GAGA TALK NOTES

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This note gives a mostly complete proof of the standard GAGA theorems for projective schemes over $\mathbb{C}$. I closely Serre's original paper [Ser56] and some OCW lecture notes by Kedlaya [Ked], the latter of which discuss the proof in a more modern language, but only in the case of projective space. Hartshorne also has a GAGA appendix, but I haven't looked at it. I tried to balance the classical and explicit perspective of the original paper with a more modern perspective. Disclaimer: I cite without proof standard (but not necessarily easy) facts from both algebraic and analytic complex geometry.

I. Analytification of spaces

The analytification functor is intuitively an extremely straightforward idea. The Zariski topology is extremely weak, but we could also give the $\mathbb{C}$-valued locus of some polynomial equations $V(f_1, \dots, f_n) \subseteq \mathbb{C}^n$ the induced topology as a subspace of $\mathbb{C}^n$. Patching these together, we have a natural notion of the analytic topology on a scheme locally of finite type over $\mathbb{C}$. Since polynomial functions are continuous in the classical topology, this should be a finer topology than the Zariski topology. If the scheme is not a smooth variety, then the resulting space will not be a complex manifold, so we need a notion of complex analytic space that can allow for such degeneracies.

Definition 1. An *affine complex analytic space* is the locally ringed space $(Z, \mathcal{O}_Z)$ consisting of a closed subspace Z of an open $U \subseteq \mathbb{C}^n$ such that Z is cut out by holomorphic functions $f_1, \dots, f_k$ on U, with structure sheaf $\mathcal{O}_U/\mathcal{I}(Z)$ (the holomorphic functions modulo the analytic ideal sheaf). A *complex analytic space* is a locally ringed space $(X^{an}, \mathcal{O}_X^{an})$ which has a cover by open subspaces which are isomorphic to affine complex analytic spaces.

What do local rings of analytic spaces look like? If $X = \mathbb{C}^n$ then

$$\mathcal{O}_{X,x}^{an} = \mathbb{C}\{z_1, \dots, z_n\}$$

is the ring of convergent power series in n variables around the point $(z_1, \dots, z_n)$. In general it is a quotient of this ring by the local defining functions of X, hence is Noetherian!

When we analytify a scheme, $\mathcal{O}_X^{an}$ should consist of continuous functions $X^{an} \to \mathbb{C}$ which are on each affine patch $V(f_1, \dots, f_k) \subseteq \mathbb{C}^n$ the restriction of holomorphic functions on $\mathbb{C}^n$. We give a coordinate-free description that boils down to the same thing, but if you would prefer a more hands-on description, see [Ser56]. The right ambient category for all of this will be locally ringed spaces so that we can work uniformly with analytic spaces and with schemes. Let $\mathcal{L}$ the category of locally ringed spaces and $\mathcal{A}$ the full subcategory of analytic spaces inside $\mathcal{L}$.

Theorem 2. *Let X a scheme locally of finite type over $\mathbb{C}$. The functor $\mathcal{A} \to \mathsf{Set}$*

$$Y \mapsto \mathrm{Hom}_{\mathcal{L}}(Y, X)$$

is representable by an analytic space X^{an}, i.e. there are natural bijections

$$\mathrm{Hom}_{\mathcal{L}}(Y, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(Y, X^{an}) = \mathrm{Hom}_{\mathcal{L}}(Y, X^{an}).$$

Moreover, X^{an} has underlying set $X(\mathbb{C})$.

Proof. We show that the class of schemes X satisfying the theorem is closed under taking open subschemes, closed subschemes and products. The idea is that we want to mimic each of the algebraic constructions in terms of analytic spaces.

Open subschemes. Assume that X satisfies the theorem with some X^{an}, and $i : U \subseteq X$ is an open subscheme. Then U^{an} should be the set $U(\mathbb{C})$, with the induced analytic space structure $\mathcal{O}_X^{an}|_U$.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{L}}(Y, X) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{A}}(Y, X^{an}) \\ \uparrow & & \uparrow \\ \mathrm{Hom}_{\mathcal{L}}(Y, U) & \dashrightarrow & \mathrm{Hom}_{\mathcal{A}}(Y, U^{an}) \end{array}$$

