COMPARISON THEOREM IN ÉTALE COHOMOLOGY

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Étale cohomology was developed by Grothendieck as a theory which interpolates between Galois cohomology of fields and singular cohomology of complex varieties. On the one hand, for varieties X over a field k, the étale cohomology groups retain actions of the absolute Galois group of k, which is a powerful tool for building Galois representations in arithmetic. Simultaneously, for varieties X over $\mathbb{C}$, this algebraic theory is capable of recovering the singular cohomology of X as a complex-analytic space.

Theorem 1 (Comparison Theorem)**.** *Let X a nonsingular variety*[1] *over $\mathbb{C}$. Let X^{an} denote the analytification of X. For any finite abelian group Λ and $r \geq 0$, we have an isomorphism*

$$H^r_{\acute{e}t}(X, \Lambda) \cong H^r(X^{an}, \Lambda).$$

where the right group is singular cohomology.

For simplicity, we retain the simplifying assumption that X is smooth/nonsingular, though the result holds regardless. All sheaves are assumed abelian throughout.

Our main sources will be Milne's lectures [3] and Daniel Litt's course on Étale cohomology and the Weil conjectures. We take for granted the fundamental definitions and statements about Grothendieck topologies and sites (as in [4], for example), but not the foundational properties of étale cohomology. We will freely use the GAGA correspondence, as in [5], some notes on which can be found on my website. Occasionally we use the more thorough, but relatively classical reference [2].

I. Analytic preliminaries

First we observe that for manifolds, singular cohomology computes sheaf cohomology.

Proposition 2. *For M a locally contractible topological space and G an arbitrary commutative ring, there are natural isomorphisms*

$$H^q_{sing}(M, G) \cong H^q_{sh}(M, \underline{G}).$$

Sketch. The idea is to take the standard singular complex over each open subset U, obtaining a presheafified version of the singular complex. These are not sheaves, but after sheafification we obtain a resolution of the constant sheaf $\underline{G}$. One shows these are flasque, so compute sheaf cohomology as desired. For details, see [6], Theorem 4.47. □

Now we introduce a site which serves as intermediary between sheaves on the analytic site and the étale site.

Definition 3. The (small) analytic-étale site on a complex manifold $M^{an-\acute{e}t}$ is the site with objects given by local analytic isomorphisms $U \to M$, morphisms being all triangles

$$\begin{array}{ccc} U & \longrightarrow & V \\ & \searrow & \downarrow \\ & & M \end{array},$$

(these are covering maps of their images) and covering families being jointly surjective families of local homeomorphisms.

[1]A variety is a separated integral scheme of finite type over a field.

For the first part of the proof, we need to show that sheaf cohomology on the analytic(-topological) and analytic-étale sites coincide. Intuitively, this comes from the fact that both of these sites refine one another: given any analytic open cover, it is also an analytic-étale cover in the natural way. Conversely, given an analytic-étale cover, we can refine it to a cover which consists of genuine analytic open immersions (just restrict to neighborhoods on which the maps are homeomorphisms around each point). Thus in the appropriate sense both topologies refine each other, and should compute the same sheaf cohomology. A fully formal argument makes this rigorous.

Proposition 4. *The inclusion of sites* $i : M^{an} \to M^{an-\acute{e}t}$ *induces an equivalence of topoi (i.e. an equivalence of categories)* $\mathrm{Sh}(M^{an}) \cong \mathrm{Sh}(M^{an-\acute{e}t})$.

