-WFL holds on $V = |X_0|$. Peelability of the regular places then tells us that the $\mathfrak{M}'$-WFL holds on the finite set $\{v\} \cup \mathrm{supp}(a^\star_{M'}\mathfrak{D}_{M'} + a^\star_{M'}\mathfrak{R}^M_{M'} + a^\star_M\mathfrak{R}^G_M) \subset |X_0|$. We have engineered our global parameter to intersect this divisor transversely at smooth points, so Lemmas 16.17, 16.18, and 16.19 imply that all places other than v are peelable. So the Peeling Lemma 16.13 implies that the $\mathfrak{M}'$-WFL for $M \subset G$ holds on $\{v\}$. Since we have engineered our global cover to satisfy $\Gamma = D_v = \Gamma_v$, as in Remark 16.11, this recovers exactly the local statement of the WFL. $\square$

CHAPTER 17

The nonstandard WFL

Work in progress!

Bibliography

[AGV73] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. *Theorie de Topos et Cohomologie Etale des Schemas I, II, III*. Vol. 269, 270, 305. Lecture Notes in Mathematics. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat. Springer-Verlag, Berlin-New York, 1973.

[AHH23] Jarod Alper, Daniel Halpern-Leistner, and Jochen Heinloth. "Existence of moduli spaces for algebraic stacks". In: *Invent. Math.* 234.3 (2023), pp. 949–1038. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s00222-023-01214-4. URL: https://doi-org.libproxy.berkeley.edu/10.1007/s00222-023-01214-4.

[AHR20] Jarod Alper, Jack Hall, and David Rydh. "A Luna étale slice theorem for algebraic stacks". In: *Ann. of Math. (2)* 191.3 (2020), pp. 675–738. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2020.191.3.1. URL: https://doi-org.libproxy.berkeley.edu/10.4007/annals.2020.191.3.1.

[Art01] James Arthur. "A stable trace formula. II. Global descent". In: *Invent. Math.* 143.1 (2001), pp. 157–220. ISSN: 0020-9910,1432-1297. DOI: 10.1007/s002220000107. URL: https://doi.org/10.1007/s002220000107.

[Art02] James Arthur. "A stable trace formula. I. General expansions". In: *J. Inst. Math. Jussieu* 1.2 (2002), pp. 175–277. ISSN: 1474-7480,1475-3030. DOI: 10.1017/S1474-748002000051. URL: https://doi.org/10.1017/S1474-748002000051.

[Art03] James Arthur. "A stable trace formula. III. Proof of the main theorems". In: *Ann. of Math. (2)* 158.3 (2003), pp. 769–873. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2003.158.769. URL: https://doi.org/10.4007/annals.2003.158.769.

[Art13] James Arthur. *The endoscopic classification of representations*. Vol. 61. American Mathematical Society Colloquium Publications. Orthogonal and symplectic groups. American Mathematical Society, Providence, RI, 2013, pp. xviii+590. ISBN: 978-0-8218-4990-3. DOI: 10.1090/coll/061. URL: https://doi.org/10.1090/coll/061.

[Art81] James Arthur. "The trace formula in invariant form". In: *Ann. of Math. (2)* 114.1 (1981), pp. 1–74. ISSN: 0003-486X. DOI: 10.2307/1971376. URL: https://doi.org/10.2307/1971376.

[Art88] James Arthur. "The invariant trace formula. I. Local theory". In: *J. Amer. Math. Soc.* 1.2 (1988), pp. 323–383. ISSN: 0894-0347,1088-6834. DOI: 10.2307/1990920. URL: https://doi-org.libproxy.berkeley.edu/10.2307/1990920.

[Art91] James Arthur. "A local trace formula". In: *Inst. Hautes Études Sci. Publ. Math.* 73 (1991), pp. 5–96. ISSN: 0073-8301,1618-1913. URL: http://www.numdam.org/item?id=PMIHES_1991__73__5_0.