Given Y an analytic space and $f : Y \to U$ a map of locally ringed spaces, composition gives a map $Y \to X$ of locally ringed spaces, hence a map $f : Y \to X^{an}$ of analytic spaces. As a continuous map, the image of f lies inside $U^{an} = U(\mathbb{C})$, so it factors as a map of spaces. On pullback of functions, we have maps of sheaves on X

$$f^\# : \mathcal{O}_X^{an} \to i_\star f_\star \mathcal{O}_Y^{an}.$$

We want to show that this factors through $\mathcal{O}_U^{an} = \mathcal{O}_X^{an}|_U$. Given an open $V \subseteq X$ and section $f^\#(s) \in H^0(f^{-1}(V), \mathcal{O}_Y^{an})$, we have $f^{-1}(V) = f^{-1}(V \cap U)$ and so $f^\#(s)$ is also the image of $s|_U$. So the bijection on morphisms to X restricts to a bijection on morphisms to U. The latter bijection is automatically functorial in Y since the former is and since the construction above is compatible with precomposition $Z \to Y$. By construction U^{an} has the right underlying set. That the last condition passes from X to U is automatic since the maps of stalks are the same.

In the same way, we can check that if X is a scheme which has an analytification X^{an} and $Z \hookrightarrow X$ is a closed subscheme, then $Z^{an} = (Z(\mathbb{C}), \mathcal{O}_X^{an}/\mathcal{I}^{an}(Z))$ represents the functor $Y \mapsto \mathrm{Hom}_{\mathcal{L}}(Y, Z)$. This has the right underlying set by construction, and the analytic and algebraic ideal sheaves are related by

$$\mathcal{I}^{an}(Z)_x = \mathcal{I}(Z)_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}^{an},$$

so if

$$\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}^{an}$$

induces isomorphisms on the completions, then since completion is right exact

$$\widehat{\mathcal{O}}_{Z,z}^{an} = \frac{\widehat{\mathcal{O}}_{X,z}^{an}}{\widehat{\mathcal{I}}(Z) \otimes_{\widehat{\mathcal{O}}_{X,z}} \widehat{\mathcal{O}}_{X,z}^{an}} \cong \frac{\widehat{\mathcal{O}}_{X,z}}{\widehat{\mathcal{I}}(Z)} = \widehat{\mathcal{O}}_{Z,x}.$$

Finally, for products, one should take $(X^{an} \times W^{an}, \mathcal{O}_X^{an} \otimes_{\mathbb{C}} \mathcal{O}_W^{an})$ as the analytification of $X \times W$. If we assume that both X and W have analytifications, then

$$\begin{aligned} \mathrm{Hom}_{\mathcal{L}}(Y, X \times W) &= \mathrm{Hom}_{\mathcal{L}}(Y, X) \times_{\mathrm{Hom}(Y, \mathrm{Spec}\,\mathbb{C})} \mathrm{Hom}_{\mathcal{L}}(Y, W) \\ &= \mathrm{Hom}_{\mathcal{A}}(Y, X^{an}) \times \mathrm{Hom}_{\mathcal{A}}(Y, W^{an}) \\ &= \mathrm{Hom}_{\mathcal{A}}(Y, X^{an} \times Y^{an}), \end{aligned}$$

because the fiber product of schemes and classical product of analytic spaces represent corresponding functors. Subtle point: the fiber product of morphisms of schemes is also the fiber product of those morphisms viewed inside $\mathcal{L}$, as in Stacks Project, tag 01JN. This has the right underlying set because $(X \times_{\mathbb{C}} Y)(\mathbb{C}) = X(\mathbb{C}) \times Y(\mathbb{C})$. As for the statement about completion, the map in question is

$$\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_W \to \mathcal{O}_X^{an} \otimes_{\mathbb{C}} \mathcal{O}_W^{an}.$$

Completion commutes with tensor products and the tensor product of isomorphisms is an isomorphism so we are done.

We have reduced to showing the theorem for $X = \mathbb{A}^1_{\mathbb{C}}$. Of course we should have $X^{an} = \mathbb{C}$. A map of locally ringed spaces $Y \to X$ is the same as a global section of Y (the usual proof for Y a scheme works for arbitrary locally ringed spaces). So $\mathrm{Hom}_{\mathcal{L}}(Y, X) = \Gamma(Y, \mathcal{O}_Y)$. But if Y is an analytic, then $\mathrm{Hom}_{\mathcal{A}}(Y, \mathbb{C}) = \Gamma(Y, \mathcal{O}_Y)$ also, so we get the result.