Proof. The inclusion of sites gives rise to a natural restriction map

$$r : \mathrm{Sh}(M^{an-\acute{e}t}) \to \mathrm{Sh}(M^{an}).$$

Note this map induces an isomorphism of global sections. We claim it is an equivalence of categories. For essential surjectivity, let $\mathcal{F}$ be a sheaf on the analytic topological site. We construct a preimage $\mathcal{G}$ as follows: for each analytic etale cover $\varphi : Y \to M$, pick a cover of Y by opens Y_α on which the map is an isomorphism, and define $\mathcal{G}(Y)$ as the equalizer of the sheaf diagram

$$\mathcal{G}(Y) \to \prod_{\alpha} \mathcal{F}(\varphi(Y_\alpha)) \to \prod_{\alpha,\beta} \mathcal{F}(\varphi(Y_\alpha \cap Y_\beta)).$$

The restriction maps are given by functoriality of equalizer diagrams. If we chose a finer cover of Y, then a diagram chase and sheaf axioms of $\mathcal{F}$ show that we get an isomorphism of equalizers, and so given two covers we can compare them each to the common refinement to see that $\mathcal{G}(Y)$ is well-defined up to natural isomorphism. So we have a well-defined presheaf $\mathcal{G}$. To show it is a sheaf: for both axioms we refine our covers so that they are actual inclusions of opens, and then the above diagram is an equalizer diagram and the sheaf properties hold.

Now we show that the functor is full and faithful.[2] Let $\psi, \pi : \mathcal{F} \to \mathcal{F}'$ be two morphisms of sheaves in $M^{an-\acute{e}t}$ which agree on their restriction to the analytic topological site. Then for any étale cover $Y \to M$, pick a refinement by topological opens Y_α and the gluing axioms for the two sheaves imply that $\mathcal{F}(Y) \to \mathcal{F}'(Y)$ are both the natural map of equalizers for the cover (both $\psi(Y)$ and $\pi(Y)$ send a section of $\mathcal{F}$ to the unique section of $\mathcal{F}'$ gluing the image of local restrictions). So $\psi = \pi$. The same functoriality of equalizer diagrams shows r is full as well as faithful. Pushforward along r induces isomorphisms of global sections, hence induces isomorphisms of sheaf cohomology. □

Incidentally, this argument horribly fails for the corresponding algebraic sites: the Zariski topology is too weak for étale maps to actually be local isomorphisms, so the étale site is strictly larger. Even over an algebraically closed field, where the two sites have the same points, there are more sheaves on the étale site.

II. Algebraic Preliminaries

By the results above, it suffices to relate the algebraic and analytic étale sites on a nonsingular variety X over $\mathbb{C}$. Write $an : X^{\acute{e}t} \to X^{an-\acute{e}t}$ for the analytification functor. This is a morphism of sites because the analytification of an étale map is an analytic local isomorphism. It induces a morphism of topoi, i.e. an adjoint pair $(an_\star, an^\star)$. Because of the directional conventions for maps of sites (a continuous map of topological spaces induces a morphism of sites in the opposite direction), these go the opposite direction one might expect:

$$an_\star : \mathrm{Sh}(X^{an-\acute{e}t}) \to \mathrm{Sh}(X^{\acute{e}t}), \quad an^\star : \mathrm{Sh}(X^{\acute{e}t}) \to \mathrm{Sh}(X^{an-\acute{e}t}).$$

So $an_\star$ is the right-adjoint, i.e. is the left-exact one, as we would expect.

Theorem 5 (Leray Spectral Sequence)**.** *Let* $F : \mathsf{C} \to \mathsf{D}$ *be a morphism of sites. There is a spectral sequence for any* $\mathcal{F} \in \mathrm{Sh}(\mathsf{D})$

$$E_2^{p,q} = H^p(\mathsf{D}, R^q F_\star \mathcal{F}) \implies H^{p+q}(\mathsf{C}, \mathcal{F})$$

[2] A high-level argument notes that since both sites refine the other, there is a natural bijection of "points" in the abstract sense of topoi, and so since a morphism of sheaves is determined by stalks at all points, the data correspond.

Proof. This is a special case of the Grothendieck composition-of-functors spectral sequence, since the global sections functor is pushforward to the site on a single point. □

This allows us to clarify the obstruction to pushforward inducing isomorphisms on cohomology.