[Art96] James Arthur. "On local character relations". In: *Selecta Math. (N.S.)* 2.4 (1996), pp. 501–579. ISSN: 1022-1824,1420-9020. DOI: 10.1007/PL00001383. URL: https://doi-org.libproxy.berkeley.edu/10.1007/PL00001383.

[Art98] James Arthur. "Canonical normalization of weighted characters and a transfer conjecture". In: *C. R. Math. Acad. Sci. Soc. R. Can.* 20.2 (1998), pp. 33–52. ISSN: 0706-1994,2816-5810.

[Art99] James Arthur. "On the transfer of distributions: weighted orbital integrals". In: *Duke Math. J.* 99.2 (1999), pp. 209–283. ISSN: 0012-7094,1547-7398. DOI: 10.1215/S0012-7094-99-09909-X. URL: https://doi-org.libproxy.berkeley.edu/10.1215/S0012-7094-99-09909-X.

[Ato+25] Hiraku Atobe et al. *Local Intertwining Relations and Co-tempered A-packets of Classical Groups.* 2025. arXiv: 2410.13504 [math.NT]. URL: https://arxiv.org/abs/2410.13504.

[BČ22] Alexis Bouthier and K estutis esnaviius. "Torsors on loop groups and the Hitchin fibration". In: *Ann. Sci. Éc. Norm. Supér. (4)* 55.3 (2022), pp. 791–864. ISSN: 0012-9593,1873-2151.

[Beh91] Kai A. Behrend. "The Lefschetz trace formula for the moduli stack of principal bundles". English. Copyright - Database copyright ProQuest LLC; ProQuest does not claim copyright in the individual underlying works; Last updated - 2023-02-23. PhD thesis. 1991, p. 148. URL: https://www.proquest.com/dissertations-theses/lefschetz-trace-formula-moduli-stack-principal/docview/303937093/se-2.

[Beh95] Kai A. Behrend. "Semi-stability of reductive group schemes over curves". In: *Math. Ann.* 301.2 (1995), pp. 281–305. ISSN: 0025-5831,1432-1807. DOI: 10.1007/BF01446630. URL: https://doi-org.libproxy.berkeley.edu/10.1007/BF01446630.

[Bei+18] Alexander Beilinson et al. "Faisceaux pervers". In: *Astérisque* 100 (2018), pp. vi+180. ISSN: 0303-1179,2492-5926.

[Bez96] Roman Bezrukavnikov. "The dimension of the fixed point set on affine flag manifolds". In: *Math. Res. Lett.* 3.2 (1996), pp. 185–189. ISSN: 1073-2780. DOI: 10.4310/MRL.1996.v3.n2.a5. URL: https://doi-org.libproxy.berkeley.edu/10.4310/MRL.1996.v3.n2.a5.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models.* Vol. 21. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1990, pp. x+325. ISBN: 3-540-50587-3. DOI: 10.1007/978-3-642-51438-8. URL: https://doi-org.libproxy.berkeley.edu/10.1007/978-3-642-51438-8.

[BT65] Armand Borel and Jacques Tits. "Groupes réductifs". fr. In: *Publications Mathématiques de l'IHÉS* 27 (1965), pp. 55–151. URL: http://www.numdam.org/item/PMIHES_1965__27__55_0/.

[CE12] Giovanna Carnovale and Francesco Esposito. "On sheets of conjugacy classes in good characteristic". In: *Int. Math. Res. Not. IMRN* 4 (2012), pp. 810–828. ISSN: 1073-7928,1687-0247. DOI: 10.1093/imrn/rnr047. URL: https://doi.org/10.1093/imrn/rnr047.

[CF] B. Conrad and T. Feng. *Reductive Groups over Fields.*

[Cha14] Pierre-Henri Chaudouard. "Geometry of the fundamental lemma". In: *Automorphic Forms and Galois Representations: Volume 1* 414 (2014), p. 188.

[CL10] Pierre-Henri Chaudouard and Gérard Laumon. "Le lemme fondamental pondéré. I. Constructions géométriques". In: *Compositio Mathematica* 146.6 (2010), pp. 1416–1506.