Finally, we check that $X^{an} \to X$ induces isomorphisms on the completion of the local rings. Since the condition is local, we may assume X (and thus X^{an}) are affine. In the case that $X = \mathbb{A}^n_{\mathbb{C}}$, the result is trivial because then $\mathcal{O}_{X,x}^{an}$ is convergent power series in the variables $\square$

By the proof of this theorem, analytification is automatically compatible with passing to open or closed subschemes.

Direction can get a bit confusing: analytification is a functor $X \mapsto X^{an}$, but there is also a map between the two as locally ringed spaces, which goes the other way: $X^{an} \to X$. We think about the map $X^{an} \to X$ as doing two things: (1) weakens the topology from analytic to Zariski, and (2) adds in scheme points.

Example 3. If $X = \mathbb{P}^r_{\mathbb{C}}$ as a scheme, then $X^{an} = \mathbb{P}^r_{\mathbb{C}}$ (projective space as a complex manifold). The morphism $X^{an} \to X$ can be explicitly described over open charts as follows: over each $X_i := D_+(x_i) \subseteq \mathbb{P}^r_{\mathbb{C}}$ (writing $\widetilde{X}_i$ likewise in analytic projective space) the morphism is locally

$$\widetilde{X}_i \to X_i$$

given by the obvious morphism on spaces and with pullbacks given by the obvious maps

$$\Gamma(X_i, \mathcal{O}_{X_i}) = \mathbb{C}[x_0/x_i, \dots x_n/x_i] \to \Gamma(\widetilde{X}_i, \mathcal{O}_{\widetilde{X}_i}).$$

We investigate now the properties of the map $X^{an} \to X$.

Proposition 4. *Write $\mathcal{O} = \mathcal{O}_{X,x}$ and $\mathcal{O}^{an} = \mathcal{O}^{an}_{X,x}$. Denote by θ the map of local rings above a point*

$$\theta : \mathcal{O} \to \mathcal{O}^{an}.$$

Since this is a local homomorphism, it passes to a homomorphism $\widehat{\theta}$ of the completions. Then

(1) $\widehat{\theta}$ is an isomorphism.
(2) If Y is a Zariski-closed subscheme of X, write I for the stalk of the algebraic ideal sheaf $I := \mathcal{I}(Y)_x$. Likewise write I^{an} for the stalk of the analytic ideal sheaf. Then I^{an} is generated by $\theta(I)$.

We prove only the special case $X = \mathbb{A}^n$ and Y a subvariety, i.e. reduced. We prove both of these first in the case $X = \mathbb{A}^n_{\mathbb{C}}$. The first part is clear, because as above $\mathcal{O}^{an}$ is the ring of convergent power series in the z_i, whose completion is of course $\mathbb{C}[[z_1, \dots, z_n]] \cong \widehat{\mathcal{O}}$.

To prove (2) for $X = \mathbb{A}^n_{\mathbb{C}}$, we will pass to the completions. Let $J \subseteq I^{an}$ denote the ideal of $\mathcal{O}^{an}$ generated by $\theta(I)$. Every ideal of $\mathcal{O}^{an}$ cuts out the germ of an analytic subspace at x, and of course J cuts out $Y^{an} \subseteq X^{an}$, the same as I^{an} does.

By the Nullstellensatz, it suffices to show that J is radical. This is basically what we will do. Let $f \in I^{an}$, so $f^r \in J$ for some r. In the completion, we have

$$f^r \in J\widehat{\mathcal{O}}^{an} = I\widehat{\mathcal{O}}^{an} = I\widehat{\mathcal{O}}.$$

It is a theorem of Chevalley that the ideal $I\widehat{\mathcal{O}}$ is still the intersection of prime ideals corresponding to the irreducible components of Y, hence $I\widehat{\mathcal{O}}$ is radical and so $f \in I\widehat{\mathcal{O}}$. For ideals in Noetherian local rings

$$(J\widehat{\mathcal{O}}) \cap \mathcal{O} = J,$$

so $f \in J$. $\square$

Corollary 5. *The universal map $X^{an} \to X$ is flat.*

Proof. We showed above the map induces isomorphisms of local completions. This implies flatness as follows. Recall that if we have a map of Noetherian local rings $A \to B$ and $B \to E$ with the latter faithfully flat, then $A \to E$ is flat iff $A \to B$ is flat. We have a commutative square with the top map and both side maps faithfully flat.

$$\begin{array}{ccc} \widehat{A} & \xrightarrow{\sim} & \widehat{B} \\ \uparrow & \nearrow & \uparrow \\ A & \longrightarrow & B \end{array}$$

The diagonal map is flat by composing along the top and the right map is faithfully flat, so $A \to B$ is flat. $\square$

In the next section, we will define an induced analytification functor on coherent sheaves. Recall that a *coherent sheaf* on any locally ringed space $(X, \mathcal{O}_X)$ is a sheaf $\mathcal{F}$ satisfying the following two axioms.