Proposition 6. *Let $\mathcal{F}$ be a sheaf on $X^{\acute{e}t}$ and assume that*

(1) the natural map $\mathcal{F} \to an_\star an^\star \mathcal{F}$ is an isomorphism, and
(2) for all $q > 0$, we have

$$(R^q an_\star) an^\star \mathcal{F} = 0.$$

Then we have a natural isomorphism for all i

$$H^i(X^{\acute{e}t}, \mathcal{F}) \cong H^i(X^{an-\acute{e}t}, an^\star \mathcal{F}).$$

Proof. The second condition implies that the E_2 page of the spectral sequence for $an^\star \mathcal{F}$ is supported in a single row, hence the spectral sequence has converged. Composing with the unit gives

$$H^i(X^{\text{ét}}, \mathcal{F}) \cong H^i(X^{\text{ét}}, an_\star an^\star \mathcal{F}) \cong H^i(X^{an-\acute{e}t}, an^\star \mathcal{F}).$$

□

So it suffices for Theorem 1 to show that for any constant sheaf $\mathcal{F}$, the two conditions above hold. Notice that the argument is somewhat more involved than in the previous part because the sites are not equivalent. For example, the half-plane exists as an analytic cover for genus $g \geq 2$ Riemann surfaces, but is not an algebraic variety. Our argument will depend, however, on the fact that finite covers *do* correspond (this explains in some sense the necessity of assuming the coefficients are finite).

Theorem 7 (Riemann Existence Theorem)**.** *For any nonsingular algebraic variety X over $\mathbb{C}$, the analytification functor induces an equivalence of categories between finite étale covers of X and of X^{an}.*

Proof Sketch. First, assume that X is projective. Then GAGA tells us that there is an equivalence of categories between coherent $\mathcal{O}_X$-modules on X and on X^{an}. But the relative Spec construction (and its analytic analog) shows that the data of a finite cover is the same as the data of a coherent $\mathcal{O}_X$-algebra, i.e. algebra objects in the category of $\mathcal{O}_X$-modules. So we have an equivalence of categories between finite covers of X and of X^{an} in the projective case. The étale and unramified covers correspond under the analytification functor, basically because nonsingularity can be checked at the level of strictly henselian or of complete local rings, and the completion factors through the strict henselization.

Now remove the assumption X is projective. One statement of resolution of singularities is that any nonsingular X has a projective model $\overline{X}$ such that the complement of X is a normal crossings divisor. One shows that every cover of X extends to a cover of $\overline{X}$, and so reduces to the projective case. This shows that the analytification functor is essentially surjective on finite covers. We omit the proof of full and faithfulness, as it uses similar techniques we discuss elsewhere. □

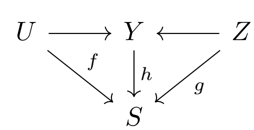
Corollary 8. The étale fundamental group of X is isomorphic to the profinite completion of the analytic fundamental group.

Proof. The above implies that the two have isomorphic finite quotients, and the étale fundamental group is profinite. □

III. Elementary Fibrations

The proof of the theorem proceeds by induction on dimension, and our primary tool for doing so is the following.

Definition 9. An elementary fibration is a map $f : U \to S$ which can be embedded in a commutative diagram



where $U \to Y$ is an open immersion which is dense in the fibers of h (in particular, dense), $Z \to Y$ its complement, h is smooth and projective with fibers which are curves, and g is finite étale with nonempty fibers.

The key result about elementary fibrations is that they exist locally everywhere on nonsingular varieties.

Theorem 10 (Artin)**.** *Let X be a nonsingular variety over $\mathbb{C}$. For any closed point x of X, there is an elementary fibration $U \to S$ for some U Zariski neighborhood of x and S nonsingular.*

Proof Sketch. Take an affine neighborhood and embed it into projective space $U \hookrightarrow \mathbb{P}^N$ for some N. One can do this in such a way that the closure of the image is normal. Taking a generic projection down onto a $\mathbb{P}^{\dim U - 1}$, one obtains a map $U \to S$ which is almost an elementary fibration, and can be made such after blowing up a bad locus. A complete proof can be found in SGA 4, XI [1]. $\square$

This classical result was an important step in the development of the theory, making possible dévissage arguments.

IV. The Main Proof

The following class of sheaves provide a well-behaved generalization of finite constant sheaves more amenable to dévissage arguments.