[CL12] Pierre-Henri Chaudouard and Gérard Laumon. "Le lemme fondamental pondéré. II. Énoncés cohomologiques". In: *Annals of Mathematics* (2012), pp. 1647–1781.

[Con14] Brian Conrad. "Reductive group schemes". In: *Autour des schémas en groupes* 1.93-444 (2014), p. 24.

[CS79] J.-L. Colliot-Thélène and J.-J. Sansuc. "Fibrés quadratiques et composantes connexes réelles". In: *Math. Ann.* 244.2 (1979), pp. 105–134. ISSN: 0025-5831,1432-1807. DOI: 10.1007/BF01420486. URL: https://doi.org/10.1007/BF01420486.

[DG67] Jean Dieudonné and Alexander Grothendieck. "Éléments de géométrie algébrique". In: *Inst. Hautes Études Sci. Publ. Math.* 4, 8, 11, 17, 20, 24, 28, 32 (1967).

[DS95] Vladimir G Drinfeld and Carlos Simpson. "B-structures on G-bundles and local triviality". In: *Mathematical Research Letters* 2.6 (1995), pp. 823–829.

[Gil15] Philippe Gille. "Sur la classification des schémas en groupes semi-simples". In: *Panoramas & Synthèses* 47 (2015), pp. 39–110.

[GKM04] Mark Goresky, Robert Kottwitz, and Robert Macpherson. "Homology of affine Springer fibers in the unramified case". In: *Duke Math. J.* 121.3 (2004), pp. 509–561. ISSN: 0012-7094,1547-7398. DOI: 10.1215/S0012-7094-04-12135-9. URL: https://doi-org.libproxy.berkeley.edu/10.1215/S0012-7094-04-12135-9.

[GKM09] Mark Goresky, Robert Kottwitz, and Robert MacPherson. "Codimensions of root valuation strata". In: *Pure Appl. Math. Q.* 5.4 (2009), pp. 1253–1310. ISSN: 1558-8599,1558-8602. DOI: 10.4310/PAMQ.2009.v5.n4.a4. URL: https://doi-org.libproxy.berkeley.edu/10.4310/PAMQ.2009.v5.n4.a4.

[GL19] Dennis Gaitsgory and Jacob Lurie. *Weil's conjecture for function fields. Vol. 1.* Vol. 199. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2019, pp. viii+311. ISBN: 978-0-691-18213-1.

[GWZ20] Michael Groechenig, Dimitri Wyss, and Paul Ziegler. "Geometric stabilisation via p-adic integration". In: *Journal of the American Mathematical Society* 33.3 (2020), pp. 807–873.

[Hal05] Thomas C. Hales. "A statement of the fundamental lemma". In: *Harmonic analysis, the trace formula, and Shimura varieties.* Vol. 4. Clay Math. Proc. Amer. Math. Soc., Providence, RI, 2005, pp. 643–658. ISBN: 0-8218-3844-X.

[Hal22] Daniel Halpern-Leistner. *On the structure of instability in moduli theory.* 2022. arXiv: 1411.0627 [math.AG]. URL: https://arxiv.org/abs/1411.0627.

[Har68] Günter Harder. "Halbeinfache Gruppenschemata über vollständigen Kurven". In: *Invent. Math.* 6 (1968), pp. 107–149. ISSN: 0020-9910,1432-1297. DOI: 10.1007/BF01425451. URL: https://doi.org/10.1007/BF01425451.

[Har69] G. Harder. "Minkowskische Reduktionstheorie über Funktionenkörpern". In: *Invent. Math.* 7 (1969), pp. 33–54. ISSN: 0020-9910,1432-1297. DOI: 10.1007/BF01418773. URL: https://doi.org/10.1007/BF01418773.

[Hei10] Jochen Heinloth. "Uniformization of G-bundles". In: *Mathematische Annalen* 347 (July 2010). DOI: 10.1007/s00208-009-0443-4.