(1) $\mathcal{F}$ is finite type over $\mathcal{O}_X$, i.e. every point has a neighborhood U such that there is a surjective morphism $\mathcal{O}^n_X|_U \to \mathcal{F}|_U$ for some n.

(2) For any surjection as above, the kernel is also of finite type.

Note that $\mathcal{O}_X$ need not itself be coherent for an arbitrary ringed space. However, it is in the settings we care about.

Proposition 6. *If V is a locally Noetherian scheme, then $\mathcal{O}_V = \mathcal{O}_X|_V$ is coherent.*

Proof. The ring $\mathcal{O}_X$ is affine-locally of the form $\widetilde{A}$, which is finitely generated over A hence coherent. $\square$

On the other hand, in analytic settings we have the following result.

Theorem 7 (Oka coherence)**.** *If X is an open subset of $\mathbb{C}^n$, then the sheaf $\mathcal{O}_X$ is coherent as a module over itself.*

Corollary 8. *The same is true for X any analytic space.*

Proof. To reduce to the above, note that the condition is local, and we can pass easily to closed subvarieties because if $X \subset U \subset \mathbb{C}^n$ is locally closed, then $\mathcal{O}_X^{an} = \mathcal{O}_U^{an}/I(X)$ and $I(X)$ is a f.g. sheaf of ideals on U (an open subset of $\mathbb{C}^n$), hence is coherent. $\square$

When $\mathcal{O}_X$ is coherent (over itself), the definition of coherence simplifies.

Proposition 9. *If $(X, \mathcal{O}_X)$ is a ringed space and $\mathcal{O}_X$ is coherent, then an $\mathcal{O}_X$-module $\mathcal{F}$ is coherent iff it is locally finitely presented, i.e. for each $x \in X$ there exists a neighborhood U and an exact sequence*

$$\mathcal{O}_X^p|_U \to \mathcal{O}_X^q|_U \to \mathcal{F}|_U \to 0$$

for some $p, q \in \mathbb{N}$.

II. Analytification of sheaves

Let X a scheme locally of finite type over $\mathbb{C}$. Now to analytify sheaves, we just pull them back along the map $h : X^{an} \to X$.

Definition 10. Given $\mathcal{F}$ a sheaf on X, define $\mathcal{F}^{an}$ as the sheaf of $\mathcal{O}_X^{an}$-modules

$$\mathcal{F}^{an} := h^\star \mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{an}.$$

Since pullback and direct image are adjoints on categories of modules, there is a universal counit map of sheaves

$$\mathcal{F} \to h_\star \mathcal{F}^{an}$$

corresponding to $\mathrm{Id} \in \mathrm{Hom}(\mathcal{F}^{an}, \mathcal{F}^{an})$. On stalks it is given by

$$\begin{aligned} \mathcal{F}_x &\mapsto \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,x}^{an} \\ f &\mapsto f \otimes 1 \end{aligned}$$

Proposition 11. *The maps on stalks $\mathcal{O}_{X,x} \to h_\star \mathcal{O}_{X,x}^{an}$ are flat.*

Proof. This is technically encompassed in the master analytification functor theorem, but we can also see it directly. $\square$

Proposition 12. *Let $\mathcal{F}$ a sheaf on X.*

(1) The analytification functor on sheaves is exact.
(2) The sheaf morphism $\mathcal{F} \to h_\star \mathcal{F}^{an}$ is injective.
(3) If $\mathcal{F}$ is coherent, then $\mathcal{F}^{an}$ is also coherent.

Proof. The first two follow from flatness of $\mathcal{O}_{X,x} \to \mathcal{O}_{X,x}^{an}$. For the third, note that $h^\star \mathcal{O}_X = \mathcal{O}_X^{an}$ by definition. Since for both X and X^{an}, coherent is the same as locally finitely presented, the third also follows from flatness. $\square$

We are now prepared to state the GAGA theorem.