Definition 11. We say that a sheaf $\mathcal{F} \in \mathrm{Sh}(X^{\text{ét}})$ is *locally constant constructible* (lcc) if it is étale-locally constant, and its stalks are finite.

By the results of the previous section, we have reduced the theorem to checking that for any lcc sheaf $\mathcal{F}$ on $X^{\text{ét}}$, conditions (1) and (2) hold.

Lemma 12. *Let $\mathcal{F}$ lcc on $X^{\acute{e}t}$. The natural map $\mathcal{F} \to an_\star an^\star \mathcal{F}$ is an isomorphism.*

Proof. The condition is certainly Zariski-local, so we may assume $\mathcal{F} = \Lambda$ is a constant sheaf corresponding to a finite abelian group. The pullback of a constant sheaf is constant, so over an arbitrary U the map of sections is

$$\Gamma(U, \Lambda) \to \Gamma(U^{an}, \Lambda).$$

Checking that this map is an isomorphism is just checking that the connected components of U and U^{an} agree, i.e. that analytification preserves the set of connected components! Shockingly, this fact has no easy proof, so we will proceed by induction on dimension, using the existence of elementary fibrations.

Splitting into connected components, we may assume U is connected. So U is connected étale over a smooth variety, hence also irreducible. Shrinking[3] U, we may assume it fits into some elementary fibration

$$\begin{array}{ccccc} U & \longrightarrow & Y & \longleftarrow & Z \\ & \searrow & \downarrow{\scriptstyle h} & \swarrow & \\ & & S & & \end{array}$$

Assume for the sake of contradiction U^{an} were not connected. Then Y^{an} is not connected since U^{an} is dense, hence the complement is of real codimension at least 2 and cannot connect or disconnect the manifold. Moreover, the disjoint union $Y^{an} = Y_1 \bigsqcup Y_2$ consists of complete fibers of h, because the theorem is clearly true for curves (whose analytifications are Riemann surfaces minus finitely many points). The morphism h is smooth (hence flat, hence open) and proper (hence closed) so $h(Y_1), h(Y_2)$ are a disjoint union decomposition of S^{an}. In this way we reduce to checking the claim for S, which is of dimension one less than U. By induction on dimension and the trivial case of curves, we are done. $\square$

This seemingly simple result about connected components holds without assuming smoothness, but is even more difficult.

Lemma 13. *For any lcc sheaf $\mathcal{F}$ on $X^{\acute{e}t}$ and all $q > 0$, we have*

$$(R^q an_\star) an^\star \mathcal{F} = 0.$$

[3] This shrinking is valid because a dense open $U' \subset U$ is irreducible, so also connected, and if the analytification of U' is connected (and dense inside U^{an}) then U^{an} is also connected by an elementary argument.

Proof. Being zero is Zariski-local, so we may assume $\mathcal{F} = \Lambda$ is constant (hence also the pullback) and restrict to a U which fits into an elementary fibration. Recall that derived pushforward is the sheafification of the cohomology presheaf

$$(V \subseteq U) \mapsto H^q(V^{an}, \Lambda),$$

where the cohomology is in the analytic étale site. So what we must show is that that the algebraic étale site is *sufficiently fine to kill cohomology classes*, i.e. that for any $s \in H^q(V^{an}, \lambda)$, there exists an algebraic cover $\{V_i' \to V\}$ such that $s|_{V_i'^{an}} = 0$ for all i. This implies that the stalks of the cohomology presheaf, and thus of the higher pushforward, vanish, giving the result.