[Hei17] Jochen Heinloth. "Hilbert-Mumford stability on algebraic stacks and applications to $\mathscr{G}$-bundles on curves". In: *Épijournal Géom. Algébrique* 1 (2017), Art. 11, 37. ISSN: 2491-6765. DOI: 10.46298/epiga.2018.volume1.2062. URL: https://doi-org.libproxy.berkeley.edu/10.46298/epiga.2018.volume1.2062.

[HK21] Paul Hamacher and Wansu Kim. "On G-isoshtukas over function fields". In: *Selecta Math. (N.S.)* 27.4 (2021), Paper No. 75, 34. ISSN: 1022-1824,1420-9020. DOI: 10.1007/s00029-021-00683-w. URL: https://doi-org.libproxy.berkeley.edu/10.1007/s00029-021-00683-w.

[HLN14] Q. P. Hồ, V. B. Lê Hùng, and B. C. Ngô. "Average size of 2-Selmer groups of elliptic curves over function fields". In: *Math. Res. Lett.* 21.6 (2014), pp. 1305–1339. ISSN: 1073-2780,1945-001X. DOI: 10.4310/MRL.2014.v21.n6.a6. URL: https://doi.org/10.4310/MRL.2014.v21.n6.a6.

[HS10] Jochen Heinloth and Alexander H. W. Schmitt. "The cohomology rings of moduli stacks of principal bundles over curves". In: *Doc. Math.* 15 (2010), pp. 423–488. ISSN: 1431-0635,1431-0643.

[Kem78] George R. Kempf. "Instability in invariant theory". In: *Ann. of Math. (2)* 108.2 (1978), pp. 299–316. ISSN: 0003-486X. DOI: 10.2307/1971168. URL: https://doi-org.libproxy.berkeley.edu/10.2307/1971168.

[Kot84] Robert E. Kottwitz. "Shimura varieties and twisted orbital integrals". In: *Math. Ann.* 269.3 (1984), pp. 287–300. ISSN: 0025-5831,1432-1807. DOI: 10.1007/BF01450697. URL: https://doi.org/10.1007/BF01450697.

[Kot85] Robert E. Kottwitz. "Isocrystals with additional structure". In: *Compositio Math.* 56.2 (1985), pp. 201–220. ISSN: 0010-437X,1570-5846. URL: http://www.numdam.org/item?id=CM_1985__56_2_201_0.

[Kot97] Robert E. Kottwitz. "Isocrystals with additional structure. II". In: *Compositio Math.* 109.3 (1997), pp. 255–339. ISSN: 0010-437X,1570-5846. DOI: 10.1023/A:1000102604688. URL: https://doi-org.libproxy.berkeley.edu/10.1023/A:1000102604688.

[Kot99] Robert E. Kottwitz. "Transfer factors for Lie algebras". In: *Represent. Theory* 3 (1999), pp. 127–138. ISSN: 1088-4165. DOI: 10.1090/S1088-4165-99-00068-0. URL: https://doi-org.libproxy.berkeley.edu/10.1090/S1088-4165-99-00068-0.

[KP23] Tasho Kaletha and Gopal Prasad. *Bruhat-Tits theory—a new approach.* Vol. 44. New Mathematical Monographs. Cambridge University Press, Cambridge, 2023, pp. xxx+718. ISBN: 978-1-108-83196-3.

[Lan75] Stacy G. Langton. "Valuative criteria for families of vector bundles on algebraic varieties". In: *Ann. of Math. (2)* 101 (1975), pp. 88–110. ISSN: 0003-486X. DOI: 10.2307/1970987. URL: https://doi-org.libproxy.berkeley.edu/10.2307/1970987.

[Lan83] R. P. Langlands. *Les débuts d'une formule des traces stable*. Vol. 13. Publications Mathématiques de l'Université Paris VII [Mathematical Publications of the University of Paris VII]. Université de Paris VII, U.E.R. de Mathématiques, Paris, 1983, pp. v+188.