Theorem 13. *Let X a closed subscheme of $\mathbb{P}_{\mathbb{C}}^r$ for some $r \geq 1$. Let $h : X^{an} \to X$ be the analytification morphism. Then the analytification functor induces an equivalence of categories between coherent sheaves on X and coherent sheaves on X^{an}. Moreover, for any coherent $\mathcal{F}$ on X, there are natural isomorphisms*

$$H^i(X, \mathcal{F}) \xrightarrow{\sim} H^i(X^{an}, h^\star \mathcal{F}).$$

III. The proof of GAGA

We follow Serre in proving the theorem in three parts.

Theorem 14. *Let X a closed subscheme of $\mathbb{P}^r_{\mathbb{C}}$ for some $r \geq 1$. Let $h : X^{an} \to X$ be the analytification morphism.*

- *(A) (Cohomology) The natural homomorphisms $H^i(X, \mathcal{F}) \to H^i(X^{an}, \mathcal{F}^{an})$ are isomorphisms for all i.*
- *(B) (Fully faithful) If $\mathcal{F}, \mathcal{G}$ are coherent sheaves on X and $f : \mathcal{F}^{an} \to \mathcal{G}^{an}$ is an analytic homomorphism between their completions, then $f = g^{an}$ for a unique $g : \mathcal{F} \to \mathcal{G}$.*
- *(C) (Essentially surjective) For all analytic coherent sheaves $\mathcal{G}$ on X^{an}, there exists an algebraic coherent sheaf $\mathcal{F}$ such that $\mathcal{F}^{an} \cong \mathcal{G}$.*

The first part's proof passes through Čech cohomology, so we need to ensure Čech and derived functor cohomology agree on both the algebraic and analytic side. The theorem applicable to both is Leray's theorem.

Theorem 15. *Let $\mathcal{F}$ be a sheaf on a topological space X and $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover of X. If $\mathcal{F}$ is acyclic over every finite intersection of elements of $\mathcal{U}$, then*

$$\check{H}^i(\mathcal{U}, \mathcal{F}) = H^i(X, \mathcal{F}).$$

On the algebraic side, the hypotheses hold for any quasi-coherent sheaf on a Noetherian separated scheme. The following result shows that this is still true upon analytification.

Proposition 16. *If $\mathcal{F}$ is a coherent sheaf on X an* affine *scheme of finite type over $\mathbb{C}$, then $\mathcal{F}^{an}$ is acyclic on X^{an}.*

Proof. Hard analysis fact: every analytic subvariety of $\mathbb{C}^n$ is a Stein space. This immediately implies the result. See for example, Hartshorne, "Ample Subvarieties of Algebraic Varieties." □

Upshot: we can compute algebraic Čech cohomology using any affine cover $\mathcal{U}$, and analytic Čech cohomology using the analytification of the same cover.

Now of course we have universal maps for every U:

$$\Gamma(U, \mathcal{F}) \to \Gamma(U, h_\star \mathcal{F}^{an}) = \Gamma(U^{an}, \mathcal{F}^{an}).$$

So analytification induces natural functorial homomorphisms

$$\check{H}^i(\mathcal{U}, \mathcal{F}) \to \check{H}^i(\mathcal{U}^{an}, \mathcal{F}^{an})$$

by interpreting an algebraic cycle as an analytic one.

Proof of (A). Write j for a closed embedding of X into $\mathbb{P}^r_{\mathbb{C}}$ for some r. Closed embeddings (immersions in the analytic case) induce the identity on all sheaf cohomology groups in both settings, so we have

$$H^i(X, \mathcal{F}) = H^i(\mathbb{P}^r, j_\star \mathcal{F}), \qquad H^i(X^{an}, \mathcal{F}^{an}) = H^i(\widetilde{\mathbb{P}}^r_{\mathbb{C}}, j_\star^{an} \mathcal{F}^{an}).$$

So we reduce to the case of $X = \mathbb{P}^r$.

One checks the theorem first for the case of $\mathcal{F} = \mathcal{O}$. In degree zero, both are the constants. In higher degree, $H^i(X, \mathcal{O}) = 0$ by a usual Čech computation. On the analytic side, an explicit Cech computation in Dolbeault cohomology (or de Rham, the Hodge decomposition, and some Kähler geometry) implies that $H^i(X^{an}, \mathcal{O}^{an}) \cong H^{(0,i)}(X) = 0$.

