Problem Set 7

The following exercises are open-ended and sometimes under-specified, sometimes intentionally so. You should always feel free to ask for help from the mentors (as well as from your fellow students). **Read all the exercises before beginning to work.** You should spend at least 1 hour thinking about at least one of the starred problems.

Fixed notation: for a field K, let $\mathcal{O} = K[[x]]$ with maximal ideal $\mathfrak{m} = (x)$ and fraction field $F = K((x))$.

1. **Infinitesimal Baby Steps.** Using the "differentiation" procedures from this morning (e.g. applying the defining equations to matrices of the form $I + \epsilon A$), find equations describing the Lie algebras of $\mathrm{SL}_n, \mathrm{Sp}_{2n}$ and O_n (this is the orthogonal group, the group of invertible matrices which satisfy

$$A^T H A = H, \qquad \text{where} \qquad H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \cdot^{\cdot^{\cdot}} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

   Use this to compute the dimensions of these groups.

2. **Your new favorite group.** Let $G = \mathrm{Sp}_4$ denote the usual symplectic group. Let $B \leq G$ denote the subgroup of matrices which are upper triangular and contained in G, and T the diagonal matrices in G. We have residual conjectures about Sp_4. Let's resolve them.
   (a) Verify that $D_8 = N(T)/T$.
   (b) We conjectured that D_8 also parameterized the $B \times B$ orbits in G. Prove this. Can you figure out a nice order on D_8 that you expect to match the orbit closure relation? Hint: what nice set of generators should replace the adjacent transpositions $(i\ i+1)$ that we used for S_n?
   (c) Can you do affine Sp_4? That is, $\mathrm{Sp}_4(\mathcal{O}) \backslash \mathrm{Sp}_4(F) / \mathrm{Sp}_4(\mathcal{O})$?

3. **Convolution confusion.** Let $\mathcal{H}_{n,q}$ be the Hecke algebra for $\mathrm{GL}_n(\mathbb{F}_q)$, i.e. the convolution algebra of functions on $B \backslash GL_n(\mathbb{F}_q)/B$. Let's write T_w for the indicator function 1_{BwB}.
   (a) Let s_i denote the transpositions $(i\ i+1)$ - we will call these the *simple reflections*. Show that $T_{s_i}^2 = (q-1)T_{s_i} + q$. (Hint: what quadratic relation did we calculate in class?)
   (b) Show that if $\ell(ww') = \ell(w) + \ell(w')$, then $T_w T_{w'} = T_{ww'}$.
   (c) Using the previous part (or otherwise), check the braid relation:

$$T_{s_i} T_{s_{i+1}} T_{s_i} = T_{s_{i+1}} T_{s_i} T_{s_{i+1}}.$$

   (d) How do these relations compare with the symmetric group? Do you think we have all the relations for $\mathcal{H}_{n,q}$?

4. **Monoidal monastery.** Let R be a commutative ring. Consult a holy text of your choice and learn enough to understand this sentence: the category R-mod of R-modules is monoidal, with monoidal structure given by tensoring over R.

5. **There can only be one!** Let V be a (finite-dimensional) vector space. A symplectic form on V is a bilinear form $\langle , \rangle$ which is alternating ($\langle v, v\rangle = 0$ for all v) and nondegenerate (for any $v \neq 0$, there exists a $w \in V$ such that $\langle v, w\rangle \neq 0$).

   (a) Prove that alternating implies skew-symmetric ($\langle v, w\rangle = -\langle w, v\rangle$) and the converse holds as long as the characteristic of k is not 2.

   (b) Recall that given any ordered basis E, a bilinear form is represented by a matrix: for some B,
   $$\langle , \rangle_E : (v, w) \mapsto v^T B w.$$
   Prove that for any symplectic form, there exists a basis which presents the symplectic form in the standard form
   $$J = \begin{pmatrix} & & & & & 1 \\ & & & & \cdot^{\cdot^{\cdot}} & \\ & & & 1 & & \\ & & -1 & & & \\ & \cdot^{\cdot^{\cdot}} & & & & \\ -1 & & & & & \end{pmatrix}$$
   Moral: up to change of basis, there is only one symplectic form. (Hint: Proceed by induction. This is essentially a Gram-Schmidt process).

6. **There can be somewhat more than one!** Given a vector space V with a bilinear form $\langle , \rangle$, define a subspace W to be *isotropic* if $\langle , \rangle|_W = 0$. That is, for any two vectors $w_1, w_2 \in W$, $\langle w_1, w_2\rangle = 0$.

   (a) Prove that a subspace $W \subset V$ is isotropic if and only if the complement $W^\perp$ contains W. (Weird!) Prove that nevertheless, as long as the bilinear form is nondegenerate, dimensions of complements work correctly: $\dim W^\perp = \dim V - \dim W$.

   (b) Assume that $\langle , \rangle$ is the standard symplectic form J. Describe the maximal isotropic subspaces of V.

   (c) Assume that $\langle , \rangle$ is the inner product
   $$\Omega_1 = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}.$$
   Describe the maximal isotropic subspaces of V.

   (d) Assume that $\langle , \rangle$ is the "dot product" inner product
   $$\Omega_2 = I.$$
   Describe the maximal isotropic subspaces of V. Combining with the previous part, conclude there does not exist a change of basis that will change Ω_1 into Ω_2.

7. **Flags of many colors.** Using the notion of isotropic from the previous problem, give an intrinsic characterization of B, the upper triangular subgroup in Sp_{2n}. Use this to give an interpretation of Sp_{2n}/B as the moduli space of certain kinds of flags.

8. **The Most Dangerous Game.** Prove any other of our conjectures from the table today. Work on your project writeup.