[LM00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*. Vol. 39. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2000, pp. xii+208. ISBN: 3-540-65761-4.

[LN08] Gérard Laumon and Bao Châu Ngô. "Le lemme fondamental pour les groupes unitaires". In: *Ann. of Math. (2)* 168.2 (2008), pp. 477–573. ISSN: 0003-486X,1939-8980. DOI: `10.4007/annals.2008.168.477`. URL: `https://doi-org.libproxy.berkeley.edu/10.4007/annals.2008.168.477`.

[LS87] R. P. Langlands and D. Shelstad. "On the definition of transfer factors". In: *Math. Ann.* 278.1-4 (1987), pp. 219–271. ISSN: 0025-5831,1432-1807. DOI: `10.1007/BF01458070`. URL: `https://doi-org.libproxy.berkeley.edu/10.1007/BF01458070`.

[MW16a] Colette Moeglin and Jean-Loup Waldspurger. *Stabilisation de la formule des traces tordue. Vol. 1.* Vol. 316. Progress in Mathematics. Birkhäuser/Springer, Cham, 2016, pp. xviii+587. ISBN: 978-3-319-30049-8. DOI: `10.1007/978-3-319-30049-8`. URL: `https://doi-org.libproxy.berkeley.edu/10.1007/978-3-319-30049-8`.

[MW16b] Colette Moeglin and Jean-Loup Waldspurger. *Stabilisation de la formule des traces tordue. Vol. 2.* Vol. 317. Progress in Mathematics. Birkhäuser/Springer, Cham, 2016, v–xxviii and 589–1315. ISBN: 978-3-319-30058-0. DOI: `10.1007/978-3-319-30058-0`. URL: `https://doi-org.libproxy.berkeley.edu/10.1007/978-3-319-30058-0`.

[Ngô06] Bao Châu Ngô. "Fibration de Hitchin et endoscopie". In: *Invent. Math.* 164.2 (2006), pp. 399–453. ISSN: 0020-9910,1432-1297. DOI: `10.1007/s00222-005-0483-7`. URL: `https://doi-org.libproxy.berkeley.edu/10.1007/s00222-005-0483-7`.

[Ngô10] Bao Châu Ngô. "Le lemme fondamental pour les algèbres de Lie". In: *Publ. Math. Inst. Hautes Études Sci.* 111 (2010), pp. 1–169. ISSN: 0073-8301,1618-1913. DOI: `10.1007/s10240-010-0026-7`. URL: `https://doi.org/10.1007/s10240-010-0026-7`.

[Poo03] Bjorn Poonen. "Squarefree values of multivariable polynomials". In: *Duke Math. J.* 118.2 (2003), pp. 353–373. ISSN: 0012-7094,1547-7398. DOI: `10.1215/S0012-7094-03-11826-8`. URL: `https://doi.org/10.1215/S0012-7094-03-11826-8`.

[Poo04] Bjorn Poonen. "Bertini theorems over finite fields". In: *Ann. of Math. (2)* 160.3 (2004), pp. 1099–1127. ISSN: 0003-486X,1939-8980. DOI: `10.4007/annals.2004.160.1099`. URL: `https://doi.org/10.4007/annals.2004.160.1099`.

[PR08] G. Pappas and M. Rapoport. "Twisted loop groups and their affine flag varieties". In: *Adv. Math.* 219.1 (2008). With an appendix by T. Haines and Rapoport, pp. 118–198. ISSN: 0001-8708,1090-2082. DOI: `10.1016/j.aim.2008.04.006`. URL: `https://doi-org.libproxy.berkeley.edu/10.1016/j.aim.2008.04.006`.

[PR24] Georgios Pappas and Michael Rapoport. "On tamely ramified $\mathscr{G}$-bundles on curves". In: *Algebr. Geom.* 11.6 (2024). With an appendix by Brian Conrad, pp. 796–829. ISSN: 2313-1691,2214-2584.