For other line bundles $\mathcal{O}(n)$, we reduce to $n = 0$ by induction on r, similarly to the proof of Serre's finiteness theorem ([Har] III.5.2). Take the SES

$$0 \to \mathcal{O}(n-1) \xrightarrow{\cdot x_r} \mathcal{O}(n) \to \mathcal{O}_H(n) \to 0$$

where $H = \{x_r = 0\} \cong \mathbb{P}^{r-1}$. This remains exact after analytifying. By considering transition functions $\mathcal{O}(n)$ is $\mathcal{O}^{an}(n)$, and analytification is compatible with passage to closed subschemes so $(\mathcal{O}_H)^{an} = (\mathcal{O}^{an})_H$. We get by naturality a ladder of long exact sequences

$$\begin{array}{ccccccccc}
\longrightarrow & H^i(X, \mathcal{O}_H(n)) & \longrightarrow & H^{i+1}(X, \mathcal{O}(n-1)) & \longrightarrow & H^{i+1}(X, \mathcal{O}(n)) & \longrightarrow & H^{i+1}(X, \mathcal{O}_H(n)) & \longrightarrow \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\longrightarrow & H^i(X^{an}, \mathcal{O}_H^{an}(n)) & \longrightarrow & H^{i+1}(X^{an}, \mathcal{O}^{an}(n-1)) & \longrightarrow & H^{i+1}(X^{an}, \mathcal{O}^{an}(n)) & \longrightarrow & H^{i+1}(X^{an}, \mathcal{O}_H^{an}(n)) & \longrightarrow
\end{array}$$

The leftmost and rightmost are both isomorphisms. For both positive and negative n, the five lemma (not fully pictured) lets us reduce to $n = 0$.

Finally, we do the general case by downward induction on i. As in the proof of the finiteness theorem, build a SES

$$0 \to \mathcal{G} \to \mathcal{E} \to \mathcal{F} \to 0$$

in which $\mathcal{E}$ is a direct sum of line bundles (twist until $\mathcal{F}(n)$ is generated by global sections, $\mathcal{O}^m \to \mathcal{F}(n) \to 0$, then twist back down). Note $\mathcal{G}$ is coherent. Looking at the ladder of LES in cohomology similar to the above, we are done by induction and a more careful application of the five lemma, using that the maps on $H^i(\mathcal{G})$ are surjective. □

Proof of (B). Observe that since the analytification functor is compatible with taking open subschemes, we can upgrade the map of hom sets to a natural map of sheaf homs:

$$h^\star \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X^{an}}(h^\star \mathcal{F}, h^\star \mathcal{G}).$$

We claim this is an isomorphism. For coherent sheaves, stalks of sheaf hom and stalks of pullbacks are what one would expect, so the map becomes on stalks

$$\mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, \mathcal{G}_x) \otimes \mathcal{O}_X^{an} \to \mathrm{Hom}_{\mathcal{O}_{X,x}^{an}}(\mathcal{F}_x \otimes \mathcal{O}_X^{an}, \mathcal{G}_x \otimes \mathcal{O}_X^{an}),$$

all tensors over $\mathcal{O}_{X,x}$. Flatness and the fact that $\mathcal{O}_{X,x}$ is Noetherian tells us this is an isomorphism, [EE95], Proposition 2.10. To prove (B) from this, we apply (A) with $i = 0$ to the coherent sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$! □

IV. Proof of GAGA (C)

Throughout this section, let $\mathcal{G}$ an *analytic* coherent sheaf on X^{an}. Uniqueness follows from (B), so we need only show existence. We reduce to the case of projective space.

Reduction of (C) to $X = \mathbb{P}^r$. Let $j : X \hookrightarrow \mathbb{P}^r_{\mathbb{C}}$ a closed embedding with corresponding ideal sheaf $\mathcal{I}$. Then $j_\star^{an}\mathcal{G}$ is a coherent analytic sheaf on $\mathbb{P}^r$. If we assume that (C) holds for projective space, then there exists an algebraic coherent sheaf $\mathcal{F}$ such that $\mathcal{F}^{an} \cong j_\star^{an}\mathcal{G}$. Any local element of the algebraic ideal sheaf $\mathcal{I}$ acts by zero on $j_\star^{an}\mathcal{G} = \mathcal{F}^{an}$ because the sheaf has zero stalks outside of X. This means that the same element acts by zero on $\mathcal{F}$ (since $\mathcal{F} \to h_\star \mathcal{F}^{an}$ is injective) so $\mathcal{F}$ is actually an $\mathcal{O}_{\mathbb{P}^r}/\mathcal{I}$-module, i.e. is actually the pushforward of a coherent sheaf on X. So restricting the isomorphism

$$j_\star^{an}\mathcal{G} = \mathcal{F}^{an}$$

to X shows that $\mathcal{G} = \mathcal{F}^{an}|_X = (\mathcal{F}|_X)^{an}$. □

Henceforth we assume $X = \mathbb{P}^r_{\mathbb{C}}$. Now we proceed by induction on r. The result is trivial for $r = 0$, so assume that (C) holds in dimensions up to $(r-1)$.

