Problem Set 8

The following exercises are open-ended and sometimes under-specified, sometimes intentionally so. You should always feel free to ask for help from the mentors (as well as from your fellow students). **Read all the exercises before beginning to work.** You should spend at least 1 hour thinking about at least one of the starred problems.

Fixed notation: for a field K, let $\mathcal{O} = K[[x]]$ with maximal ideal $\mathfrak{m} = (x)$ and fraction field $F = K((x))$. Write v for the additive valuation function on F.

1. ***Flags of many colors.** Using the notion of isotropic from the previous problem, give an intrinsic characterization of B, the upper triangular subgroup in Sp_{2n}. Use this to give an interpretation of Sp_{2n}/B as the moduli space of certain kinds of flags.

2. **Heavy Lifting.** In this exercise, we will study how polynomials $f \in \mathcal{O}[\lambda]$ factor. Denote $\overline{f} \in k[\lambda]$ the polynomial of constant terms, that is, the image of f under the map $\mathcal{O}[\lambda] \to (\mathcal{O}/m)[\lambda] = k[\lambda]$.

   (a) First, let f be a monic quadratic polynomial. Assume the characteristic of k is not 2. Assume that we have a factorization

   $$\overline{f} = (\lambda - \overline{a})(\lambda - \overline{b})$$

   with $\overline{a} \neq \overline{b} \in k$. Prove that f admits a factorization $(\lambda - a)(\lambda - b)$ such that a has constant term $\overline{a}$ and b has constant term $\overline{b}$.

   Here are some hints, as this can be tricky. The constants $\overline{a} = a_0, \overline{b} = b_0$ are "approximate zeros" of f, in the sense that $f(a_0), f(b_0)$. You want to show that you can inductively add higher-order modifications to build power series $a_n, b_n \in \mathcal{O}$ such that $f(a_n)$ and $f(b_n)$ get closer to 0, i.e. are contained in higher and higher powers of $\mathfrak{m}$. Take inspiration from Newton's method for approximating zeroes of real polynomials.[1]

   (b) Can you extend this to more general kinds of factorizations? For example, what if f is degree n and $\overline{f} = \prod_i (\lambda - \overline{a}_i)$ factors into distinct linear factors?

   (c) Can you figure out what goes wrong if $\overline{a}, \overline{b}$ are not distinct in k? Can you strengthen hypotheses to match this?

   (d) Application: It is a fact from basic number theory that -1 is a square in $\mathbb{F}_p$ if and only if $p \equiv 1, 2 \mod 4$. Using this, for which fields $F = \mathbb{F}_p((t))$ is -1 a square?

3. **Further afield.** In this exercise, we will start to investigate the field theory of the Laurent series field $F = \mathbb{F}_q((x))$.

   (a) Let K/F be a finite field extension, i.e. $K = F[x]/f(x)$ for some $f \in F[x]$ irreducible. Show that there exists a unique valuation v_K valued in $\mathbb{Q}$ with the usual properties (ultrametric inequality, $v_K(xy) = v_K(x) + v_K(y)$, $v_K(0) = \infty$) such that $v_K|_F = v$.

   (b) We call an extension K/F *unramified* if v_K still has image in $\mathbb{Z}$ – that is, if 1 is still the smallest possible valuation in K. Otherwise, we call K/F *ramified*. Construct examples of specific f generating ramified and unramified quadratic extensions of $\mathbb{F}_3((x))$. (The previous problem may inspire you).

[1] https://en.wikipedia.org/wiki/Newton%27s_method

4. ***It's semi-simple!** Let $A \in \mathrm{Mat}_{n\times n}(k)$ for any field k. We say that A is *semisimple* if every A-stable subspace has a complement. That is, for every $W \subseteq k^n$ such that $A(W) \subset W$, there exists a W' such that $A(W') \subset W'$ and $W \oplus W' = k^n$ is a direct sum decomposition.

   (a) Consider the real-valued matrix R corresponding to a quarter-rotation in the plane. Show that R is semisimple.

   (b) Prove that diagonalizable matrices are semisimple.

   (c) Prove in fact a stronger biconditional characterization: Let $\overline{k}$ be an algebraically closed field containing k. Prove that a matrix $A \in \mathrm{Mat}_{n\times n}(k)$ is semisimple if and only if, when we view A as a matrix with $\overline{k}$-entries, it is diagonalizable. (Hint: Jordan canonical form works over any algebraically closed field $\overline{k}$).

   (d) Call A *regular semisimple* if, when viewed in $\overline{k}$, A is diagonalizable with distinct eigenvalues. Show that a matrix is regular semisimple if and only if its centralizer in $\mathrm{GL}_n(\overline{k})$ is isomorphic to $(\overline{k}^{\times})^n$.

   (e) What is the centralizer of the matrix R from part (a) in $\mathrm{GL}_2(\mathbb{R})$?

5. ***The Incidence Incident.** Recall that GL_n/B can be interpreted as the moduli space of complete flags $0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$, where $\dim V_i = i$. We know that the left B-orbits on G/B are parameterized by W. Can you characterize geometrically which flags lie in the orbit corresponding to w? (It might be good to start with GL_2 and/or with small-length elements of W).

6. **Kuala Lumpur.** Let $\mathcal{H} = \mathrm{Fun}(B\backslash \mathrm{GL}_n/B) = \mathbb{C}\langle 1_w\rangle_{w\in W}$, where we write 1_w for the indicator function of the $B \times B$ orbit containing w. In this exercise, we specialize to GL_2 and GL_3, so the Hecke algebra is 2-dimensional or 6-dimensional. Do this problem first for GL_2, then for GL_3.

   (a) The basic convolution relation satisfied by the indicator functions 1_w is that if w, w' satisfy $l(w) + l(w') = l(ww')$, then

   $$1_w \star 1_{w'} = 1_{ww'}.$$

   (It is only when the elements "overlap" that the convolution is complicated). Taking this as given, show 1_e is the unit for convolution. Show also that $(1_s + 1_e)^2 = (q+1)(1_s + 1_e)$ for each simple reflection.

   (b) Using these identities, find convolution inverses for 1_w for each $w \in W$. That is, find functions 1_w^{-1} such that $1_w \star 1_w^{-1} = 1_e$. (Hint: start with doing this for the simple reflections).

   (c) Consider the involution $D : \mathcal{H} \to \mathcal{H}$ defined by the formula

   $$D(1_w) = (1_{w^{-1}})^{-1}.$$

   We would like to find a skewed basis for $\mathcal{H}$ which is closely related to the indicator basis but is fixed by D. To be precise, find a new basis $\{f_w\}_{w\in W}$ for $\mathcal{H}$ with the following properties:

   - $D(f_w) = f_w$, and
   - the change of basis matrix from $\{1_w\}_{w\in W}$ to $\{f_w\}_{w\in W}$ is upper triangular, has 1's on the diagonal, and everything above the diagonal is a polynomial in q which is divisible by q.