[Ric17] Simon Riche. "Kostant section, universal centralizer, and a modular derived Satake equivalence". In: *Math. Z.* 286.1-2 (2017), pp. 223–261. ISSN: 0025-5874,1432-1823. DOI: `10.1007/s00209-016-1761-3`. URL: `https://doi.org/10.1007/s00209-016-1761-3`.

[Sch15] Simon Schieder. "The Harder-Narasimhan stratification of the moduli stack of G-bundles via Drinfeld's compactifications". In: *Selecta Math. (N.S.)* 21.3 (2015), pp. 763–831. ISSN: 1022-1824,1420-9020. DOI: `10.1007/s00029-014-0161-y`. URL: `https://doi-org.libproxy.berkeley.edu/10.1007/s00029-014-0161-y`.

[Ser68] Jean-Pierre Serre. *Corps locaux*. Vol. No. VIII. Publications de l'Université de Nancago. Deuxième édition. Hermann, Paris, 1968, p. 245.

[Ser88] Jean-Pierre Serre. *Algebraic groups and class fields*. Vol. 117. Graduate Texts in Mathematics. Translated from the French. Springer-Verlag, New York, 1988, pp. x+207. ISBN: 0-387-96648-X. DOI: `10.1007/978-1-4612-1035-1`. URL: `https://doi.org/10.1007/978-1-4612-1035-1`.

[Ser94] Jean-Pierre Serre. *Cohomologie galoisienne*. Vol. 5. Springer Science & Business Media, 1994.

[SGAI] Michele Raynaud. *Revêtements Étales et Groupe Fondamental: Séminaire de Géométrie Algébrique du Bois Marie 1960/61 (SGA 1)*. Vol. 224. Springer, 2006.

[SGAIII] M Demazure et al. *Séminaire de géométrie algébrique du Bois Marie 1962-64. Schémas en groupes (SGA 3)*. 2011.

[Sol20] Maarten Solleveld. "Conjugacy of Levi subgroups of reductive groups and a generalization to linear algebraic groups". In: *J. Pure Appl. Algebra* 224.6 (2020), pp. 106254, 16. ISSN: 0022-4049,1873-1376. DOI: 10.1016/j.jpaa.2019.106254. URL: https://doi.org/10.1016/j.jpaa.2019.106254.

[Sta22] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2022.

[Ste75] Robert Steinberg. "Torsion in reductive groups". In: *Advances in Mathematics* 15.1 (1975), pp. 63–92.

[Tit59] Jacques Tits. "Sur la classification des groupes algébriques semi-simples". In: *C. R. Acad. Sci. Paris* 249 (1959), pp. 1438–1440. ISSN: 0001-4036.

[Tot13] Burt Totaro. "Pseudo-abelian varieties". In: *Ann. Sci. Éc. Norm. Supér. (4)* 46.5 (2013), pp. 693–721. ISSN: 0012-9593,1873-2151. DOI: 10.24033/asens.2199. URL: https://doi-org.libproxy.berkeley.edu/10.24033/asens.2199.

[Wal11] J.-L. Waldspurger. "À propos du lemme fondamental tordu". In: *On the stabilization of the trace formula*. Vol. 1. Stab. Trace Formula Shimura Var. Arith. Appl. Int. Press, Somerville, MA, 2011, pp. 181–225. ISBN: 978-1-57146-227-5.

[Wal97] J.-L. Waldspurger. "Le lemme fondamental implique le transfert". In: *Compositio Math.* 105.2 (1997), pp. 153–236. ISSN: 0010-437X,1570-5846. DOI: 10.1023/A:1000103112268. URL: https://doi-org.libproxy.berkeley.edu/10.1023/A:1000103112268.

[Wan23] Xiao Wang. *Multiplicative Hitchin Fibration and Fundamental Lemma*. Thesis (Ph.D.)–The University of Chicago. ProQuest LLC, Ann Arbor, MI, 2023, p. 341. ISBN: 979-8379-70433-9. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqm&rft_dat=xri:pqdiss:30311649.