We prove this claim by induction on dimension. Consider the analytification of an elementary fibration for V:

$$\begin{array}{ccccc} V^{an} & \longrightarrow & Y^{an} & \longleftarrow & Z \\ & \searrow^{f} & \downarrow & \swarrow & \\ & & S^{an} & & \end{array}$$

The Leray spectral sequence for $f : V^{an} \to S^{an}$ in the analytic étale topology is

$$H^i(S^{an}, R^j f_\star \Lambda) \implies H^{i+j}(V^{an}, \Lambda).$$

Given a class $s \in H^q(V^{an}, \Lambda)$, we can test whether it is zero in the associated graded module of the filtration, which consists of a direct sum

$$s \in H^q(S^{an}, f_\star \Lambda) \oplus \bigoplus_{j>0} H^{q-j}(S^{an}, R^j f_\star \Lambda).$$

Now $R^j f_\star \Lambda$ is the sheafification of $W \mapsto H^j(f^{-1}(W), \Lambda)$, and we know by the inductive hypothesis if $\dim V > 1$ that we can étale-locally on S kill analytic cohomology classes with algebraic covers, hence the preimages of these opens in V are a cover on which we can kill the contribution coming from all terms with $j > 0$.

There remains the contribution from the $j = 0$ term. We claim that since f is an analytic fibration by smooth curves (Riemann surfaces minus finitely many distinct points which move continuously), that $f_\star \Lambda$ is lcc.[4] So $f_\star \Lambda$ is locally constant with value groups still finite, and the contribution of the $j = 0$ term is now our original problem, but on S. Induction tells us we can kill the cohomology classes algebraically, as long as $\dim U > 1$.

It remains to prove the base case, the result for curves. So let $\mathcal{F}$ an lcc sheaf on $U^{an-\acute{e}t}$, a not-necessarily compact Riemann surface. We must show that for any $s \in H^i(U^{an-\acute{e}t}, \Lambda)$, there exists an algebraic étale cover of U on which s is trivial. This cohomology is only nontrivial for $i = 1, 2$. For $i = 2$, we can use any affine Zariski cover: these affine curves are all Riemann surfaces with a point missing, hence contract onto their standard 1-skeleta as polygon quotients and have no 2-cohomology for any group Λ. For $i = 1$, elements of $H^1(U^{an-\acute{e}t}, \Lambda)$ correspond to analytic covering spaces with Galois group Λ (finite!), and the Riemann existence theorem tells us any such is already algebraic! "Torsors split themselves," i.e. pulling back our element of H^1 along the algebraic cover it corresponds to gives zero. This completes the case of curves and the proof. $\square$

V. Applications and Closing Remarks

The comparison theorem provides the most powerful method of computing étale cohomology groups. We remark that while the nonsingularity of X can be relaxed at the cost of much more effort, the assumption that our coefficients are finite cannot. This reflects essentially that the algebraic category is ill-equipped to handle infinite covers, and can be traced back to the fact that GAGA holds only for coherent $\mathcal{O}_X$-algebras/finite covers. The following example is typical.

[4] Indeed, whenever we restrict to a ball in S^{an}, the fibration by punctured Riemann surfaces f is homeomorphic to a trivial fiber bundle, so the sections over a connected open are just sections of Λ over unions of full fibers, which are still connected since the fibers are, and restriction maps are the identity maps since they are identity maps upstairs.

Example 14. The fundamental domain for the complex exponential is an analytic covering space of $\mathbb{G}_m = \mathbb{C} \setminus \{0\}$ with Galois group G, but is not an algebraic variety (nor the complex exponential an algebraic map). Indeed, one can show that

$$H^1(X^{\text{ét}}, \mathbb{Z}) = 0, \quad H^1(X^{an-ét}, \mathbb{Z}) = \mathbb{Z}.$$

Combined with the smooth and proper base change theorem in étale cohomology, we can use the above comparison to compute analytically the $\mathbb{F}_l$ and $\mathbb{Q}_l$ cohomology of varieties over fields of positive characteristic not l. For example, the étale cohomology of a projective algebraic variety over $\mathbb{F}_p$ can be computed by interpreting the defining equations of the variety as equations over $\mathbb{Z}$, and thus over $\mathbb{C}$.

Combined with the Grothendieck-Lefschetz trace formula, this allows us to relate questions of rational points on curves over finite fields to the topological structure of lifts of those curves to $\mathbb{C}$, a powerful correspondence between arithmetic and topology which provides a starting point towards the Weil conjectures.

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