We need to check in the analytic setting a fact which is always true algebraically.

Lemma 17. *If (C) holds in dimensions up to $(r-1)$, then for any coherent sheaf $\mathcal{F}$ on X^{an} and any $z \in X^{an}$, there exists an integer n_0 depending on $\mathcal{F}$ and on z such that for $n \geq n_0$, the stalk $\mathcal{F}(n)_z$ is generated by global sections of $\mathcal{F}(n)$.*

Proof. Choose $x_r \in H^0(X, \mathcal{O}(1))$ vanishing at z, and let E be the hyperplane $x_r = 0$ with embedding $j : E \to X$. Then

$$0 \to \mathcal{O}(-1) \xrightarrow{\cdot x_r} \mathcal{O} \to \mathcal{O}_E \to 0$$

as usual. Tensoring by a coherent sheaf isn't always left-exact, but we still have

$$\mathcal{F}(-1) \to \mathcal{F} \to j_\star \mathcal{F}_E \to 0.$$

Let $\mathcal{H}$ be the kernel of the left side. Twisting gives

$$0 \to \mathcal{H}(n) \to \mathcal{F}(n-1) \to \mathcal{F}(n) \to (j_\star \mathcal{F}_E)(n) \to 0.$$

Note that both $j_\star \mathcal{F}_E$ and $\mathcal{H}$ are supported on E, so the inductive hypothesis applies to them.

The sequence above splits as two short exact sequences

$$0 \to \mathcal{H}(n) \to \mathcal{F}(n-1) \to \mathcal{R} \to 0,$$

and

$$0 \to \mathcal{R} \to \mathcal{F}(n) \to (j_\star \mathcal{F}_E)(n) \to 0.$$

The associated long exact sequences in cohomology give

$$H^1(X^{an}, \mathcal{F}(n-1)) \to H^1(X^{an}, \mathcal{R}) \to H^2(X^{an}, \mathcal{H}(n))$$

and

$$H^1(X^{an}, \mathcal{R}) \to H^1(X^{an}, \mathcal{F}(n)) \to H^1(X^{an}, (j_\star \mathcal{F}_E)(n)).$$

By the inductive hypothesis on $(r-1)$, the last terms of these two sequences are zero for $n >> 0$. So both of these are surjections for n sufficiently large.

Analytic fact: the analytic sheaf cohomology spaces are finite dimensional. So we have for $n >> 0$

$$\dim H^1(X^{an}, \mathcal{F}(n-1)) \geq \dim H^1(X^{an}, \mathcal{R}) \geq \dim H^1(X^{an}, \mathcal{F}(n)).$$

So the function $n \mapsto \dim H^1(\mathcal{F}(n))$ is eventually a weakly decreasing function which is always ≥ 0. It follows that it stabilizes! So for $n >> 0$ these inequalities are equalities and

$$H^0(X^{an}, \mathcal{F}(n)) \to H^0(X^{an}, (j_\star \mathcal{F}_E)(n)) \to H^1(X^{an}, \mathcal{R}) \to H^1(X^{an}, \mathcal{F}(n)) \to 0$$

is actually an isomorphism in the last map. By exactness the left map is surjective.

Since we know $\mathcal{F}_E$ is algebraic by induction, applying (A), for n sufficiently large $(\mathcal{F}_E(n))_z$ is generated by global sections. Now a Nakayama's lemma argument will show that this implies the same for $\mathcal{F}(n)$. Write $A = \mathcal{O}_{X,x}^{an}$, $M = \mathcal{F}(n)_x$, and $p = \mathcal{I}_x(E)$. Let $N \subseteq M$ be the sub-A-module generated by global sections. We have

$$\mathcal{F}_E(n)_x = M \otimes_A A/p = M/pM.$$

The above surjectivity implies that the image of N in M/pM generates M/pM. So $M = N + pM = N + mM$ for m the maximal ideal of the local ring. By Nakayama, $M = N$. □

A compactness argument lets us upgrade the lemma so that n_0 doesn't depend on z.

