[Usual preamble omitted for space.]

Fixed notation: for a field K, let $\mathcal{O} = K[[x]]$ with maximal ideal $\mathfrak{m} = (x)$ and fraction field $F = K((x))$. Write v for the additive valuation function on F.

1. ****The Incidence Incident.** Recall that GL_n / B can be interpreted as the moduli space of complete flags $0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$, where $\dim V_i = i$. We know that the left B-orbits on G/B are parameterized by W. Can you characterize geometrically which flags lie in the orbit corresponding to w? (It might be good to start with GL_2 and/or with small-length elements of W).

2. ****Klotho, Spinner of Fate.** Let $\mathcal{H} = \mathrm{Fun}(B\backslash \mathrm{GL}_n / B) = \mathbb{C}\langle 1_w \rangle_{w \in W}$, where we write 1_w for the indicator function of the $B \times B$ orbit containing w. In this exercise, we specialize to GL_2 and GL_3, so the Hecke algebra is 2-dimensional or 6-dimensional. Do this problem first for GL_2, then for GL_3.

   (a) The basic convolution relation satisfied by the indicator functions 1_w is that if w, w' satisfy $l(w) + l(w') = l(ww')$, then
   $$1_w \star 1_{w'} = 1_{ww'}.$$
   Taking this as given, show 1_e is the unit for convolution. Assume also that $(1_s + 1_e)^2 = (q+1)(1_s + 1_e)$ for each simple reflection.

   (b) Using these identities, find convolution inverses for 1_w for each $w \in W$. That is, find functions 1_w^{-1} such that $1_w \star 1_w^{-1} = 1_e$. (Hint: start with doing this for the simple reflections).

   (c) Consider the involution $D : \mathcal{H} \to \mathcal{H}$ defined by the formula
   $$D(1_w) = (1_{w^{-1}})^{-1}, \quad D(q^{1/2}) = q^{-1/2}.$$
   Find a skewed basis for $\mathcal{H}$ which is closely related to the indicator basis but is fixed by D. To be precise, find a basis $\{f_w\}_{w \in W}$ for $\mathcal{H}$ such that:

   - $D(f_w) = f_w$, and
   - $f_w = q^{-\ell(w)/2} \sum_{y \leq w} P_{y,w}(q) 1_y$, where $P_{y,w}(q)$ is a polynomial in q with integer coefficients.

3. **If you can't solve your problem, cut it in half!** Let V a vector space with symplectic form ω. Construct a reasonable bijection between (1) complete flags $V_0 \subset V_1 \subset \cdots \subset V_{2n}$ with $\dim V_i = i$ which are *symmetric* in the sense that $V_i^{\perp} = V_{2n-i}$, and (2) flags consisting of *isotropic* subspaces $V_0 \subset \cdots \subset V_n$ with $\dim V_i = i$. This gives an equivalent way to describe the flag variety Sp_{2n} / B.

4. **In which Connor begrugingly agrees that maybe the students should at some point look at a textbook.**

   (a) Read section 3.2 of Getz-Hahn's "Introduction to Automorphic Representations."[1] Pay special attention to Prop. 3.2.1 and Theorem 3.2.2.

[1] https://sites.duke.edu/jgetz/files/2022/04/Graduate_Text.pdf

(b) Prove that if G is compact (Hausdorff), it is necessarily unimodular (Hint: think about the modular character and subgroups of $\mathbb{R}_{>0}$).

5. **I said the REAL Iwasawa decomposition.** Prove that $\mathrm{GL}_n(F) = B(F)I$, that is, that every matrix in $\mathrm{GL}_n(F)$ can be written as a product of an upper triangular matrix and a matrix in I. This should be similar to your existing algorithms. Deduce that $\mathrm{GL}_n(F) = B(F)\,\mathrm{GL}_n(\mathcal{O})$.

6. **The arc of history is like, a parabola or something.** Pick n, and fix an ordered partition $n_1 + \cdots + n_k = n$. Let P denote the corresponding "block upper-triangular matrix group," and M the corresponding "block-diagonal matrix group." For example, if our partition is $3 + 1 = 4$, then the corresponding subgroups of GL_4 is

$$P = \begin{pmatrix} \star & \star & \star & \star \\ \star & \star & \star & \star \\ \star & \star & \star & \star \\ 0 & 0 & 0 & \star \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} \star & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & 0 \\ 0 & 0 & 0 & \star \end{pmatrix}.$$

Find, for any such P and M, a normal subgroup $N \trianglelefteq P$ such that M and N are complementary, i.e. $N \cap M = \{I\}$ and $MN = P$.

7. **Off to orbit!** (Harder but ultimately *; uses the previous three problems) Let $G = \mathrm{GL}_2$ and consider the matrix

$$\gamma = \begin{pmatrix} a_1 x^k & 0 \\ 0 & a_2 x^k \end{pmatrix}$$

where $a_1 \neq a_2 \in k$. The goal of this problem is to compute the orbital integral

$$\int_{G_\gamma(F)\backslash G(F)} 1_{G(\mathcal{O})}(g^{-1}\gamma g)\,dg.$$

Note that when $k = 0$ this is the computation I did in class (the answer was 1 with respect to a smart choice of normalizations for the measures on $G(F)$ and $G_\gamma(F)$). Don't worry, we'll do it in a smarter way here.

(a) What is the centralizer $G_\gamma(F)$ of γ? That is, which elements in $G(F)$ commute with γ?

(b) Describe Haar measures on G and G_γ. Find a compactly supported function f on $G(F)$ such that

$$\int_{h \in G_\gamma(F)} f(gh)\,dh = 1_{G(\mathcal{O})}(g^{-1}\gamma g).$$

Using this, rewrite the orbital integral as an integral over $G(F)$.

(c) Using the decomposition $G(F) = B(F)G(\mathcal{O})$, write the orbital integral as an iterated integral $\int_{B(F)} \int_{G(\mathcal{O})}$. Is it with respect to the left or right measure on $B(F)$? (Hint: use results from Getz-Hahn section 3.2).

(d) Now using the decomposition $B(F) = T(F)N(F)$ from the previous problem, write the orbital integral as an iterated integral $\int_{T(F)} \int_{N(F)} \int_{G(\mathcal{O})}$.

(e) If you've done the previous parts in a reasonable way, the innermost and outermost integrals will be really easy (the functions will be constant). Thus, the orbital integral reduces to an integral just over $N(F)$. By explicitly thinking about which matrices in $N(F)$ conjugate γ to still have integer entries, compute the value of the orbital integral. Your answer should be a function of k, but interestingly will be independent of a_1, a_2 as long as they are distinct.