Corollary 18. *For any coherent sheaf $\mathcal{F}$ on X^{an}, there exists an integer n_0 depending only on $\mathcal{F}$ such that for any $n \geq n_0$ and any $z \in X^{an}$, the stalk $\mathcal{F}(n)_z$ is generated by global sections.*

Proof of (C). Let $\mathcal{F}$ as above. By the corollary, there exists an n such that each stalk of $\mathcal{F}(n)$ is generated by the space of sections $H^0(X^{an}, \mathcal{F}(n))$, which is finite-dimensional over $\mathbb{C}$. This finite generation gives a surjection

$$\mathcal{O}_{X^{an}}^{\oplus m} \to \mathcal{F}(n) \to 0,$$

which we can twist down to a surjection

$$0 \to \mathcal{R} \to \mathcal{O}_{X^{an}}(-n)^{\oplus m} \to \mathcal{F} \to 0.$$

We can also apply the same argument to $\mathcal{R}$, as it is also coherent, giving an exact sequence

$$\mathcal{O}_{X^{an}}(-n')^{\oplus m'} \to \mathcal{O}_{X^{an}}(-n)^{\oplus m} \to \mathcal{F} \to 0.$$

The twisting sheaves are algebraic (by definition). By (B) the morphism between them is also algebraic. So the analytic cokernel $\mathcal{F}$ is isomorphic to the analytification of the algebraic cokernel. □

V. Conclusions and Applications

Chow's lemma on proper and projective morphisms lets one weaken the hypothesis of projective to proper. This is the best you can do, (A) is totally false for $\mathbb{A}^r$ and $i = 0$.

Corollary 19 (Chow's Theorem)**.** *Every closed analytic subvariety of $\widetilde{\mathbb{P}^r}$ is algebraic.*

There are many other things that are preserved under analytification.

Theorem 20. *Let $f : X \to Y$ a morphism of schemes locally of finite type over $\mathbb{C}$. Then f is separated iff f^{an} is separated. In particular, X is separated iff X^{an} is Hausdorff.*

Theorem 21. *Let X be a of finite type over $\mathbb{C}$. Then X^{an} is compact iff $X \to \operatorname{Spec} \mathbb{C}$ is proper.*

One or both of these is a good exercise.

GAGA is also foundational in the development of the étale fundamental group.

Theorem 22 (Grothendieck)**.** *Let X be a smooth proper scheme over $\mathbb{C}$. Any finite covering map $Y \to X^{an}$ of topological spaces corresponds to a finite étale cover of X in the category of schemes.*

So analytic covering maps are actually algebraic. So the étale fundamental group, appropriately defined, captures all the information of topological covering maps on X^{an}!

Finally, there is an application to non-abelian group cohomology that I didn't fully understand (section 20). If G is an group scheme and X is an algebraic variety, then germs of morphisms $X \to G$ form an *a priori* noncommutative sheaf of groups, $\mathcal{G}$. The group structure on noncommutative group cohomology breaks down after H^0, but one can still recover $H^1(X, \mathcal{G})$ as a pointed set. The elements of the set correspond to classes of algebraic principal G-bundles over X. There is a natural way to analytify $\mathcal{G}$, and we have a map of pointed sets

$$\varepsilon : H^1(X, \mathcal{G}) \to H^1(X^{an}, \mathcal{G}^{an}).$$

If X is compact, this map is injective. Surjectivity can be more complicated: it holds for the additive group (which is basically $\mathcal{O}_X$), and for $\mathrm{GL}_n(\mathbb{C})$ (since principal GL_n-bundles are all frame bundles of rank-n vector bundles, whose sheaves of sections are all coherent so covered by GAGA above.

Proposition 23. *Let G an algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$ satisfying the condition (R): there exists a rational section $\mathrm{GL}_n(\mathbb{C})/G \to \mathrm{GL}_n(\mathbb{C})$. Then the map ε above is bijective for every projective variety X.*

This holds for G solvable, $G = \mathrm{SL}_n(\mathbb{C}), \mathrm{Sp}_n(\mathbb{C})$. Serre then conjectured at the end of the paper that the condition (R) holds for each semisimple, simply connected G.

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